Contextual Semantics: From Quantum Mechanics to Logic, Databases, Constraints, and Complexity Lecture 2

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Recap of Basic Mathematical Setting

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A setting for **contextual probability**.

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The model is **contextual** if there is no such global section.

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The effect of applying this to a probabilistic model is exactly to produce the boolean model corresponding to its support: for each context C, the probability distribution $e_C \in \mathcal{D}_{\mathbb{R}_{\geq 0}}(O^C)$ is mapped to the finite non-empty subset

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This means that no-go theorems proved at the possibilistic level are stronger (in fact, **strictly** stronger) than those proved at the probabilistic level.

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As we shall now see, they arise very directly in a number of familiar CS settings.

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branch-name	**account-no**	**customer-name**	**balance**
Cambridge	10991-06284	Newton	£2,567.53
Hanover	10992-35671	Leibniz	€11,245.75
...	...	...	...

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Does this look familiar?

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The compatibility condition for an instance is **projection consistency**:

$$R_A|_{A \cap B} = R_B|_{A \cap B}$$

means that the two relations have the same projections onto their common set of attributes.

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Proposition

Let $(R_1, \ldots, R_k)$ be an instance for the schema $\Sigma = \{A_1, \ldots, A_k\}$. Define $R := \bowtie_{i=1}^{k} R_i$. Then a universal relation for the instance exists if and only if $R|_{A_i} = R_i$, $i = 1, \ldots, k$, and in this case R is the largest relation in $\mathcal{R}(\bigcup_i A_i)$ satisfying the condition for a global section.

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set of attributes defining a relation table	compatible set of measurements
database schema	measurement cover
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We can also consider probabilistic databases and other generalisations;
cf. provenance semirings.

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These formulas define possibilistic models; and their satisfying assignments

$$v : \mathcal{X} \longrightarrow O$$

correspond exactly to **global sections** of these models.

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Proof

From the previous Proposition, it is clear that HV(n) is defined by the following second-order formula interpreted over finite structures (U, e):

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By standard quantifier manipulations, this can be brought into an equivalent Σ^1_1 form, and hence $\mathsf{HV}(n)$ is in NP. □

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- Main results: ROBUST 3-COLOURABILITY and ROBUST 2-SAT are NP-complete.
- These are used to show that HV(n), $n > 2$, is NP-complete; smaller instances are in PTIME.

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- These are used to show that HV(n), $n > 2$, is NP-complete; smaller instances are in PTIME.
- The robust paradigm is an interesting and non-trivial extension of current theory, and worthy of further study.

Contextual Semantics

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The **Contextual semantics hypothesis**: we can find common mathematical structure in all these diverse manifestations, and develop a widely applicable theory.

Valuation Algebras

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Subsequently understood to be applicable to a very wide range of inference problems. E.g. *Generic Inference* by Marc Pouly and Jürg Kohlas, Wiley 2011;

> *The inference task can then be described for general valuation algebras, which leads to a single computational problem that abstracts numerous important and seemingly different applications in computer science as for example query answering in databases, the evaluation of Bayesian and Gaussian networks, the solution of constraint, equation and inequality systems, satisfiability and theorem proving in logics, smoothing and filtering in linear dynamic systems and hidden Markov chains, the computation of discrete Fourier and Cosine transforms, various applications of path problems and coding schemes, sparse matrix techniques or numerical and symbolic partial differentiation. The properties of valuation algebras enable the efficient solution of all these problems with a single generic algorithm that exploits so-called tree-decomposition techniques.*

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In practice, take $D \subseteq \mathcal{P}(\mathrm{Var})$, where Var is a set of variables.

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Operations:

- **Domain** $d : \Phi \to D$; $\phi \mapsto d(\phi)$.
- **Projection** $\Phi \times D \to \Phi$; $(\phi, x) \mapsto \phi^{\downarrow x}$, for $x \subseteq d(\phi)$.
- **Combination** $\Phi \times \Phi \to \Phi$; $(\phi, \psi) \mapsto \phi \otimes \psi$.

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Axioms:

- (A1) $(\Phi, \otimes)$ forms a commutative semigroup.
- (A2) $d(\phi \otimes \psi) = d(\phi) \cup d(\psi)$.
- (A3) $d(\phi^{\downarrow x}) = x$.
- (A4) $(\phi^{\downarrow y})^{\downarrow x} = \phi^{\downarrow x}, \quad (x \subseteq y \subseteq d(\phi))$.
- (A5) $(\phi \otimes \psi)^{\downarrow z} = \phi \otimes \psi^{\downarrow z \cap d(\psi)}, \quad (d(\phi) \subseteq z \subseteq d(\phi) \cup d(\psi))$.
- (A6) $\phi^{\downarrow d(\phi)} = \phi$.

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In most applications, valuations with domain x are distributions on assignments $\Pi_{X \in x} V_X$ valued in a semiring R. Projection is marginalisation.

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A basic form of this localisation is **variable elimination**:

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This leads to the fusion rule: if $x = \{X_1, \ldots, X_n\}$, then

$$(\phi_1 \otimes \cdots \otimes \phi_n)^{\downarrow x} \;=\; \bigotimes \mathsf{Fus}_{X_n}(\cdots(\mathsf{Fus}_{X_1}(\{\phi_1, \ldots, \phi_n\}))\cdots)$$

where

$$\mathsf{Fus}_Y(\{\phi_1, \ldots, \phi_n\}) \;=\; \{\psi^{-Y}\} \cup \{\phi_i \, : \, Y \notin d(\phi_i)\}.$$

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The Shafer-Shenoy architecture is the most basic, and works for any valuation algebra. The L-S and HUGIN architectures achieve more efficiency, assuming additional properties of the valuation algebra (divisibility), while logical inference can be performed if we assume idempotence.

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  The properties of this map given by (A1) and (A5) say that it is a natural transformation making $\mathcal{F}$ into a **symmetric monoidal functor**, with respect to the monoidal structure on D as a join semilattice, and $(\mathbf{Set}, \times)$ as a monoidal category.

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- In particular, the crucial axiom (A5) is (essentially) **naturality**.

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This may suggest a novel extension of DL with combination "built in", which may be well adapted to studying generic inference as captured by valuation algebras.

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However, using the cover

$$f_1 : x \mapsto a, \quad f_2 : y \mapsto b, \quad f_3 : z \mapsto b$$

we do have a gluing:

$$s = \{John(a), Man(a), donkey(b), \neg Man(b), grey(b)\}.$$

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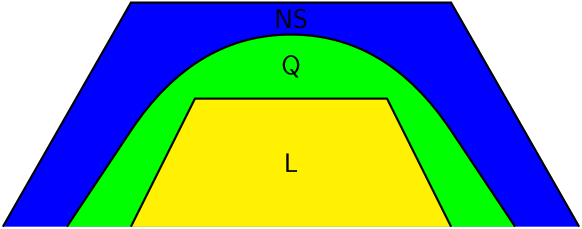

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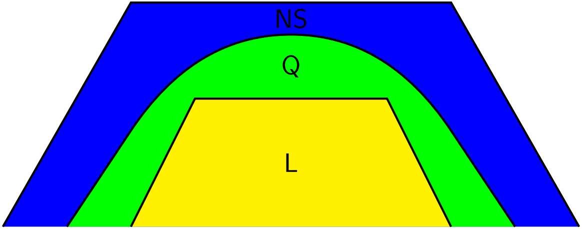

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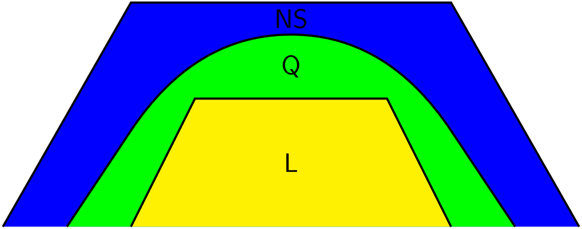

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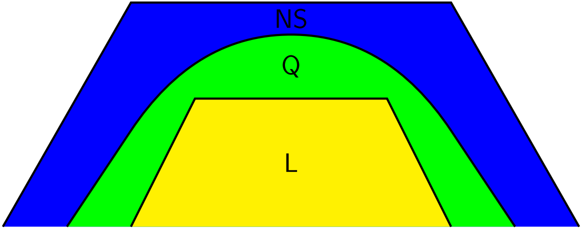

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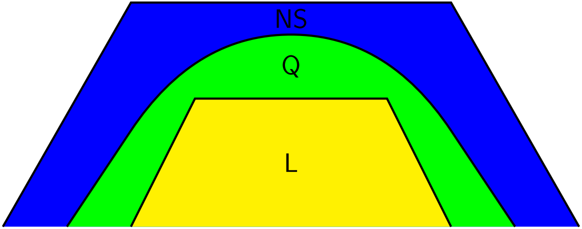

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- Quantum measures on global assignments give quantum models?

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For each choice of measurement $\overline{m} \in M$, and outcome $\overline{o} \in O$, the usual 'statistical algorithm' of quantum mechanics defines a probability $p_{\overline{m}}(\overline{o})$ for obtaining outcome $\overline{o}$ from performing the measurement $\overline{m}$ on ρ:

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We define a relational empirical model $e \subseteq M \times O$ by

$$e(\overline{m}, \overline{o}) \;\equiv\; p_{\overline{m}}(\overline{o}) > 0.$$

Thus e arises as the 'possibilistic collapse' of the usual quantum formalism.

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We consider the two-qubit system, with X_2 and Y_2 measurement in the computational basis. We take $R = 0$, $G = 1$. The eigenvectors for X_1 are taken to be

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The possibilistic collapse of this model is thus a Hardy model.

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Proof Outline

The condition for quantum realization of a relational model can be written as the existence of a list of complex matrices satisfying some algebraic conditions. These can be written in terms of the entries of the matrices, and we can use the standard representation of complex numbers as pairs of reals.

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This fragment has PSPACE complexity (Canny). Moreover, the sentence can be constructed in polynomial time from the given relational empirical model. Hence membership of QM is in PSPACE.

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Seems hard . . .

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