

Minimal Weighted Clones with Boolean Support

Peter G. Jeavons, Andrius Vaicenaivičius and Stanislav Živný

Department of Computer Science, University of Oxford, UK

{peter.jeavons, andrius.vaicenavicius, standa.zivny}@cs.ox.ac.uk

Abstract—We study algebraic structures called **weighted clones**. These structures characterise the computational complexity of discrete optimisation problems of special form, known as **valued constraint satisfaction problems**. We identify all *minimal weighted clones* for every Boolean support clone.

I. INTRODUCTION

The Valued Constraint Satisfaction Problem (VCSP) is a general framework of discrete optimisation problems [7], [26]. It is a generalisation of the Constraint Satisfaction Problem (CSP), a framework that captures fundamental decision problems from artificial intelligence, database theory, logic, and graph theory [9]–[11], [17]. The VCSP framework, in addition to the problems above, captures many discrete optimisation problems from combinatorial optimisation and computer vision [7], [26].

In analysing the computational complexity of the CSP with restricted constraint types (the so-called *non-uniform* CSP), the algebraic approach was one of the main tools [1]–[3], [5], [13]. This approach was obtained by establishing the link between CSPs and universal algebra in [4], [15], [16]. Recently, the link between VCSPs and the algebraic structures, called *weighted clones*, has also been established [6]. This algebraic approach has been used to classify the complexity of various fragments of the VCSP [12], [19], [22] and to reduce the question of whether the complexity of the VCSP has a *dichotomy* to the well-established Algebraic CSP Dichotomy Conjecture [18], [20].

Weighted clones are known to capture the computational complexity of the VCSP with any specified set of valued constraints [6]. In particular, weighted clones with Boolean support clone capture the complexity of the VCSP on the Boolean domain. In this paper we classify all *minimal weighted clones* for each possible Boolean support clone. These minimal weighted clones correspond to maximal weighted relational clones and hence to maximal sets of possible valued constraints. We refer the reader to the recent survey [14] that provides motivation for this work and connects it to the computational complexity of the VCSP. Other recent work on weighted clones includes [23]–[25].

II. PRELIMINARIES

Let D be finite a set. A function $f : D^k \rightarrow D$ is called a k -ary **operation** on D . We denote by \mathbf{O}_D the set of all finitary operations on D and by $\mathbf{O}_D^{(k)}$ the k -ary operations in \mathbf{O}_D .

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Definition II.1. The k -ary **projections** on D are the operations $\pi_i^{(k)} : D^k \rightarrow D$ such that $\pi_i^{(k)} : (x_1, \dots, x_k) \mapsto x_i$, where $i \in \{1, \dots, k\}$ and $(x_1, \dots, x_k) \in D^k$.

Definition II.2. Let $f \in \mathbf{O}_D^{(k)}$ and $g_1, \dots, g_k \in \mathbf{O}_D^{(\ell)}$. Define the **superposition** of f and g_1, \dots, g_k to be the ℓ -ary operation $f[g_1, \dots, g_k] : D^\ell \rightarrow D$ such that for any $(x_1, \dots, x_\ell) \in D^\ell$

$$(x_1, \dots, x_\ell) \mapsto f(g_1(x_1, \dots, x_\ell), \dots, g_k(x_1, \dots, x_\ell)).$$

Definition II.3. A set $C \subseteq \mathbf{O}_D$ is called a **clone** of operations on D if it contains all the projections on D and is closed under superposition. For each $F \subseteq \mathbf{O}_D$ we define $\text{Clone}(F)$ to be the smallest clone containing F . We denote by $C^{(k)}$ the k -ary operations in clone C .

Example 1. For any set D , the set \mathbf{O}_D , and the set \mathbf{J}_D containing all projections on D (of all arities), are both clones.

Example 2. For any totally ordered set D , the clone $C := \text{Clone}(\{\wedge\})$, where \wedge is the binary minimum operation on D , consists of all operations on D which return the minimum element of some (non-empty) subset of their arguments. In particular, $C^{(2)}$, the set of *binary* operations in C , contains precisely three operations, namely, $\pi_1^{(2)}, \pi_2^{(2)}$, and \wedge .

Definition II.4. We define a k -ary **weighting** of a clone C to be a function $\omega : C^{(k)} \rightarrow \mathbb{Q}$ such that $\sum_{f \in C^{(k)}} \omega(f) = 0$, and

$\omega(f) < 0$ only if f is a projection. We denote by $\mathbf{W}_C^{(k)}$ the set of all k -ary weightings of C , and by \mathbf{W}_C the set of all possible weightings of C (of arbitrary arity).

The notation $(\alpha_1, \dots, \alpha_m \mid f_1, \dots, f_m)$, where $\alpha_i \in \mathbb{Q}$ and $f_i \in C^{(k)}$, will denote the function $\omega : C^{(k)} \rightarrow \mathbb{Q}$ given by

$$\omega(f) = \sum_{\substack{i \in \{1, \dots, m\} \\ f_i = f}} \alpha_i, \text{ for every } f \in C^{(k)}. \quad (1)$$

Note that some of the functions f_i might be equal, and $\omega(f) = 0$ when the summation is empty.

Example 3. The following functions are weightings of $\text{Clone}(\{\wedge\})$, which was described in Example 2:

$$\left(-1, 1 \mid \pi_1^{(2)}, \wedge\right) \text{ and } \left(-2, 1, 1 \mid \pi_1^{(2)}, \pi_2^{(2)}, \wedge\right).$$

The following functions are *not* weightings:

$$\left(1, -1 \mid \pi_1^{(2)}, \wedge\right) \text{ and } \left(-1, -1, 3 \mid \pi_1^{(2)}, \pi_2^{(2)}, \wedge\right)$$

as the first assigns a negative weight to a non-projection and in the second the weights do not add up to zero.

Definition II.5. For any $\omega = (\alpha_1, \dots, \alpha_m \mid f_1, \dots, f_m) \in \mathbf{W}_C^{(k)}$ and any $g_1, \dots, g_k \in C^{(\ell)}$, we define the **superposition** of ω and g_1, \dots, g_k to be the function $\omega[g_1, \dots, g_k] : C^{(\ell)} \rightarrow \mathbb{Q}$ given by $(\alpha_1, \dots, \alpha_m \mid f_1[g_1, \dots, g_k], \dots, f_m[g_1, \dots, g_k])$.

If the result of the superposition is a valid weighting, then it is said to be a **proper superposition**.

Definition II.6. A **weighted clone**, W , is a non-empty set of weightings of some fixed clone C which is closed under:

- (i) nonnegative scaling by rational numbers;
- (ii) addition of weightings of equal arity; and
- (iii) proper superposition with operations from C .

The clone C is called the **support** of W . For any $S \subseteq \mathbf{W}_C$, we define $\langle S \rangle$ to be the smallest weighted clone containing S and say that $\langle S \rangle$ is **generated** by S . We write $\langle \omega_1, \dots, \omega_n \rangle$ for the weighted clone $\langle \{\omega_1, \dots, \omega_n\} \rangle$.

For a weighting $\omega \in \mathbf{W}_C^{(k)}$, we say that ω is a **zero-valued** weighting if $\omega(f) = 0$ for all $f \in C^{(k)}$. Otherwise, we say that ω is a **non-zero** weighting. A weighted clone that contains a non-zero weighting is said to be **non-trivial**.

Example 4. For any clone C , the set \mathbf{W}_C^0 containing all zero-valued weightings of C , and the set \mathbf{W}_C containing all possible weightings of C , are both weighted clones with support C .

For any two weighted clones W' and W'' with support C , we define $W' \wedge W'' := W' \cap W''$ and $W' \vee W'' := \langle W' \cup W'' \rangle$. For any clone C on a finite set D , the set of all weighted clones with support C ordered by inclusion forms a complete lattice, with the meet and join operations as defined above, and with the least element \mathbf{W}_C^0 and the greatest element \mathbf{W}_C . We denote this lattice by $\text{Lat}(C)$ and call it the **lattice** of a clone C .

Recall that an **atom** is a minimal non-trivial element in a lattice. A **minimal weighted clone** is a weighted clone W such that for any non-zero weighting $\omega \in W$ we have $\langle \omega \rangle = W$. Note that a weighted clone W is minimal if, and only if, it is an atom of $\text{Lat}(C)$, where C is the support of W .

We will use the following two results. The first gives a uniform way to obtain any weighting in a weighted clone.

Lemma II.7 ([25, Corollary 3.13], [24, Corollary 2.2.4]). *Let C be a clone and $S \subseteq \mathbf{W}_C$ be a set of weightings. Let W be the smallest weighted clone containing S , that is, $W = \langle S \rangle$. Then for each k -ary weighting $\omega \in W$ there are weightings $\omega_1, \dots, \omega_n \in S$ of arities k_1, \dots, k_n respectively, scalars $c_1, \dots, c_n \in \mathbb{Q}_{\geq 0}$, and k -ary operations $g_1^1, \dots, g_{k_n}^n \in C^{(k)}$, such that*

$$\omega = \sum_{i=1}^n c_i \cdot \omega_i[g_1^i, \dots, g_{k_i}^i]. \quad (2)$$

Note that some of the ω_i can be equal and improper superposition is allowed in (2).

The second result gives a sufficient condition for a weighting to generate the weighted clone of all weightings, \mathbf{W}_C .

Lemma II.8 ([25, Theorem 3.2]). *Let ω be a non-zero weighting of a clone C that assigns positive weight to some projection. Then ω generates the weighted clone containing all weightings of C , that is, $\langle \omega \rangle = \mathbf{W}_C$.*

III. MAIN RESULTS

In this paper we focus on **Boolean** clones, consisting of operations defined on the domain $B := \{0, 1\}$, with the usual order $0 < 1$. We will use the following six standard Boolean operations: $\mathbf{0}, \mathbf{1}, \neg, \wedge, \vee, \text{mjr}, \text{mnr}$, which we call **key operations**. They are defined, respectively, to be the unary constant 0 operation, the unary constant 1 operation, the unary inverse operation given by $x \mapsto 1 - x$, the binary operation returning the smaller of its two arguments, the binary operation returning the larger of its two arguments, the unique ternary operation on B such that $\text{mjr}(x, x, y) = \text{mjr}(x, y, x) = \text{mjr}(y, x, x) = x$ for any $x, y \in B$ (known as the *majority* operation), and the unique ternary operation on D such that $\text{mnr}(x, x, y) = \text{mnr}(x, y, x) = \text{mnr}(y, x, x) = y$ for any $x, y \in B$ (known as the *minority* operation).

The next result identifies a special set of weightings, called **key weightings** and shows that every non-trivial weighted Boolean clone *must* contain a key weighting. This implies that every minimal weighted clone is *generated* by a key weighting. Since there are only nine key weightings, for any Boolean clone C , $\text{Lat}(C)$ has at most nine candidates for atoms.

Theorem III.1 ([6, Theorem 8.1]). *Let $C \neq \mathbf{J}_B$ be a clone on B . Then any non-trivial weighted clone W with support C must contain at least one of the following weightings:*

- 1) $(-1, 1 \mid \pi_1^{(1)}, f)$, where $f = \mathbf{0}, \mathbf{1}$ or \neg ;
- 2) $(-1, -1, 2 \mid \pi_1^{(2)}, \pi_2^{(2)}, f)$, where $f = \wedge$ or \vee , or $(-1, -1, 1, 1 \mid \pi_1^{(2)}, \pi_2^{(2)}, \wedge, \vee)$;
- 3) $(-1, -1, -1, 3 \mid \pi_1^{(3)}, \pi_2^{(3)}, \pi_3^{(3)}, f)$, where $f = \text{mjr}$ or mnr , or $(-1, -1, -1, 2, 1 \mid \pi_1^{(3)}, \pi_2^{(3)}, \pi_3^{(3)}, \text{mjr}, \text{mnr})$.

Our notations for these key weightings, and the weighted clones they generate, are listed in Table I. We add a subscript

TABLE I
WEIGHTED CLONES FROM WEIGHTINGS IN THEOREM III.1

Weighted clone	Generating key weighting
$W^{\mathbf{0}}$	$\omega^{\mathbf{0}} := (-1, 1 \mid \pi_1^{(1)}, \mathbf{0})$
$W^{\mathbf{1}}$	$\omega^{\mathbf{1}} := (-1, 1 \mid \pi_1^{(1)}, \mathbf{1})$
W^{\neg}	$\omega^{\neg} := (-1, 1 \mid \pi_1^{(1)}, \neg)$
W^{\wedge}	$\omega^{\wedge} := (-1, -1, 2 \mid \pi_1^{(2)}, \pi_2^{(2)}, \wedge)$
W^{\vee}	$\omega^{\vee} := (-1, -1, 2 \mid \pi_1^{(2)}, \pi_2^{(2)}, \vee)$
$W^{\wedge, \vee}$	$\omega^{\wedge, \vee} := (-1, -1, 1, 1 \mid \pi_1^{(2)}, \pi_2^{(2)}, \wedge, \vee)$
W^{mjr}	$\omega^{\text{mjr}} := (-1, -1, -1, 3 \mid \pi_1^{(3)}, \pi_2^{(3)}, \pi_3^{(3)}, \text{mjr})$
W^{mnr}	$\omega^{\text{mnr}} := (-1, -1, -1, 3 \mid \pi_1^{(3)}, \pi_2^{(3)}, \pi_3^{(3)}, \text{mnr})$
$W^{\text{mjr}, \text{mnr}}$	$\omega^{\text{mjr}, \text{mnr}} := (-1, -1, -1, 2, 1 \mid \pi_1^{(3)}, \pi_2^{(3)}, \pi_3^{(3)}, \text{mjr}, \text{mnr})$

C to denote a key weighting of a clone C , and the corresponding weighted clone with support C that it generates. Note that a key weighting exists as a weighting of C if, and only if, the operations to which it assigns a positive value are contained in C . For example, ω_C^0 and W_C^0 do not exist if $\mathbf{0} \notin C$.

The next result is a simple consequence of Theorem III.1.

Lemma III.2. *Let C be a Boolean clone, ω_C be a key weighting of C and let $W_C = \langle \omega_C \rangle$. If ω_C is the only key weighting contained in W_C , then W_C is an atom of $\text{Lat}(C)$.*

By analysing the structure of weightings in the respective weighted clones, we will establish the following results.

Theorem III.3. *Let C be a Boolean clone with $\mathbf{0}, \mathbf{1}$ or $\neg \in C$. Then W_C^0 , W_C^1 or W_C^\neg is an atom of $\text{Lat}(C)$, respectively.*

Theorem III.4. *Let C be a Boolean clone. If $\wedge \in C$, then W_C^\wedge is an atom of $\text{Lat}(C)$ if, and only if, $\mathbf{0} \notin C$. Similarly, if $\vee \in C$, then W_C^\vee is an atom of $\text{Lat}(C)$ if, and only if, $\mathbf{1} \notin C$.*

As our main result, we show the following result.

Theorem III.5. *Let C be a Boolean clone with $\wedge, \vee \in C$. Then $W_C^{\wedge, \vee}$ is an atom of $\text{Lat}(C)$.*

The weighted clone $W_C^{\wedge, \vee}$ corresponds, via the Galois connection established in [6], to the set of valued constraints defined by submodular functions, which are a fundamental concept in combinatorial optimisation [21]. Hence Theorem III.5 implies that the corresponding valued constraint languages (which are strictly contained in the class of all general-valued submodular functions if C is strictly larger than $\text{Clone}(\{\wedge, \vee\})$) are *maximal*. This was previously shown by different means for $C = \text{Clone}(\{\wedge, \vee\})$ and $C = \mathbf{O}_D$ (where D is any finite set) in [7].

We complete the picture of minimal Boolean weighted clones by establishing the following results.

Theorem III.6. *Let C be a Boolean clone. Then W_C^{mjr} is an atom of $\text{Lat}(C)$ if, and only if, mjr is the only key operation in C , and W_C^{mnr} is an atom of $\text{Lat}(C)$ if, and only if, mnr is the only key operation in C .*

Theorem III.7. *Let C be a Boolean clone with mjr, mnr $\in C$. Then $W_C^{\text{mjr}, \text{mnr}}$ is an atom of $\text{Lat}(C)$ if, and only if, $\wedge, \vee \notin C$.*

Note that these results are sufficient to establish the minimal weighted clones with support C for every Boolean clone C . Some of the proofs we give below also fully characterise the weightings in these minimal weighted clones. Obvious directions for future work are to try to extend these results to non-Boolean domains, and to obtain complete descriptions of all possible weighted clones.

IV. PROOF OF THEOREM III.3

For any $\alpha \in D$, we define a corresponding unary constant operation $\alpha : D \rightarrow D$, where $\alpha : x \mapsto \alpha$ for all $x \in D$. For any clone C with $\alpha \in C$, we define the weighting $\omega_C^\alpha := (-1, 1 \mid \pi_1^{(1)}, \alpha)$ and the weighted clone $W_C^\alpha := \langle \omega_C^\alpha \rangle$.

Proposition IV.1. *Let $\alpha \in D$ and C be a clone with $\alpha \in C$. A weighting ω of C is an element of W_C^α if, and only if, ω is of the form*

$$\left(-q_1, \dots, -q_n, \sum_{i=1}^n q_i \mid \pi_1^{(k)}, \dots, \pi_n^{(k)}, \alpha \right),$$

where $q_i \geq 0$ for each $i \in \{1, \dots, n\}$.

Proof. By Lemma II.7, $\omega \in W_C^\alpha$ if, and only if,

$$\omega = \sum_{i=1}^n c_i \cdot \omega_C^\alpha[g_i] = \sum_{i=1}^n c_i \cdot (-1, 1 \mid g_i, \alpha),$$

where each $g_i \in C^{(k)}$. Since weightings assign negative values only to projections, all of the g_i must be k -ary projections. \square

The following proposition shows that the W_C^α are atoms if $\alpha \in C$, thus establishing the first two cases of Theorem III.3.

Proposition IV.2. *Let C be a clone with $\alpha \in C$. Then W_C^α is an atom of $\text{Lat}(C)$.*

Proof. Let ω be a non-zero k -ary weighting in W_C^α . By Proposition IV.1, only α gets positive weight in ω and we have

$$\omega[\underbrace{\pi_1^{(1)}, \dots, \pi_1^{(1)}}_{k \text{ times}}] = (-q, q \mid \pi_1^{(1)}, \alpha),$$

with $q > 0$. Since $\omega[\pi_1^{(1)}, \dots, \pi_1^{(1)}]$ is equal to $q \cdot \omega^\alpha$ we have that ω generates ω_C^α and hence W_C^α , so W_C^α is minimal. \square

For an operation g , we denote by $\neg g$ the operation $\neg[g]$. The following proposition is essentially [25, Theorem 4.1].

Proposition IV.3. *Let C be a clone with $\neg \in C$. A weighting ω of C is an element of W_C^\neg if, and only if, ω is of the form*

$$\left(-q_1, \dots, -q_k, q_1, \dots, q_k \mid \pi_1^{(k)}, \dots, \pi_k^{(k)}, \neg\pi_1^{(k)}, \dots, \neg\pi_k^{(k)} \right),$$

where $q_i \geq 0$ for each $i \in \{1, \dots, k\}$.

Proof. By Lemma II.7, $\omega \in W_C^\neg$ if, and only if,

$$\begin{aligned} \omega &= \sum_{i=1}^k c_i \cdot \omega_C^\neg[g_i] \\ &= \sum_{i=1}^k c_i \cdot (-1, 1 \mid g_i, \neg g_i), \end{aligned} \quad (3)$$

where each $g_i \in C^{(k)}$. As ω can assign negative weight only to projections, the terms in (3) give the required form. \square

The following proposition shows that W_C^\neg is an atom if $\neg \in C$, thus establishing the third case of Theorem III.3.

Proposition IV.4. *Let C be a clone with $\neg \in C$. Then W_C^\neg is an atom of $\text{Lat}(C)$.*

Proof. Let ω be a non-zero k -ary weighting in W_C^\neg . By Proposition IV.3,

$$\omega[\underbrace{\pi_1^{(1)}, \dots, \pi_1^{(1)}}_{k \text{ times}}] = (-q, q \mid \pi_1^{(1)}, \neg\pi_1^{(1)}),$$

where $q > 0$. As $\omega[\pi_1^{(1)}, \dots, \pi_1^{(1)}] = q \cdot \omega_C^-$, the weighting ω generates ω_C^- and hence W_C^- , so W_C^- is minimal. \square

V. PROOF OF THEOREM III.4

We now give a description of the weightings in the weighted clones W_C^\wedge and W_C^\vee and use it to prove Theorem III.4. A description of weightings in $W_{\Lambda P}^\wedge$, where $\Lambda P = \text{Clone}(\{\wedge\})$ was obtained in [25] using Farkas lemma. For our more general treatment of W_C^\wedge and W_C^\vee , we use the notion of *dependence* [24], which is motivated by the following observation.

Lemma V.1. *Suppose that $f_1, f_2 \in \mathbf{O}_D^{(k)}$ are operations and $\alpha, \beta \in D$ such that for any $\mathbf{x} \in D^k$ we have*

$$f_1(\mathbf{x}) = \alpha \implies f_2(\mathbf{x}) = \beta.$$

Then for any $g_1, \dots, g_k \in \mathbf{O}_D^{(\ell)}$ and $\mathbf{y} \in D^\ell$, we have

$$f_1[g_1, \dots, g_k](\mathbf{y}) = \alpha \implies f_2[g_1, \dots, g_k](\mathbf{y}) = \beta.$$

Definition V.2. If $f_1, f_2 \in \mathbf{O}_D^{(k)}$ are *distinct* operations and $\alpha, \beta \in D$ such that, for all $\mathbf{x} \in D^k$, we have

$$f_1(\mathbf{x}) = \alpha \implies f_2(\mathbf{x}) = \beta,$$

then we say that f_2 is $(\alpha \rightarrow \beta)$ -**dependent** on f_1 .

Moreover, we say that $f \in \mathbf{O}_D^{(k)}$ is $(\alpha \rightarrow \beta)$ -**dependent on** $T \subseteq \{1, \dots, k\}$ if there is $i \in T$ such that f is $(\alpha \rightarrow \beta)$ -dependent on $\pi_i^{(k)}$ and f is not a projection.

We now give a characterisation of the weightings in W_C^\wedge . Note that, for any k -ary non-projection operation f , the operation $\pi_i^{(k)} \wedge f$ is $(0 \rightarrow 0)$ -dependent on $\pi_i^{(k)}$. This is a crucial observation for the description of W_C^\wedge .

Proposition V.3. *Let C be a clone with $\wedge \in C$. A k -ary weighting ω of C is an element of the weighted clone W_C^\wedge if, and only if,*

- ω assigns positive weights only to non-projections that are $(0 \rightarrow 0)$ -dependent on some k -ary projection;
- for every subset $T \subseteq \{1, \dots, k\}$,

$$-\sum_{i \in T} \omega(\pi_i^{(k)}) \leq \sum_{\substack{f \in C^{(k)} \\ (0 \rightarrow 0)\text{-dependent on } T}} \omega(f). \quad (4)$$

Proof. First, suppose that $\omega \in W_C^\wedge$. We define

$$\omega_C^{\wedge 01} := \omega_C^\wedge[\pi_1^{(2)}, \wedge] = (-1, 1 \mid \pi_1^{(2)}, \wedge).$$

We have $\omega_C^{\wedge 01}[\pi_1^{(2)}, \pi_2^{(2)}] + \omega_C^{\wedge 01}[\pi_2^{(2)}, \pi_1^{(2)}] = \omega^\wedge$, and thus

$$\langle \omega_C^\wedge \rangle = \langle \omega_C^{\wedge 01} \rangle = W_C^\wedge.$$

Hence, by Lemma II.7, there are k -ary operations $g_1^j, g_2^j \in C$ and constants $c_j \in \mathbb{Q}_{\geq 0}$ such that

$$\begin{aligned} \omega &= \sum_{j=1}^n c_j \cdot \omega_C^{\wedge 01}[g_1^j, g_2^j] \\ &= \sum_{j=1}^n c_j \cdot (-1, 1 \mid g_1^j, g_1^j \wedge g_2^j). \end{aligned} \quad (5)$$

We will now show that there is a subset $T \subseteq \{1, \dots, k\}$ and for each $i \in T$ an indexing set T_i such that

$$\omega = \sum_{i \in T} \sum_{j \in T_i} c_{ij} \cdot (-1, 1 \mid \pi_i^{(k)}, \pi_i^{(k)} \wedge f_j^i), \quad (6)$$

for some $f_j^i \in C$, which proves that ω is of the required form.

First note that we can rewrite (5) as

$$\omega = \sum_{i \in S} \sum_{j \in S_i} c'_{ij} \cdot (-1, 1 \mid g_i, g_i \wedge h_j^i), \quad (7)$$

where $g_i, h_j^i \in C$ with g_i distinct, and S and S_i are indexing sets.

If for each $i \in S$ the operation g_i is a projection, then the expression in (7) is of the form (6) and we are done. Otherwise there is some $m \in S$ such that g_m is a non-projection. In this case the sum of the *positive* and *negative* weights assigned to g_m by the terms in (7) has to be non-negative, as otherwise $\omega(g_m) < 0$, so ω would not be a valid weighting.

We can cancel out positive and negative weights assigned to g_m and preserve the form of expression (7). Suppose g_ℓ and h_r^ℓ are such that $g_m = g_\ell \wedge h_r^\ell$. Note that we can find an operation h_s^m with $s \in S_m$ such that $c'_{ms} \leq c'_{lr}$, if necessary we split the weighting with coefficient c'_{ms} into two. We replace the weightings $c'_{lr} \cdot (-1, 1 \mid g_\ell, g_\ell \wedge h_r^\ell)$ and $c'_{ms} \cdot (-1, 1 \mid g_m, g_m \wedge h_s^m)$ by

$$(c'_{lr} - c'_{ms}) \cdot (-1, 1 \mid g_\ell, g_\ell \wedge h_r^\ell)$$

and

$$c'_{ms} \cdot (-1, 1 \mid g_\ell, g_\ell \wedge (h_r^\ell \wedge h_s^m)).$$

After identifying all operations assigned non-negative weights and performing all possible eliminations of this kind, we obtain a weighting of the form (6).

For the other direction suppose that $\omega \in W_C^{(k)}$ satisfies the stated conditions. To show that ω is in the weighted clone W_C^\wedge we shall show that ω corresponds to a flow in a certain network, and all such flows correspond to elements of W_C^\wedge .

For any $k \geq 2$, we construct a directed network G_k , with vertices $V_1 \cup V_2 \cup \{s, t\}$, where V_1 is the set of all k -ary projections on B , V_2 is the set of all k -ary non-projections on B that are $(0 \rightarrow 0)$ -dependent on some projection, and s and t are two additional source and sink vertices. The edges of G_k are all possible edges of the following three types: either $(s, \pi_i^{(k)})$, where $\pi_i^{(k)} \in V_1$, or $(\pi_i^{(k)}, f)$, where $\pi_i^{(k)} \in V_1$ and $f \in V_2$ is $(0 \rightarrow 0)$ -dependent on $\pi_i^{(k)}$, or (f, t) , where $f \in V_2$.

The network G_2 is shown in Figure 1, where the binary operations are defined in Table II.

We set the capacity on each edge into $\pi_i^{(k)}$ to be $-\omega(\pi_i^{(k)})$, which is nonnegative by the first inequality in (4), the capacity on each edge out of $f \in V_2$ to be $\omega(f)$, and all other edge capacities to be infinite. Then by the second inequality in (4), the minimum cut in the network has capacity $-\sum_{i=1}^k \omega(\pi_i^{(k)}) = \sum_{f \in V_2} \omega(f)$. Hence, by the max-flow min-cut theorem there is a flow where each edge out of s and each edge into t carries its maximum capacity.

TABLE II
BINARY OPERATIONS (0→0)-DEPENDENT ON SOME $\pi_i^{(2)}$

input	\wedge	op ₁	op ₂	$\mathbf{0}$
(0,0)	0	0	0	0
(0,1)	0	1	0	0
(1,0)	0	0	1	0
(1,1)	1	0	0	0

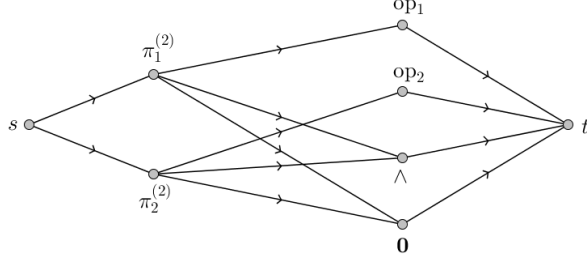


Fig. 1. Directed network G_2

Finally, we note that any valid flow on G_k is determined by the flows along the edges of the form $(\pi_i^{(k)}, f)$, where $\pi_i^{(k)} \in V_1$ and $f \in V_2$. Hence any flow in G_k can be associated with a weighting in W_C^\wedge by adding nonnegative scalings of weightings of the form $(-1, 1 | \pi_i^{(k)}, f) = \omega_C^{\wedge 01}[\pi_i^{(k)}, f]$, where each nonnegative scaling factor is given by the flow on the edge $(\pi_i^{(k)}, f)$. \square

We note that, dually, a similar description holds for the weighted clone W_C^\vee with (0→0)-dependence replaced by (1→1)-dependence. A similar proof technique was used in [8].

The corollary below follows from Proposition V.3 and Proposition IV.1.

Corollary V.4. *Let C be a clone. We have that*

- 1) $W_C^{\mathbf{0}} \subsetneq W_C^\wedge$ if the support clone C contains $\mathbf{0}$ and \wedge , and
- 2) $W_C^{\mathbf{1}} \subsetneq W_C^\vee$ if the support clone C contains $\mathbf{1}$ and \vee .

Finally, we prove Theorem III.4, which tells us that W_C^\wedge is an atom of $\text{Lat}(C)$ if, and only if, $\wedge \in C$ and $\mathbf{0} \notin C$.

Proof of Theorem III.4. First we show that the condition is necessary for W_C^\wedge to be an atom. If $\wedge \notin C$, then W_C^\wedge would not exist. If $\mathbf{0} \in C$ then $W_C^{\mathbf{0}} \in \text{Lat}(C)$ and, by Corollary V.4, $W_C^{\mathbf{0}} \subsetneq W_C^\wedge$, which means that W_C^\wedge is not an atom.

Now we show that the condition is sufficient. The only key operations that are (0→0)-dependent on projections are $\mathbf{0}$ and \wedge . Thus, by Proposition V.3, the only key weightings that might belong to W_C^\wedge are $\omega_C^{\mathbf{0}}$ and ω_C^\wedge . Since $\mathbf{0} \notin C$, $\omega_C^{\mathbf{0}}$ is not a weighting of C . Therefore, ω_C^\wedge is the only key weighting in W_C^\wedge , and so by Lemma III.2 W_C^\wedge is an atom of $\text{Lat}(C)$.

An analogous argument gives the second case of Theorem III.4. \square

VI. PROOF OF THEOREM III.5

The following proposition gives a property that all weightings in $W_C^{\wedge, \vee}$ satisfy, and this will be sufficient to show that $W_C^{\wedge, \vee}$ is an atom.

Proposition VI.1. *Let C be a clone containing \wedge and \vee , and let $\omega \in W_C^{\wedge, \vee}$ be a non-zero k -ary weighting of C . Then for every projection $\pi_i^{(k)}$ with $\omega(\pi_i^{(k)}) < 0$, there are operations $g, h \in C^{(k)}$ with $\omega(g), \omega(h) > 0$ such that g is (0→0)-dependent on $\pi_i^{(k)}$ and h is (1→1)-dependent on $\pi_i^{(k)}$.*

Proof. Let $\pi_i^{(k)}$ be a projection with $\omega(\pi_i^{(k)}) < 0$. Note that such a projection exists as ω is non-zero and assigns negative weights only to projections. We shall show that there exists an operation g with $\omega(g) > 0$ that is (0→0)-dependent on $\pi_i^{(k)}$. We will find such a function by iteration.

By Lemma II.7, for some $g_{i,1}, g_{i,2} \in C^{(k)}$,

$$\omega = \sum_{i=1}^m c_i \cdot \omega_C^{\wedge, \vee}[g_{i,1}, g_{i,2}]. \quad (8)$$

As $\omega(\pi_i^{(k)}) < 0$, a non-zero term $\omega_C^{\wedge, \vee}[\pi_i^{(k)}, g'_0]$ appears in the summation (8) for some $g'_0 \in C^{(k)}$. Let $g_1 = \pi_i^{(k)} \wedge g'_0$. If $\omega(g_1) > 0$, as g_1 is (0→0)-dependent on $\pi_i^{(k)}$, then we have found a suitable operation g . Otherwise, $\omega(g_1) = 0$ and so there is an operation $g'_1 \in C^{(k)}$ such that a non-zero term $\omega_C^{\wedge, \vee}[g_1, g'_1]$ appears in the summation (8). For $i > 1$ we define $g_i = g_{i-1} \wedge g'_{i-1}$. If $\omega(g_i) > 0$, as the operation g_i is (0→0)-dependent on g_{i-1} and so on $\pi_i^{(k)}$, then we have found a suitable operation g . Otherwise, $\omega(g_i) = 0$ and there is an operation g'_i such that a non-zero term $\omega[g_i, g'_i]$ appears in the summation (8). We repeat this process until we find an operation g_i with $\omega(g_i) > 0$.

Note that for each $i \geq 0$ we have that g_{i+1} is (0→0)-dependent on g_i and hence g_i assigns strictly more zeros than g_{i+1} to tuples in D^k . The above process must terminate as $C^{(k)}$ is finite and no two operations among g_i can be equal.

Similarly, we can show the existence of an operation h with $\omega(h) > 0$ that is (1→1)-dependent on $\pi_i^{(k)}$. \square

Proof of Theorem III.5. First note that the condition is necessary for $W_C^{\wedge, \vee}$ to be an atom. If \wedge or \vee is not in C , then $W_C^{\wedge, \vee}$ does not exist and so $W_C^{\wedge, \vee}$ is not an atom of $\text{Lat}(C)$.

Now we show that the condition is sufficient. If \wedge and \vee are in C , then $\omega_C^{\wedge, \vee}$ is a weighting of C . We claim that every other possible key weighting violates properties listed in Proposition VI.1 and so $\omega_C^{\wedge, \vee}$ is the only key weighting in $W_C^{\wedge, \vee}$. Hence, by Lemma III.2 $W_C^{\wedge, \vee}$ is an atom of $\text{Lat}(C)$.

To establish this claim note that the key weighting $\omega_C^\neg \notin W_C^{\wedge, \vee}$ as \neg is neither (0→0) nor (1→1)-dependent on $\pi_1^{(1)}$. The key weightings $\omega_C^{\mathbf{0}} \notin W_C^{\wedge, \vee}$ and $\omega_C^{\mathbf{1}} \notin W_C^{\wedge, \vee}$ as these weightings do not assign a positive weight to any (1→1) and (0→0)-dependent operation, respectively. The key weightings $\omega_C^\wedge \notin W_C^{\wedge, \vee}$ and $\omega_C^\vee \notin W_C^{\wedge, \vee}$ as these weightings do not assign a positive weight to any (1→1) and (0→0)-dependent operation, respectively. The key weightings $\omega_C^{\text{mjr}}, \omega_C^{\text{mnr}}$, and $\omega_C^{\text{mjr, mnr}} \notin W_C^{\wedge, \vee}$ as neither mjr nor mnr is (0→0) or (1→1)-dependent on any ternary projection. \square

VII. PROOF OF THEOREM III.6

First we show that weighted clones $W^{\text{mjr}}, W^{\text{mnr}}$ are always equal to the top element of the lattice of weighted clones.

Proposition VII.1. *Let C be a clone. If C contains mjr, then $W_C^{\text{mjr}} = \mathbf{W}_C$. If C contains mnr, then $W_C^{\text{mnr}} = \mathbf{W}_C$.*

Proof. Note that

$$\omega_C^{\text{mjr}}[\pi_1^{(2)}, \pi_2^{(2)}, \pi_2^{(2)}] = (-1, 1 \mid \pi_1^{(2)}, \pi_2^{(2)})$$

and

$$\omega_C^{\text{mnr}}[\pi_1^{(2)}, \pi_1^{(2)}, \pi_2^{(2)}] = (-1, 1 \mid \pi_1^{(2)}, \pi_2^{(2)}).$$

Both of these weightings assign a positive weight to a projection operation. So by Lemma II.8, we have that ω_C^{mjr} and ω_C^{mnr} , when they exist, generate \mathbf{W}_C . \square

Proof of Theorem III.6. First we show that the condition is necessary for W_C^{mjr} to be an atom. Suppose, that mjr is not the only key operation in C . First, note that $\text{mjr} \in C$ as otherwise W_C^{mjr} would not exist. If C also contains a key operation f equal to one of $\mathbf{0}, \mathbf{1}, \neg, \wedge$ or \vee , then we have that $W_C^f \neq \mathbf{W}_C$, by our earlier results, and so $W_C^f \subsetneq W_C^{\text{mjr}}$ by Proposition VII.1. If $f = \text{mnr}$, then as $W_C^{\text{mjr}} = \mathbf{W}_C$ we have that $\omega_C^{\text{mjr, mnr}} \in W_C^{\text{mjr}}$, but $W_C^{\text{mjr, mnr}} \neq \mathbf{W}_C$ by the proof of Theorem III.7, below. Hence in all cases W_C^{mjr} is not an atom.

Now we show that the condition is sufficient. Suppose that mjr is the only key operation in C . Then ω_C^{mjr} is the only key weighting in W_C^{mjr} and so by Lemma III.2 we have that W_C^{mjr} is an atom of $\text{Lat}(C)$.

An analogous argument establishes the result for W_C^{mnr} . \square

VIII. PROOF OF THEOREM III.7

We prove the following lemma that is needed to give the conditions for $W_C^{\text{mjr, mnr}}$ to be an atom.

Lemma VIII.1. *Let C be a clone that contains \wedge, \vee, mjr and mnr . Then $W_C^{\wedge, \vee} \subsetneq W_C^{\text{mjr, mnr}}$.*

Proof. We have that $W_C^{\wedge, \vee} \neq W_C^{\text{mjr, mnr}}$ as $\omega_C^{\text{mjr, mnr}} \notin W_C^{\wedge, \vee}$ by Proposition VI.1. Also, note that $\omega_C^{\text{mjr, mnr}}[\pi_1^{(2)}, \pi_2^{(2)}, \vee] = (-1, -1, 1, 1 \mid \pi_1^{(2)}, \pi_2^{(2)}, \wedge, \vee) = \omega_C^{\wedge, \vee}$. Hence $\omega_C^{\wedge, \vee} \in \langle \omega_C^{\text{mjr, mnr}} \rangle$ and so $W_C^{\wedge, \vee} \subseteq W_C^{\text{mjr, mnr}}$. \square

Proof of Theorem III.7. First note that by Lemma VIII.1 the condition is necessary for $W_C^{\text{mjr, mnr}}$ to be an atom of $\text{Lat}(C)$.

Now we show that the condition is sufficient. We will show that if $\text{mjr}, \text{mnr} \in C$ and $\wedge, \vee \notin C$, then $W_C^{\text{mjr, mnr}}$ contains no other key weightings apart from $\omega_C^{\text{mjr, mnr}}$, and so is an atom of $\text{Lat}(C)$ by Lemma III.2. Note that only two non-zero unary weighted maps (not weightings) may be obtained by superposition from $\omega_C^{\text{mjr, mnr}}$; these are $(-1, 1, 1, -1 \mid \pi_1^{(1)}, \mathbf{0}, \mathbf{1}, \neg)$, and $(1, -1, -1, 1 \mid \pi_1^{(1)}, \mathbf{0}, \mathbf{1}, \neg)$. Therefore by Lemma II.7 $W_C^{\text{mjr, mnr}}$ contains no unary weightings. So $\omega_C^{\mathbf{0}}, \omega_C^{\mathbf{1}}$ and ω_C^{\neg} are never contained in $W_C^{\text{mjr, mnr}}$. This also implies that $W_C^{\text{mjr, mnr}} \neq \mathbf{W}_C$. Therefore ω_C^{mjr} and ω_C^{mnr} are not contained in $W_C^{\text{mjr, mnr}}$.

Note that if C contains mjr, mnr, and at least one of \wedge and \vee , then C contains all four operations. In such a case, $W_C^{\wedge, \vee} \subsetneq W_C^{\text{mjr, mnr}}$, by Lemma VIII.1, and so $W_C^{\text{mjr, mnr}}$ is not an atom. Hence, ω_C^{\wedge} and ω_C^{\vee} are not contained in $W_C^{\text{mjr, mnr}}$. \square

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