On rainbow-free colourings of uniform hypergraphs*

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Abstract

We study rainbow-free colourings of k-uniform hypergraphs; that is, colourings that use k colours but with the property that no hyperedge attains all colours. We show that $p^* = (k-1)(\ln n)/n$ is the threshold function for the existence of a rainbow-free colouring in a random k-uniform hypergraph.

1 Introduction

A k-uniform hypergraph H consists of a set of vertices $V(H)$ and a collection $E(H)$ of k-element subsets of $V(H)$, called hyperedges. For a k-uniform hypergraph H, a map $c : V(H) \to [k]$ is called a *k-colouring* of H, where $[k] := \{1, \ldots, k\}$. The colouring c is called *rainbow-free* if for every hyperedge $e = (v_1, \ldots, v_k) \in E(H)$ we have $c(e) = \{c(v_1), \ldots, c(v_k)\} \neq [k]$ and for every $i \in [k]$ there is $v \in V(H)$ with $c(v) = i$.

The *k-rainbow-free* problem is to determine whether a given k-uniform hypergraph is rainbow-free colourable with k colours.[1]

Contributions We initiate the study of k-rainbow-free colourings on random hypergraphs. We consider a natural generalisation of Erdős-Rényi random graphs to random (k-uniform) hypergraphs: each possible hyperedge is present with a fixed probability, independently of the other hyperedges. In Section 3, we find a threshold function for the event that a random hypergragraph of the first kind is rainbow-free colourable (Theorem 7). The proof uses a second moment argument for the lowerbound and a first moment argument with an analysis of possible types of rainbow-free colourings for the upperbound.

Related work The k-rainbow-free problem is a special case of colouring mixed hypergraphs, introduced by Voloshin [11] and further extended by Král', Kratochvíl, Proskurowski, and Voss [10]. A mixed hypergraph is a triple (V, C, D) where V is the vertex set and C and D are collections of subsets of V. A colouring of the vertices of a mixed hypergraph (V, C, D) is called proper if each hyperedge in C contains two vertices of the same colour and each hyperedge in D contains two vertices of different colours. The *strict k-colouring* problem is to determine whether a given mixed hypergraph is properly colourable with exactly k colours. The strict k-colouring

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[1]The *k-rainbow-free* problem is called *k-no-rainbow-colouring* in [2]. For $k = 2$, a graph is rainbow-free 2-colourable if and only if it is disconnected (cf. Remark 4).

problem restricted to k-uniform mixed hypergraphs with $D = \emptyset$, so-called co-hypergraphs, is precisely the k-rainbow-free problem. The strict k-colouring of co-hypergraphs was later identified, under the name of *k-no-rainbow-colouring*, in the survey by Bodirsky, Kára, and Martin [2] as an interesting case of unknown complexity of *surjective* constraint satisfaction problems on a three-element domain.

Constraint satisfaction problems (CSPs) are generalisations of graph homomorphisms [9]. A graph homomorphism from G to H is a map from the vertex set of G to the vertex set of H that preserves all edges (but not necessarily non-edges). For a fixed graph H, the *H-coulouring* problems is to determine whether a given graph G admits a homomorphism to H. For instance, taking $H = K_3$ to be the complete graph on 3 vertices, H-colouring is the well known 3-colouring problem. Hell and Nešetřil established that, unless H contains a loop or is a bipartite graph, the H-colouring problem is NP-complete [8].

In an influential paper, Feder and Vardi conjectured that a similar dichotomy holds for every *digrapgh* H, or equivalently, for every finite relational structure (such as hypergraphs) [7]. This conjecture, known as the CSP dichotomy conjecture, was confirmed by two independent papers by Bulatov [4] and Zhuk [12], respectively. While the recent progress on the CSP dichotomy conjecture (and various CSP variants) relied heavily on the so-called algebraic approach [5], this method does not seem direclty amenable to *surjective* CSPs, in which we require the homomorphism be surjective. A dichotomy theorem is known to hold for surjective CSPs on two-element domains by the work of Creignou and Hébrard [6]. The k-rainbow-free problem is equivalent to a surjective CSP on a k-element domain $[k]$ with a single k-ary relation $[k]^k - \{(x_1, \ldots, x_k) : x_1, \ldots, x_k \text{ distinct}\}$. Very recently, Zhuk has announced NP-hardness of the k-rainbow-free problem for $k \geq 3$ [13].

2 Preliminaries

If k is clear from the context, we will call a k-colouring simply a colouring. For a colouring c of a k-uniform hypergraph, we denote the colour classes by $C_i := c^{-1}(i)$, $i \in [k]$.

We state now same basic properties of rainbow-free colourings.

Definition 1. Given a k-uniform hypergraph H and a subset of vertices $S \subseteq V(H)$, we define the *induced subhypergraph* H_S as the $(k-1)$-uniform hypergraph with vertices $V(H_S) := V(H) \setminus S$ and hyperedges $E(H_S) := \{e \cap (V(H) \setminus S) \mid e \in E(H) \text{ and } |e \cap S| = 1\}$.

For up to k disjoint sets $S_1, \ldots S_\ell \subseteq V(H)$ we write $H_{S_1,\ldots,S_\ell} := ((H_{S_1}) \ldots)_{S_\ell}$ for the repeated induced subhypergraph.

For $k = 3$ in Definition 1, H_S will be a graph. Furthermore, note that the order of the subscripts in the definition of the repeated induced subhypergraph does not matter.

This notion of induced subhypergraphs is useful because of the following proposition.

Proposition 2. *Let $k \geq 2$ be an integer. A k-uniform hypergraph H is rainbow-free k-colourable if and only if there exists a non-empty subset of vertices $\emptyset \neq S \subsetneq V(H)$ such that the $(k-1)$-uniform hypergraph H_S is rainbow-free $(k-1)$-colourable. In particular, this implies the existence of a colouring c of H with $C_k = S$.*

Proof. First suppose that H has a rainbow-free k-colouring c. Let S be $C_k \neq \emptyset$ and consider H_S. Write c' for the colouring c restricted to $V(H_S) = V(H) \setminus C_k$. We will show that c' is indeed a rainbow-free $(k-1)$-colouring of H_S. First note that $C'_1 \cup \cdots \cup C'_{k-1} = V(H_S)$, and hence every vertex of H_S has a well-defined colour in $[k-1]$. Now consider a hyperedge $e' \in E(H_S)$. By definition we have $e' = V(H_S) \cap e$ for some $e \in E(H)$. We will use a proof by contradiction to show that $c'(e') \neq [k-1]$, so assume that $c'(e') = [k-1]$. This implies $[k-1] = c(e') \subseteq c(e)$. Furthermore we know that $e \cap S = e \cap C_k \neq \emptyset$, and hence $k \in c(e)$. This implies that $c(e) = [k-1] \cup \{k\} = [k]$, which is a contradiction. We conclude that $c'(e) \neq [k-1]$ and hence C' is a rainbow-free colouring of H_S, as required.

For the other direction assume that $\emptyset \neq S \subsetneq V(H)$ is such that H_S has a rainbow-free $(k-1)$-colouring c'. Now extend c' to a k-colouring c of H by setting $c(v) = k$ for all $v \in S$. Thus, we have $C_k = S$. Let e be a hyperedge in H. We wish to show that $c(e) \neq [k]$, so that c is a rainbow-free colouring indeed. If $|e \cap S| = 0$ we have $k \notin c(S)$. In the $|e \cap S| = 1$ case we have $e' := e \cap (V(H) \setminus S) \in E(H_S)$. Since c' is a rainbow-free $(k-1)$-colouring of H_S we know that $c(e') = c'(e') \neq [k-1]$. Adding in the one vertex v of e that is in $S = C_k$, we get $c(e) = c(e' \cup \{v\}) \neq [k-1] \cup \{k\} = [k]$, as required. If $|e \cap S| \geq 2$ there are at most $k-2$ vertices that have a colour in $[k-1]$. Since $k-2 < |[k-1]|$ we know that $c(e')$ can not attain all colours in $[k-1]$. Hence, this case also implies $c(e) \neq [k]$. □

By induction, it follows that we can apply multiple steps of Proposition 2 at once.

Corollary 3. *Let $2 \leq \ell < k$ be integers. A k-uniform hypergraph H is rainbow-free k-colourable if and only if there exist disjoint non-empty subsets $S_1, \ldots, S_\ell$ of $V(H)$ such that $H_{S_1,\ldots,S_\ell}$ is rainbow-free $(k-\ell)$-colourable.*

Remark 4. We remark that Proposition 2 also applies to the corner case of $k = 2$. In particular, a graph H is rainbow-free 2-colourable if and only if there is a subset $S \subsetneq V(H)$ with no outgoing edges; in other words, H is disconnected.

If all possible rainbow-free hyperedges are given, not only do we know that the rainbow-free colouring is unique, but we can also easily find it.

Proposition 5. *Suppose that H is a k-uniform hypergraph with a surjective colouring $c : V(H) \to [k]$. Furthermore assume that $E = \{e \in V^{(k)} \mid c(e) \neq [k]\}$ consists of all rainbow-free hyperedges. Write $\overline{E} := V^{(k)} \setminus E$ for the set of rainbow hyperedges with $c(e) = [k]$. The colour classes of c are determined by $C_{c(v)} := \{v\} \cup \{u \in V \mid \forall e \in \overline{E} : \{u, v\} \nsubseteq e\}$.*

Proof. If $\{u, v\} \subseteq e$ for some $e \in \overline{E}$, we have that $c(u) \neq c(v)$, since e would be a rainbow-free hyperedge otherwise.

For the other direction assume that $c(u) \neq c(v)$. By surjectivity of c, all colour classes are non-empty and hence there exists a vertex x_j for every colour j in $[k] - \{c(u), c(v)\}$. Using these vertices x_j together with u and v yields a rainbow hyperedge, which is an element of $\overline{E}$. Hence there exists a rainbow hyperedge $e \in \overline{E}$ containing both u and v. This implies that the condition from the statement of the proposition is both sufficient and necessary. □

3 Random hypergraphs

The following definition of random hypergraphs is a direct generalisation of the Erdős-Rényi random graph model: every possible hyperedge is added with a given probability.

Definition 6. Let $p : \mathbb{N} \to [0, 1]$ be a given probability function. A *random k-uniform hypergraph* $H^k_{n,p}$ is a k-uniform hypergraph created by the following process:

- Start with a set of vertices $V(H^k_{n,p}) := V$ with $|V| = n$.
- For each hyperedge $e \in V^{(k)}$, add e to $E(H^k_{n,p})$ with probability $p = p(n)$.

Let A be a hypergraph property (in our case being rainbow-free colourable). We write $\mathbf{Pr}[H^k_{n,p} \models A]$ for the probability that $H^k_{n,p}$ satisfies A. A function $r(n)$ is called a *threshold function* for a hypergraph property A if (i) when $p(n) \ll r(n)$, $\lim_{n\to\infty} \mathbf{Pr}[H^k_{n,p} \models A] = 0$, (ii) when $p(n) \gg r(n)$, $\lim_{n\to\infty} \mathbf{Pr}[H^k_{n,p} \models A] = 1$, or vice versa.

Our main result is the following theorem.

Theorem 7. *The function $p^* = (k-1)(\ln n)/n$ is a threshold function for the event that a random k-uniform hypergraph $H^k_{n,p}$ is rainbow-free colourable.*

The two parts of the proof, one for small p and one for large p, are covered by the following two lemmas. The result is well known for $k = 2$ [3, Theorem VII.9] and corresponds to disconnectedness (cf. Remark 4). Hence we will assume $k \geq 3$.

Lemma 8. *For $k \geq 3$, the random hypergraph $H^k_{n,p}$ is rainbow-free colourable with high probability if $p \leq D\frac{\ln n}{n}$ for some $D < k-1$.*

Lemma 9. *If $p \geq D(\ln n)/n$ with $D > k-1$ and $k \geq 3$, the random hypergraph $H^k_{n,p}$ is not rainbow-free colourable with high probability.*

In order to prove Lemma 8, we use the second moment method; i.e, use the second moment of a random variables to bound the probability that the variable is far from its mean.

Let X be a nonnegative integer-valued random variable such that $X = \sum_{i=1}^m X_i$, where X_i is the indicator variable for event E_i. For indices i, j write $i \sim j$ if $i \neq j$ and the events E_i and E_j are *not* independent. We set (the sum is over ordered pairs)

$$\Delta = \sum_{i \sim j} \mathbf{Pr}[E_i \wedge E_j].$$

Proposition 10 ([1, Corollary 4.3.4])**.** *If $\mathbf{E}[X] \to \infty$ and $\Delta = o(\mathbf{E}[X]^2)$ then $\mathbf{Pr}[X > 0] \to 1$.*

Proof of Lemma 8. Let $H^k_{n,p}$ be a random hypergraph and let X be the number of rainbow-free colourings of $H^k_{n,p}$ with only one colour class of size larger than one. Our goal is to show that $X > 0$ with high probability, and thus $H^k_{n,p}$ is rainbow-free colourable with high probability. We will do so by invoking Proposition 10.

We first show that $\mathbf{E}[X]$ goes to infinity.

Let c be a colouring of $H^k_{n,p}$ that uses all k colours and has only one colour class of size greater than 1. We assume that $|C_i| = 1$ for $1 \leq i \leq k-1$ and $|C_k| = n-k+1$. This colouring c is rainbow-free if and only if there are no hyperedges covering all k colour classes. There are $1 \cdot \dots \cdot 1 \cdot (n-k+1) = n-k+1$ hyperedges with this property, and hence

$$\mathbf{Pr}[c \text{ is a rainbow-free colouring}] = (1-p)^{n-k+1} = \Theta((1-p)^n).$$

Since $\ln(1+x) = x + O(x^2)$ for small x, we have $1 - p = e^{-p+O(p^2)}$ and thus

$$\mathbf{Pr}[c \text{ is a rainbow-free colouring}] = \Theta\left(e^{-pn+O(p^2 n)}\right) = \Theta\left(e^{-D\ln n + O\left(D^2(\ln n)^2/n\right)}\right) = \Theta(n^{-D}).$$

The number of colourings c with one large colour class of size $n-k+1$ is $\binom{n}{n-k+1} = \Theta(n^{k-1})$. The expected number of such colourings that are rainbow-free is now given by

$$\mathbf{E}[X] = \binom{n}{n-k+1}(1-p)^{n-k+1} = \Theta(n^{k-1}n^{-D}) = \Theta(n^{k-1-D}).$$

Since $D < k-1$, this implies that $\mathbf{E}[X] \to \infty$ when $n \to \infty$.

Enumerate all possible colourings c (up to permutations of colours) satisfying $|C_k| = n-k+1$ by c^1 up to c^ℓ. We write $i \sim j$ if $i \neq j$ and $|C^i_k \cap C^j_k| = n-k$. To every colouring c^i we associate the event E_i that c^i is rainbow-free.

Consider the quantity

$$\Delta = \sum_{i \sim j} \mathbf{Pr}[E_i \wedge E_j]. \tag{1}$$

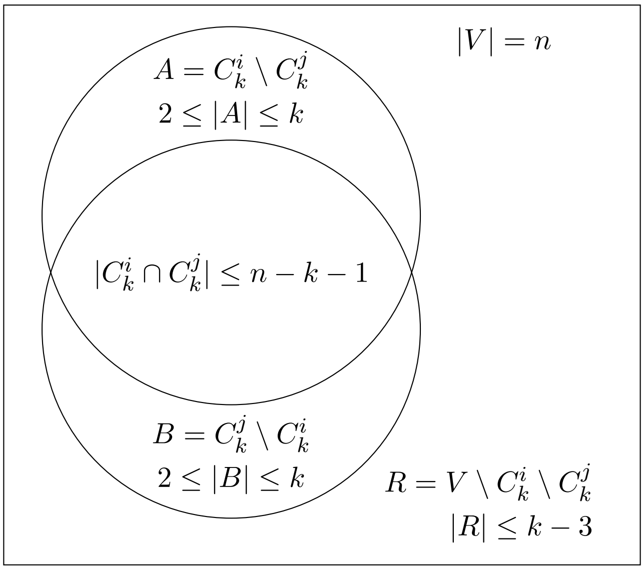

Figure 1: This Venn diagram shows the sets C_k^i (the upper circle) and C_k^j (the lower circle) from the proof of Lemma 8, along with the definitions of A, B, and R. We have $|C_k^i| = |C_k^j| = n-k+1$ and $|C_\ell^i| = |C_\ell^j| = 1$ for all $1 \leq \ell < k$.

We will prove that $\Delta = o(\mathbf{E}[X]^2)$ and and thus finish the proof by Proposition 10. In order for Proposition 10 to be applicable, we need that (for $i \neq j$) $i \sim j$ if the events E_i and E_j are not independent.

By the definition of $\sim$, we have

$$\Delta = \sum_i \sum_{\substack{j \neq i \\ |C_k^i \cap C_k^j| = n-k}} \mathbf{Pr}[E_i \wedge E_j].$$

We claim that the event E_i is independent from E_j if $i \neq j$ and $i \nsim j$. In this case, the overlap between C_k^i and C_k^j is at most $n-k-1$, since an overlap of $n-k$ implies $i \sim j$ and an overlap of $n-k+1$ implies equality. Write $A = C_k^i \setminus C_k^j$, $B = C_k^j \setminus C_k^i$, and $R = V(H_{n,p}^k) \setminus C_k^i \setminus C_k^j$, as is illustrated in Figure 1. The colouring c^i is rainbow-free if all hyperedges of the form $e_1 = B \cup R \cup \{x\}$ for $x \in C_k^i$ are not present. On the other hand, the colouring c^j is rainbow-free if all hyperedges $e_2 = A \cup R \cup \{y\}$ for $y \in C_k^j$ are not present. We have that $|A| = |C_k^i| - |C_k^i \cap C_k^j| \geq (n-k+1) - (n-k-1) = 2$. Similarly we have $|B| \geq 2$. Since A is disjoint from B, we now know that the hyperedges e_1 and e_2 can not be equal. Hence, the colourings c^i and c^j depend on different hyperedges being present, and thus these events are independent indeed.

Let i and j be such that $i \sim j$; i.e., $i \neq j$ and $|C_k^i \cap C_k^j| = n-k$. In this case, we have $|A| = |B| = 1$. The hyperedges that the events E_i and E_j depend on are of the form $A \cup R \cup \{x\}$ for $x \in C_k^i$ and $B \cup R \cup \{y\}$ for $y \in C_k^j$ respectively. We count $2 \cdot (n-k+1)$ hyperedges in total, but the hyperedge $A \cup R \cup B$ is counted twice. Hence, the probability that c^i and c^j are both rainbow-free colourings is

$$\mathbf{Pr}[E_i \wedge E_j] = (1-p)^{2(n-k+1)-1} \leq e^{-p(2n-2k+1)} = \Theta(e^{-2pn}).$$

Given c^i with $|C_k^i| = n-k+1$, the number of colourings c^j such that the large colour classes overlap in $n-k$ positions is $(n-k+1)(k-1)$. Putting this back in Δ gives

$$\begin{aligned}\Delta &= \sum_i (n-k+1)(k-1)(1-p)^{2n-2k+1} \\ &\le \binom{n}{n-k+1}(n-k+1)(k-1)e^{-p(2n-2k+1)} \\ &\le n^{k-1}\cdot n\cdot k\cdot e^{-2D\ln n+O((\ln n)/n)} \\ &= O(n^k\cdot n^{-2D}) = O(n^{k-2D}).\end{aligned}$$

Since $k\ge 3$ we have $0<k-2$ and hence $k-2D<2k-2-2D=2(k-1-D)$. We conclude that

$$\Delta = O(n^{k-2D}) = o(n^{2(k-1-D)})$$

and thus $\Delta = o(\mathbf{E}[X]^2)$. □

We will now prove the bound in the other direction, Lemma 9.

Proof of Lemma 9. We use the first moment method to show that the expected number of rainbow-free colourings of $H^k_{n,p}$ goes to 0. We identify a colouring by the sequence $(s_1,\dots,s_k)$ where $s_i=|C_i|$ and $s_1\le\cdots\le s_k$. We divide the set of all possible sequences into five types:

1. $(s_i)_i=(1,\dots,1,n-k+1)$. There is one such sequence.
2. $(s_i)_i=(1,\dots,1,2,n-k)$. There is one such sequence.
3. $(s_i)_i=(1,\dots,1,x,n-k+2-x)$ with $x\ge 3$. This case contains $O(n)$ sequences.
4. $2\le s_{k-2}\le s_{k-1}$ and $s_1+\cdots+s_{k-1}\le 6k$. This case contains $O(1)$ sequences, since k is a constant.
5. $2\le s_{k-2}\le s_{k-1}$ and $s_1+\cdots+s_{k-1}>6k$. This case contains $O(n^{k-1})$ sequences.

In each case we will show that the expected number of rainbow-free colourings of the relevant type is $o(1)$, from which it follows that the probability that $H^k_{n,p}$ is rainbow-free colourable is $o(1)$.

Before starting calculations, we introduce some notation. We write $\Sigma=s_1+\cdots+s_{k-1}$ so that $s_k=n-\Sigma\ge n/k$, and we write $\Pi=s_1\cdots s_{k-1}$.

A colouring is rainbow-free if none of the $s_1\cdots s_k$ hyperedges that span all colour classes is present. This happens with probability

$$\mathbf{Pr}[c\text{ is rainbow-free}\mid (s_i)_i] = (1-p)^{s_1\cdots s_k}\le e^{-ps_1\cdots s_k}\le n^{-D/n\cdot\Pi(n-\Sigma)}.$$

Since the number of colourings with a given sequence $(s_i)_i$ is upper-bounded by $n^{s_1}\cdots n^{s_{k-1}}=n^{\Sigma}$, the expected number of rainbow-free colourings with a given sequence $(s_i)_i$ is bounded by

$$\mathbf{E}[\text{number of rainbow-free colourings}\mid (s_i)_i]\le n^{\Sigma-D/n\cdot\Pi(n-\Sigma)}. \quad (2)$$

In each of the cases below we will bound the exponent of n in (2).

Write $D=k-1+\delta$ for some $\delta>0$.

Case 1: We have $\Sigma=k-1$ and $\Pi=1$. Putting this into (2) gives an exponent of

$$\Sigma-D/n\cdot\Pi(n-\Sigma)=(k-1)-D\cdot(1-(k-1)/n)\to-\delta.$$

This is less than $-\delta/2$ if n is large enough. Hence, this case is $o(1)$.

Case 2: Here we have $\Sigma=k$ and $\Pi=2$. The exponent of n in (2) becomes

$$\Sigma-D/n\cdot\Pi(n-\Sigma)=k-(k-1+\delta)\cdot 2\cdot(1-k/n)\to -k+2-2\delta. \quad (3)$$

Since this converges to a negative number, it will be less than $-1/2$ for all large enough n. Hence, this case is $o(1)$ as well.

Case 3: There are $O(n)$ sequences in this case, so each of them must give an expected value that is $o(n^{-1})$. The variables are $\Sigma = k + x - 2$ and $\Pi = x$. The exponent in (2) is a quadratic function of x:

$$\Sigma - (k-1+\delta)/n \cdot \Pi(n-\Sigma) = k + x - 2 - (k-1+\delta) \cdot x \cdot (1 - (k+x-2)/n). \tag{4}$$

Since the leading coefficient is positive, and we want to prove an upper bound, it suffices to check the boundaries $x = 3$ and $x = n/2$. (The maximal possible value of x is actually even smaller, but overestimating doesn't hurt.) For $x = 3$ we get

$$k + 1 - 3(k-1+\delta)(1 - (k+1)/n) \to -2k + 4 - 3\delta < -1. \tag{5}$$

Since this converges to something less than -1, we know that the expected value for $x = 3$ is $o(n^{-1})$ for n large enough.

Since the value of (4) goes to $-\infty$ if $x = n/2$ and $n \to \infty$, the upper bound (5) on the exponent in (2) works for the $x = n/2$ case as well.

Case 4: We are given that $\Sigma \leq 6k$. Furthermore we have $s_{k-2} \geq 2$. The minimal value of Π is attained if $s_1 = \cdots = s_{k-3} = 1$ and $s_{k-1} = \Sigma - (k-3) - 2 = \Sigma - k + 1$. Thus, we have $\Pi \geq 2(\Sigma - k + 1)$. Since Σ is a sum of $k-1$ terms, of which the last two are at least 2, we also have $\Sigma \geq k + 1$.

$$\begin{aligned} \Sigma - (k-1+\delta)/n \cdot \Pi(n-\Sigma) &\leq \Sigma - (k-1+\delta)2(\Sigma - k + 1)(1 - \Sigma/n) \\ &\to \Sigma - (k-1+\delta)2(\Sigma - k + 1). \end{aligned}$$

The step where we take the limit is allowed because Σ is bounded, and hence the term divided by n goes to 0 indeed. We continue

$$\begin{aligned} \Sigma - (k-1+\delta)2(\Sigma - k + 1) &= \Sigma(1 - 2(k-1+\delta)) + 2(k-1)(k-1+\delta) \\ &\leq (k+1)(-2k+1-2\delta) + 2(k-1)(k-1+\delta) \\ &= -2k^2 - k + 1 - 2k\delta - 2\delta + 2k^2 - 4k + 2 + 2k\delta - 2\delta \\ &= -3k + 1 - 4\delta < 0. \end{aligned} \tag{6}$$

As before this converges to something negative, and hence it will be $o(1)$.

Case 5: We are now ready for the only remaining case. Here we have $\Sigma \geq 6k$ and as before this implies $\Pi \geq 2(\Sigma - k + 1)$.

$$\Sigma - (k-1+\delta)/n \cdot \Pi(n-\Sigma) \leq \Sigma - (k-1+\delta)/n \cdot 2(\Sigma - k + 1)(n - \Sigma).$$

Using that $s_k = n - \Sigma \geq n/k$ and doing some rewriting gives

$$\begin{aligned} \Sigma - (k-1+\delta)/n \cdot \Pi(n-\Sigma) &\leq \Sigma - (k-1+\delta)/n \cdot 2(\Sigma - k + 1)\frac{n}{k} \\ &= \Sigma - \frac{k-1+\delta}{k} \cdot 2(\Sigma - k + 1) \\ &= (1 - 2(k-1+\delta)/k)\Sigma + 2(k-1)(k-1+\delta)/k \\ &= (-1 + 2/k - 2\delta/k)\Sigma + 2(1 - 1/k)(k-1+\delta). \end{aligned}$$

We are now at the point where we can use $\Sigma \geq 6k$. Because $-1 + 2/k - 2\delta/k < 0$ we get

$$\begin{aligned} \Sigma - (k-1+\delta)/n \cdot \Pi(n-\Sigma) &\leq (-1 + 2/k - 2\delta/k) \cdot 6k + 2(1 - 1/k)(k-1+\delta) \\ &= -6k + 12 - 12\delta + 2k - 2 + 2\delta - 2 + 2/k - 2\delta/k \\ &\leq -4k + 8 - 10\delta + 2/k \leq -4k + 9 - 10\delta. \end{aligned} \tag{7}$$

This last value is strictly less than $-k + 1$, which is just what we needed. We conclude that the total expected number of rainbow-free colourings in this case is $o(1)$ as well, and hence the random hypergraph $H^k_{n,p}$ is not rainbow-free colourable with high probability. □

Lemma 9 can be made a bit stronger with respect to the the colourings of type $(1,\dots,1,n-k+1)$.

Proposition 11. *If a random hypergraph* $H^k_{n,p}$*, with* $k \geq 3$*,* $p = D(\ln n)/n$*, and* $D > k-1$ *is rainbow-free colourable then with high probability it has a colouring of type* $(1,\dots,1,n-k+1)$*.*

Proof. The proof depends heavily on the claims established in the proofs of Lemmas 8 and 9.

Let X_i be the number of rainbow-free colourings in Case i of the proof of Lemma 9. Since $n^{1/n} = e^{(\ln n)/n} \to 1$, we know that the convergence of exponents in (2) in the proof of Lemma 9 implies that n raised to the limit of the exponent is off by at most a constant factor. Hence,

$$\mu := \mathbf{E}[X_1] = \Theta(n^{k-1-D}) = \Theta(n^{-\delta}),$$

where $D = k-1+\delta$. In Cases 2 to 5 of the proof of Lemma 9, Equations (3), (5), (6), and (7) imply that the expected number of rainbow-free colourings in each case is bounded by

$$\begin{aligned}
\mathbf{E}[X_2] &= O(n^{-k+2-2\delta}),\\
\mathbf{E}[X_3] &= O(n)\cdot O(n^{-2k+4-3\delta}) = O(n^{-2k+5-3\delta}),\\
\mathbf{E}[X_4] &= O(n^{-3k+1-4\delta}),\\
\mathbf{E}[X_5] &= O(n^{k-1})\cdot O(n^{-4k+9-10\delta}) = O(n^{-3k+8-10\delta}).
\end{aligned}$$

Since $k \geq 3$, each of these terms is $o(n^{-1-2\delta})$. Hence for $2 \leq i \leq 5$ we have $\mathbf{Pr}[X_i > 0] \leq \mathbf{E}[X_i] = o(n^{-1-2\delta})$. To show that almost all random rainbow-free colourable hypergraphs are rainbow-free colourable with a colouring of the first type indeed, all we have to show is that $\mathbf{Pr}[X_1 > 0] = \Theta(n^{-\delta})$.

As in the proof of Lemma 8 enumerate all colourings by c^1 to c^ℓ and suppose that c^i is a rainbow-free colouring. The probability that there is another rainbow-free colouring c^j is bounded by

$$\begin{aligned}
\sum_{j\sim i}\mathbf{Pr}[c^j \mid c^i] + \sum_{j\not\sim i,\, j\neq i}\mathbf{Pr}[c^j] &\leq n\cdot k\cdot e^{-p(n-k)} + n^{k-1}e^{-p(n-k+1)}\\
&= O(n^{k-1}n^{-(k-1+\delta)}) = O(n^{-\delta}).
\end{aligned}$$

Hence, the probability that the number of rainbow-free colourings is exactly 1 is at least

$$\sum_i \mathbf{Pr}[c^i](1-O(n^{-\delta})) \sim \sum_i \mathbf{Pr}[c^i] = \Theta(n^{k-1-D}) = \Theta(n^{-\delta}).$$

This implies that the probability that $H^k_{n,p}$ is rainbow-free colourable is at least $\Theta(n^{-\delta})$. □

Proposition 11 implies that checking colourings of the type $(1,\dots,1,n-k+1)$ is sufficient to find a colouring in $H^k_{n,p}$ with high probability *if* we know that the hypergraph is rainbow-free colourable.

4 Conclusions

We showed that a threshold function of the event that a random k-uniform hypergraph is rainbow-free colourable is $(k-1)(\ln n)/n$. Our results do not say anything about the case when the hyperedge probability p is close to the threshold. As far as we know, the behaviour of the rainbow-free colourings of a random hypergraph in this case is open.

References

[1] N. Alon, J. Spencer, The Probabilistic Method, John Wiley, 2004.

[2] M. Bodirsky, J. Kára, B. Martin, The complexity of surjective homomorphism problems - a survey, Discrete Applied Mathematics 160 (12) (2012) 1680–1690. doi:10.1016/j.dam.2012.03.029.

[3] B. Bollobás, Modern Graph Theory, Springer, 1998.

[4] A. Bulatov, A dichotomy theorem for nonuniform CSPs, in: Proceedings of the 58th Annual IEEE Symposium on Foundations of Computer Science (FOCS'17), IEEE, 2017, pp. 319–330. doi:10.1109/FOCS.2017.37.

[5] A. Bulatov, P. Jeavons, A. Krokhin, Classifying the Complexity of Constraints using Finite Algebras, SIAM Journal on Computing 34 (3) (2005) 720–742. doi:10.1137/S0097539700376676.

[6] N. Creignou, J. Hébrard, On generating all solutions of generalized satisfiability problems, Informatique Théorique et Applications 31 (6) (1997) 499–511. Available from: http://www.numdam.org/article/ITA_1997__31_6_499_0.pdf.

[7] T. Feder, M. Y. Vardi, The computational structure of monotone monadic SNP and constraint satisfaction: A study through datalog and group theory, SIAM Journal on Computing 28 (1) (1998) 57–104. doi:10.1137/S0097539794266766.

[8] P. Hell, J. Nešetřil, On the complexity of H-coloring, Journal of Combinatorial Theory, Series B 48 (1) (1990) 92–110. doi:10.1016/0095-8956(90)90132-J.

[9] P. Hell, J. Nešetřil, Graphs and Homomorphisms, Oxford University Press, 2004.

[10] D. Král', J. Kratochvíl, A. Proskurowski, H. Voss, Coloring mixed hypertrees, Discrete Applied Mathematics 154 (4) (2006) 660–672. doi:10.1016/j.dam.2005.05.019.

[11] V. I. Voloshin, The mixed hypergraphs, The Computer Science Journal of Moldova 1 (1) (1993) 45–52. Available from: http://www.math.md/publications/csjm/issues/v1-n1/7747/.

[12] D. Zhuk, A proof of CSP dichotomy conjecture, in: Proceedings of the 58th Annual IEEE Symposium on Foundations of Computer Science (FOCS'17), IEEE, 2017, pp. 331–342. doi:10.1109/FOCS.2017.38.

[13] D. Zhuk, No-Rainbow Problem is NP-Hard, Tech. rep. (2020). arXiv:2003.11764.