Programming Research Group

A PARTIAL ORDER ON CLASSICAL AND QUANTUM STATES

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Abstract

We introduce a partial order on classical and quantum mixed states which reveals that these *sets* are actually *domains*: Directed complete partially ordered sets with an intrinsic notion of approximation. The operational significance of the partial orders involved conclusively establishes that physical information has a natural domain theoretic structure. For example, the set of maximal elements in the domain of quantum states is precisely the set of *pure states,* while the completely mixed ensemble I/n is the order theoretic least element $\perp$.

In the same way that the order on a domain provides a rigorous qualitative definition of information, a special type of mapping on a domain called a *measurement* provides a formal account of the intuitive notion 'information content.' Not only is physical information domain theoretic, but so too is physical entropy: Shannon entropy is a measurement on the domain of classical states; von Neumann entropy is a measurement on the domain of quantum states.

These results yield a foundation from which one can (a) reason qualitatively about probability, (b) derive the lattices of Birkhoff and von Neumann, confirming that this setting achieves a more complete picture of physical reality than is currently possessed, and (c) develop new techniques for studying phenomena like noise and entanglement. Along the way, new lines of investigation open up within various subdisciplines of physics, mathematics and theoretical computer science.

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A ship in harbour is safe, but that is not what ships are built for.
— William Shedd.

1 Introduction

One of the great lessons of the differential and integral calculus is that we can conquer the infinite, and in particular, the continuous, by means of the discrete. An infinite sum may be understood as a limit of finite sums, the area beneath a curve as the limit of areas of approximating rectangles, the line tangent to a curve at a point is the limit of the secant lines joining points nearby.

The philosophy espoused is unambiguous: The *ideal* can be realized as a *limit* of the *partial;* the abstract, as a limit of the concrete; the continuous, a limit of the discrete, and so on. And this powerful ideology, as it arises in the context of recursive functionals, is part of what the axioms of domain theory are intended to capture. But even in Scott's prelude to the subject, it is difficult to keep the imagination from wandering beyond the confines of computation [16]:

> "Maybe it would be better to talk about *information;* thus, $x \sqsubseteq y$ means that x and y want to approximate the same entity, but y gives more information about it. This means we have to allow *incomplete* entities, like x, containing only *partial* information."

In its purest interpretation, domain theory is a branch of mathematics which offers an exclusively qualitative account of information: A proposal for how we might find information structured in a universe where all things arise as a limit of the partial.

Physics of course is also the study of information. But with one caveat: In physics, the term "information" normally manages to escape rigorous mathematical definition, and in those cases where it does not, its formulations tend toward the purely *quantitative.* But what is self-evident is that only qualitative accounts of physical phenomena are capable of imparting "structural laws of general validity." From A. Einstein,

> "I do not believe in micro- and macro-laws, but only in (structural) laws of general validity."

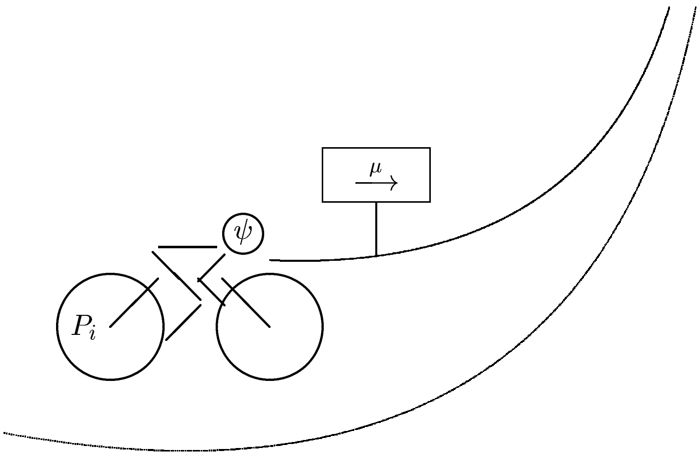
Now this is not to say that physics ought to be done in laboratories without numbers, simply that our *understanding* of physical reality should be mathematically expressible in such a way that the laws of nature are clearly delineated from the conventions of man.

Thus, at least on the surface, there is a good match between what domain theory offers, and what physics needs: Domain theory can provide the structures of reality, physics in turn can explicate the reality of the structures. A research program in this direction begins with the demonstration contained herein that the density operator formulation of quantum mechanics is an instance of domain theory: Its partial elements are the mixed states, its total or idealized elements are the pure states.

The route to this discovery passes through *the measurement formalism,* a theory [8] which allows for the quantitative expression $\mu : D \to [0,\infty)^*$ of the qualitative notion captured by a domain $(D,\sqsubseteq)$. In doing so, it yields an indispensable methodology for uncovering the structural aspects of information which often enough seem to appear in purely quantitative disguise. Such is the case with classical and quantum information, for instance, which are normally formulated in terms of Shannon and von Neumann entropy.

Our method of transport is a philosophy still advocated in certain studies on the foundations of physics [3, 9, 13]: Every formal idea should represent a meaningful physical notion, and each successive mathematical development ought to have a clear counterpart in physical reality. To illustrate, the partial order on classical states is defined inductively in terms of Bayesian state update, which corresponds to the process by which an observer looks for an object and updates his knowledge according to what he finds. Similarly, the partial order on quantum states relies on the physical notion of a measurement process.

On our way, this vehicle escorts classical and quantum probability into a genuine formal realization of the Bayesian ideal, elegantly captured by F. P. Ramsey [14]: "Probability is the logic of partial belief."



Concretely, we introduce the domain of classical states, which has Shannon entropy as a measurement. The partial order on classical states extends to yield a domain of quantum states with von Neumann entropy as a measurement. As already mentioned, the operational significance of the partial orders involved unquestionably demonstrates that physical information has a natural domain theoretic structure. By recognizing this structure, the present work achieves unity across various subdisciplines of physics and information theory. For example, the Birkhoff-von Neumann contrast, between classical and quantum, which arises in the logical aspect, is in perfect harmony with Shannon and von Neumann entropy, which arises in more "pragmatic" pursuits. All of these are part of a single, and it would appear, more complete, picture of physical reality.

2 Classical states

The information an observer has a priori about the result of an event in which one of n different outcomes is possible can be described by a function $x : \{1, \ldots, n\} \to [0,1]$ that assigns a probability x_i indicating the degree to which outcome i is likely. These are called classical states.

Definition 2.1 The *classical n-states* are

$$\Delta^n := \{x \in [0,1]^n : \sum_{i=1}^{n} x_i = 1\},$$

where $x = (x_1, \ldots, x_n)$ and $n \geq 1$.

In this section we will introduce a natural partial order on classical states that is probably best referred to as the *Bayesian order.* Before doing so, here is a brief indication of how this order was discovered and our original motivation for studying it.

In contrast to a classical n-state, a *quantum n-state* is a self-adjoint, positive, trace one, linear operator $\rho : \mathcal{H}^n \to \mathcal{H}^n$ on a n dimensional complex Hilbert space $\mathcal{H}^n$. In particular, ρ is an $n \times n$ matrix of complex numbers whose n eigenvalues $\lambda_i \geq 0$ for $1 \leq i \leq n$ add up to one. Thus, to each quantum state ρ we can associate a classical state $\mathrm{spec}(\rho) = (\lambda_1, \ldots, \lambda_n)$.

Thus, *if* we have a partial order $\sqsubseteq$ on Δ^n, we might be able to use the connection between quantum and classical given above to derive a natural candidate for a partial order on quantum states as follows:

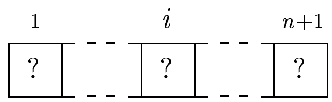
$$\rho \sqsubseteq \sigma \Leftrightarrow \mathrm{spec}(\rho) \sqsubseteq \mathrm{spec}(\sigma) \text{ and (insert magic here).}$$

And then the questions start: (i) Can we really order matrices by ordering their eigenvalues? (ii) How exactly do we form the *list* $(\lambda_1, \ldots, \lambda_n)$, when in actuality the eigenvalues $\mathrm{spec}(\rho)$ of ρ only form a *set*? (iii) How do we order classical states?

The first two questions will be answered in the next section, but the short answers are: (i) Yes, *if* we have the right order on Δ^n, and (ii) quantum measurement. Let us then get on with the answering of (iii).

2.1 Two states and the parabola

Begin by imagining $n+1$ boxes



In one of these boxes, there lies a tenured position in the land of free expression. There are two observers searching frantically for its location. The knowledge an observer has about its location is a classical state $x \in \Delta^{n+1}$, formed by assigning a probability x_i which indicates the likelihood that the tenured position is located in box i:

$$\begin{array}{rccccc} & 1 & & i & & n{+}1 \\ \text{knowledge } x :: & \boxed{x_1} & \cdots & \boxed{x_i} & \cdots & \boxed{x_{n+1}} \end{array}$$

For example, if the observer is frustrated beyond belief because he has no earthly idea which box contains the tenured position, then his knowledge would be the *completely mixed state*

$$\perp = (1/(n+1), \ldots, 1/(n+1)),$$

indicative of the fact that he regards all boxes as equally likely:

$$\begin{array}{ccccc} 1 & & i & & n{+}1 \\ \boxed{\frac{1}{n+1}} & \cdots & \boxed{\frac{1}{n+1}} & \cdots & \boxed{\frac{1}{n+1}} \end{array}$$

On the other hand, if the observer knows the tenured position is located in box i, then his knowledge would be the *pure state*

$$e_i = (0, \ldots, 1, \ldots, 0),$$

where the one occurs at index i:

$$\begin{array}{ccccc} 1 & & i & & n{+}1 \\ \boxed{0} & \cdots & \boxed{1} & \cdots & \boxed{0} \end{array}$$

In general, the actual location is always represented by a pure state. This much is independent of all observers.

Let k be the actual location of the tenured position, $x \in \Delta^{n+1}$ represent the knowledge of the first observer and $y \in \Delta^{n+1}$ the knowledge of the second observer:

$$\begin{array}{rccccc} & 1 & & k & & n{+}1 \\ \text{knowledge } x :: & x_1 & \cdots & x_k & \cdots & x_{n+1} \\ \text{knowledge } y :: & y_1 & & y_k & & y_{n+1} \\ \text{actual position} :: & 0 & \cdots & 1 & \cdots & 0 \end{array}$$

In the interest of holding the reader's attention, let $x \neq y$. If $\sqsubseteq$ is a partial order on Δ^{n+1} that expresses what it means for one state to be more informative than another, and in this order we have $x \sqsubseteq y$, then observer one knows less about the location of the tenured position than observer two.

But now suppose that each observer looks into box i only to discover that it does not contain the tenured position. Then $x_i < 1$ and $y_i < 1$. In addition, the knowledge of the first observer changes to

$$p_i(x) = \frac{1}{1 - x_i}(x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1}) \in \Delta^n,$$

while the state of the second observer's knowledge updates to $p_i(y)$:

$$
\begin{array}{r|c|c|c|c|c|}
 & \multicolumn{1}{c}{1} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{i(\neq k)} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{n+1} \\
\text{knowledge } x \;:: & \frac{x_1}{1-x_i} & \cdots & 0 & \cdots & \frac{x_{n+1}}{1-x_i} \\
\text{knowledge } y \;:: & \frac{y_1}{1-y_i} & \cdots & 0 & \cdots & \frac{y_{n+1}}{1-y_i} \\
\text{actual position} \;:: & & \cdots & \mathbf{0} & \cdots & \\
\end{array}
$$

Because the second observer knew more than the first before observation, and because they have both increased their knowledge by the same amount (they both now additionally know that it is not in box i), we must conclude that the second *still* knows more than the first. That is,

$$p_i(x) \sqsubseteq p_i(y)$$

whenever $i \neq k$. But this reasoning should apply in *all* situations, i.e., it should not depend on the actual location of the tenured position: We should allow for the reality that k could be any of the values in $\{1, \ldots, n+1\}$. Thus, we arrive at a potential definition of $(\Delta^{n+1}, \sqsubseteq)$ in terms of $(\Delta^n, \sqsubseteq)$:

$$x \sqsubseteq y \Leftrightarrow (\forall i)(x_i, y_i < 1 \Rightarrow p_i(x) \sqsubseteq p_i(y)).$$

This leaves just one question: How do we order Δ^2? The answer appears when we imagine that the order $\sqsubseteq$ on Δ^n is known, and then use it to *formally* express some of the well-known *intuitions* used in physics when reasoning about classical states as information:

- The *completely mixed state* should be the *least element* of $(\Delta^n, \sqsubseteq)$,

$$(\forall x) \perp \sqsubseteq x,$$

- The set of *pure states* should be the set of *maximal elements*,

$$\max(\Delta^n) = \{e_i : 1 \leq i \leq n\},$$

- The observer's a priori uncertainty, *Shannon entropy*

$$\mu x = -\sum_{i=1}^{n} x_i \log x_i,$$

  should be a *measurement* in the sense of domain theory [8]. In particular, as states become more informative, uncertainty should decrease:

$$x \sqsubseteq y \Rightarrow \mu x \geq \mu y,$$

  i.e., as a map from the poset Δ^n to the poset $[0, \infty)^*$ of nonnegative reals in their opposite order, it should be *monotone*.

- **The mixing law** should be respected by $\sqsubseteq$:

$$x \sqsubseteq y \text{ and } p \in [0,1] \Rightarrow x \sqsubseteq (1-p)x + py \sqsubseteq y.$$

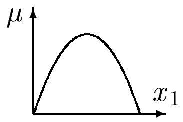
The state $(1-p)x+py$ is a mixture of x and y whose composition consists of $(1-p)$ percent x and p percent y. Thus, the mixing law says that if y is more informative than x, then any mixture of the two is more informative than x, but less informative than y.

This leaves only one way of ordering Δ^2.

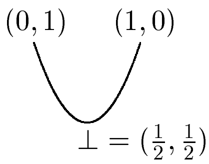
Definition 2.2 For $x, y \in \Delta^2$, we order classical two states by

$$(x_1, x_2) \sqsubseteq (y_1, y_2) \Leftrightarrow (y_1 \leq x_1 \leq 1/2) \text{ or } (1/2 \leq x_1 \leq y_1).$$

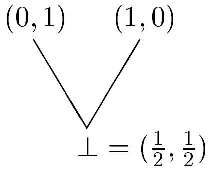
We will prove the uniqueness of this order after explaining its derivation. For the latter, look at the graph of Shannon entropy μ on two states:



Remembering that the order $\sqsubseteq$ on Δ^2 should be defined so that Shannon entropy $\mu : \Delta^2 \to [0, \infty)^*$ is a measurement in the sense of domain theory (and hence *monotone*), a natural candidate for $\sqsubseteq$ appears when we flip the parabola upside down:



The order suggested by this picture is simply a copy of $[0, 1/2]^*$ and $[1/2, 1]$ joined at $1/2$,



which is exactly how we defined the order on Δ^2 (Definition 2.2).

For its uniqueness, first realize that there are at least two reasonable interpretations of the mixing law: (i) (Informatically) When two comparable states are mixed, a loss of information is experienced from one point of view that is simultaneously a gain of information from the other, (ii) (Geometrically) The line connecting two comparable states moves up in the order.

Lemma 2.3 *A partial order $\sqsubseteq$ on Δ^n respects the mixing law iff the map $f : [0,1] \to \Delta^n$ given by $f(t) = (1-t)x + ty$ is monotone for each pair of comparable states $x \sqsubseteq y$.*

Proof. The monotonicity of f implies the mixing law. For the converse, let $s < t$. By the mixing law, $x \sqsubseteq f(t) \sqsubseteq y$, so applying the mixing law again to $x \sqsubseteq f(t)$, gives

$$x \sqsubseteq \left(1 - \frac{s}{t}\right) x + \frac{s}{t} \cdot f(t) \sqsubseteq f(t),$$

which finishes the proof since $f(s) = (1 - \frac{s}{t})x + \frac{s}{t} f(t)$. $\Box$

Now the uniqueness of $(\Delta^2, \sqsubseteq)$ is transparent.

Theorem 2.4 *There is a unique partial order on Δ^2 which satisfies the mixing law and has $\perp = (1/2, 1/2)$. It is the order on classical two states.*

Proof. Let $\leq$ be *any* partial order on Δ^2 which respects the mixing law and has least element $\perp = (1/2, 1/2)$.

Because $\perp \leq e_1 = (1, 0)$, Lemma 2.3 implies that the straight line path f_1 from $f_1(0) = \perp$ to $f_1(1) = e_1$ is monotone. Similarly, the line f_2 from $f_2(0) = \perp$ to $f_2(1) = e_2 = (0, 1)$ is monotone. Thus, $\sqsubseteq \subseteq \leq$.

To prove $\leq \subseteq \sqsubseteq$, suppose $x \leq y$. First, we must have either $x_1, y_1 \leq 1/2$ or $1/2 \geq x_1, y_1$: Otherwise, the line f from $f(0) = x$ to $f(1) = y$ passes through $\perp$, and since f is monotone by the mixing law, we have $x = \perp \sqsubseteq y$. But this means that either $x \sqsubseteq y$ or $y \sqsubseteq x$. In the first case, the proof is done. In the latter, we must have $y \leq x$, which by the antisymmetry of $\leq$, gives $x = y$, and hence $x \sqsubseteq y$. $\Box$

Thus far, we have not defined terms like 'domain' and 'measurement.' At this stage, there is no need to. Let us simply point out that Δ^2 is a domain with Shannon entropy μ as a measurement such that

$$\ker \mu = \max(\Delta^2) = \{e_1, e_2\}.$$

The precise definitions of these terms will become apparent as we proceed.

2.2 A partial order on classical states

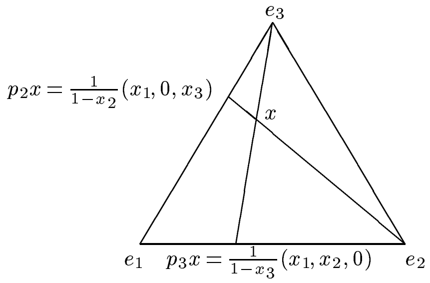
When an observer looks in box i and discovers that the object of his desire is not there, the classical state x representing his knowledge of its location collapses to one $p_i(x)$ in a lower dimension as follows.

Definition 2.5 Let $n \geq 2$. The projection which collapses the i^{th} outcome is the partial map $p_i : \Delta^{n+1} \rightharpoonup \Delta^n$ given by

$$p_i(x) = \frac{1}{1 - x_i}(x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1})$$

for $1 \leq i \leq n + 1$ and $0 \leq x_i < 1$. It is defined on $\mathrm{dom}(p_i) = \Delta^{n+1} \setminus \{e_i\}$.

In the way of needless (but fun) geometric illustration, consider the case of the *triangle* Δ^3. If $x = (x_1, x_2, x_3)$, then although $p_i(x)$ is technically a member of Δ^2, we can still picture its effect on x as follows:



Recalling the definition of $(\Delta^2, \sqsubseteq)$ from the last section, we can now completely specify the order on classical states.

Definition 2.6 Let $n \geq 2$. For $x, y \in \Delta^{n+1}$, we define

$$x \sqsubseteq y \Leftrightarrow (\forall i)(x, y \in \mathrm{dom}(p_i) \Rightarrow p_i(x) \sqsubseteq p_i(y)),$$

where i ranges over the set $\{1, \ldots, n+1\}$.

To be perfectly clear, notice that $x, y \in \mathrm{dom}(p_i)$ iff $x_i, y_i < 1$. The following operators on classical states will prove indispensable in what follows.

Definition 2.7 Let $n \geq 2$. For $x \in \Delta^n$, we set

$$x^+ := \max_{1 \leq i \leq n} x_i \quad \text{and} \quad x^- := \min_{1 \leq i \leq n} x_i.$$

We have $x^- \in [0, 1/n]$ and $x^+ \in [1/n, 1]$.

For example, a state x is pure iff $x^+ = 1$, while $\perp$ is the unique classical state x with $x^+ = x^-$.

Lemma 2.8 *Let $x, y \in \Delta^n$ for $n \geq 2$. Then*

(i) *If $x \sqsubseteq y$ with $x_i = 1$, then $y_i = 1$.*

(ii) *If $x \sqsubseteq \perp$, then $x = \perp$.*

Proof. (i) Assume for $n \geq 2$. For $n+1$, suppose $x_i = 1$. First, we claim there is some $k \neq i$ with $y_k = 0$. If not, then because $n + 1 \geq 3$, there is some $k \neq i$ such that

$$0 < y_k < \sum_{k \neq i} y_k = 1 - y_i \leq 1,$$

as the sum above involves at least two positive numbers. Because $k \neq i$, $x_k = 0$, so the inductive hypothesis applied to $p_k(x) \sqsubseteq p_k(y)$ gives

$$\frac{y_i}{1 - y_k} = \frac{x_i}{1 - x_k} = 1 \implies y_k = 1 - y_i,$$

which contradicts $y_k < 1 - y_i$. Thus, there is $k \neq i$ with $x_k = y_k = 0$, and the inductive hypothesis applied to $p_k(x) \sqsubseteq p_k(y)$ yields

$$y_i = \frac{y_i}{1 - y_k} = \frac{x_i}{1 - x_k} = 1,$$

finishing the proof.

(ii) We know $x^+ < 1$, since otherwise by (i) we would have $\perp^+ = 1$. Now the proof is a trivial induction: Since $x \sqsubseteq \perp$, we have $p_i(x) \sqsubseteq p_i(\perp) = \perp_n$, and by the inductive hypothesis, $p_i(x) = \perp_n$ for all $i \in \{1, \ldots, n+1\}$. The only possibility is $x = \perp$. □

Lemma 2.9 *For classical n-states $x \sqsubseteq y$, either $x = \perp$, $y = \perp$ or there is an index $k \in \{1, \ldots, n\}$ such that $x_k \leq y_k$, $x_k > x^-$ and $y_k > y^-$.*

Proof. The result is true for $n = 2$. Assume it for n. To prove it for $n+1$, we start with $x, y \neq \perp$. Immediately, we have $x \neq y$ (otherwise, $x \neq \perp \Rightarrow x_k = y_k = x^+ > x^-$), and by virtue of Lemma 2.8(i), $x^+ < 1$.

Now let i be an index with $x_i \geq y_i$. Throughout, $y_i < 1$, since otherwise $x = y$. Then either (1) $p_i(x) \neq \perp_n$ or (2) $p_i(x) = \perp_n$.

In case (1), we cannot have $p_i(y) = \perp_n$ (Lemma 2.8(ii)), so the inductive hypothesis applies, yielding an index of $p_i(x)$ and $p_i(y)$ which we can relabel as an index k of x and y with

$$\frac{x^-}{1 - x_i} \leq p_i(x)^- < \frac{x_k}{1 - x_i} \leq \frac{y_k}{1 - y_i} > p_i(y)^- \geq \frac{y^-}{1 - y_i}.$$

Then $x_k > x^-$ and $y_k > y^-$. In addition, since $x_i \geq y_i$,

$$x_k \leq \frac{1 - x_i}{1 - y_i} \cdot y_k \leq 1 \cdot y_k,$$

which finishes the proof in case (1).

In case (2), $p_i(x) = \perp$. It helps to picture x as the $(n+1)$-state

$$\left(\frac{1 - x_i}{n}, \ldots, x_i, \ldots, \frac{1 - x_i}{n}\right),$$

though our proof does not depend on this informal remark. Because $x \neq \perp$, $x_i \neq (1 - x_i)/n$, so either $(1 - x_i)/n > x_i$ or $x_i > (1 - x_i)/n$.

The case $(1 - x_i)/n > x_i$ is simple: Since $x_i \geq y_i$, there must exist $k \neq i$ with $x_k \leq y_k$, or else we could derive $x_i < y_i$. Then we have

$$y_k \geq x_k = \frac{1 - x_i}{n} > x_i \geq y_i,$$

which also makes it clear that $x_k > x^- = x_i$ and $y_k > y_i \geq y^-$.

For the last case, we have $x_i > (1 - x_i)/n$. First we eliminate the possibility $y^+ = 1$. If $y^+ = 1$, then there is an index j with $y_j = 0$. Delicately, we can take $j \neq i$ because

$n+1 \geq 3$. Then $p_j(x) \neq \perp_n$ and $p_j(y) \neq \perp_n$, so the inductive hypothesis applies to yield an index k

$$\frac{x^-}{1-x_j} \leq p_j(x)^- < \frac{x_k}{1-x_j} \leq \frac{y_k}{1-y_j} > p_j(y)^- \geq \frac{y^-}{1-y_j}.$$

But $x_k > x^-$ implies that $k = i$. In addition, we have known from the start that $y_i < 1$, which means $y_i = 0$ because $y^+ = 1$. But then $0 = y_i = y_k \geq x_k = x_i = 0$, which contradicts $x_i > (1-x_i)/n \geq 0$.

To finish case (2), we have $x^+, y^+ < 1$ and $x_i = x^+ > (1-x_i)/n$. What we will prove is that $x_i > x^-$, $y_i > y^-$ and $x_i \leq y_i$. The first of these is clear. For the other two, let k be *any* index different from i. Then $p_k(x) \neq \perp_n$, which means $p_k(y) \neq \perp_n$ since $p_k(x) \sqsubseteq p_k(y)$. By the inductive hypothesis, there is an index j such that

$$\frac{x^-}{1-x_k} \leq p_k(x)^- < \frac{x_j}{1-x_k} \leq \frac{y_j}{1-y_k} > p_k(y)^- \geq \frac{y^-}{1-y_k}.$$

Again, $x_j > x^-$ implies $j = i$. Hence, $y_i > y^-$. But this also gives us $x_i(1-y_k) \leq y_i(1-x_k)$, for *all* $k \neq i$, which enables

$$x_i \sum_{k \neq i}(1-y_k) \leq y_i \sum_{k \neq i}(1-x_k) \implies \frac{x_i}{y_i} \leq \frac{n-1+x_i}{n-1+y_i},$$

ending in $(n-1)x_i + x_i y_i \leq (n-1)y_i + x_i y_i$. $\square$

Lemma 2.10 *If $x \sqsubseteq y$ in Δ^n for $n \geq 2$, then there is an index $i \in \{1, \ldots, n\}$ such that $x_i = x^- \geq y^- = y_i$.*

Proof. If $x = \perp$ the claim is trivial; thus, $y \neq \perp$, by Lemma 2.8(ii). By Lemma 2.9, there is an index $k \in \{1, \ldots, n\}$ such that $x_k \leq y_k$, $x_k > x^-$ and $y_k > y^-$.

If $x_k = 1 \leq y_k$, then $x = y$ and the proof is done. If $y_k = 1$, then let i be an index where $x_i = x^-$. We cannot have $i = k$ since $x_k > x^-$. Thus, $x_i = x^- \geq y_i = 0$.

Assume for n. For $n+1$, $p_k(x) \sqsubseteq p_k(y)$, and the inductive hypothesis applies to yield an index i of x and y with

$$p_k(x)^- = \frac{x_i}{1-x_k} \geq \frac{y_i}{1-y_k} = p_k(y)^-.$$

Because $x_k \leq y_k$, $x_i \geq y_i$. But since $x_k > x^-$ and $y_k > y^-$, we have $x_i = x^-$ and $y_i = y^-$. $\square$

Now that we understand the behavior of minima, the nature of the maxima is immediate (and fundamental).

Proposition 2.11 *Let $x, y \in \Delta^n$ and e_i be the pure states in Δ^n.*

(i) *If $x \sqsubseteq y$, then there is an index i such that $x_i = x^+ \leq y^+ = y_i$.*

(ii) *For any i, $x_i = x^+$ if and only if $x \sqsubseteq e_i$.*

(iii) *If $x \sqsubseteq y$ and $x^+ = y^+$, then $x = y$.*

Proof. All of these statements are proved by induction. The arguments below all assume that the respective claims are true for n and give the argument for the $n+1$ case. That they are true for $n = 2$ is clear.

(i) By Lemma 2.10, there is an index k with $x_k = x^- \geq y^- = y_k$, so we apply the inductive hypothesis to $p_k(x) \sqsubseteq p_k(y)$ to obtain an index i such that

$$p_k(x)^+ = \frac{x_i}{1 - x^-} \leq \frac{y_i}{1 - y^-} = p_k(y)^+.$$

Since $x_i \geq x^- = x_k$ and $x_i \geq x_j$ for all $j \neq k$ using $p_k(x)^+ = x_i/(1 - x^-)$, we have $x_i = x^+$. Similarly, $y_i = y^+$. That $x_i \sqsubseteq y_i$ now follows from

$$x_i \leq \frac{1 - x^-}{1 - y^-} \cdot y_i \leq 1 \cdot y_i$$

since $x^- \geq y^-$.

(ii) Let i be an index where $x_i = x^+$ and $e_i \in \Delta^{n+1}$ be the associated pure state whose value at index i is one. To prove that $x \sqsubseteq e_i$, we must show that $p_k(x) \sqsubseteq p_k(e_i)$ for all $k \neq i$. Fix an arbitrary $k \neq i$.

First, let j be the index of $p_k(x)$ corresponding to index i in x. This index exists because $k \neq i$. The value of $p_k(x)$ at index j is

$$p_k(x)^+ = \frac{x_i}{1 - x_k}.$$

Second, $p_k(e_i)$ is a pure state in Δ^n whose value at index j is one. By the inductive hypothesis, $p_k(x) \sqsubseteq p_k(e_i)$, for all $k \neq i$, which means $x \sqsubseteq e_i$.

For the converse, suppose $x \sqsubseteq y := e_i$. By (i), there is an index k such that $x_k = x^+$ and $y_k = y^+$. But y is pure, so we must have $k = i$, which means $x_i = x_k = x^+$.

(iii) Starting with $x \sqsubseteq y$ and $x^+ = y^+$, we use Lemma 2.10 to project away the minima $x_k = x^- \geq y^- = y_k$, obtaining $p_k(x) \sqsubseteq p_k(y)$. Applying (i) to this pair yields an index i with

$$p_k(x)^+ = \frac{x_i}{1 - x^-} \leq \frac{y_i}{1 - y^-} = p_k(y)^+.$$

As in the proof of (i), $x_i = x^+$ and $y_i = y^+$. But since $x_i = y_i > 0$ and $p_k(x)^+ \leq p_k(y)^+$, we have $x^- \leq y^-$, which gives $x^- = y^-$. This means $p_k(x)^+ = p_k(y)^+$, and since $p_k(x) \sqsubseteq p_k(y)$, the inductive hypothesis applies, leaving $p_k(x) = p_k(y)$. Because we also have $x_k = y_k$, the states x and y are equal. $\square$

Prop. 2.11(ii) shows that an outcome with maximum probability in a classical state has a certain *qualitative* character to it. In general, it is the *only* outcome we can say this about.

Theorem 2.12 *Δ^n is a partially ordered set for each $n \geq 2$. Its maximal elements are the pure states,*

$$\max(\Delta^n) = \{x \in \Delta^n : x^+ = 1\},$$

and its least element is the completely mixed state $\perp := (1/n, \ldots, 1/n)$.

Proof. The proof is by induction. It is true for $n = 2$. Assume the result for n. Then for $n + 1$, the reflexivity and transitivity are clear.

For antisymmetry, let $x \sqsubseteq y$ and $y \sqsubseteq x$. By Prop. 2.11(i), we have $x^+ \leq y^+$ and $y^+ \leq x^+$. By Prop. 2.11(iii), $x = y$.

That the least element is $\perp$ follows from $p_i(\perp) = \perp_n$ for all i. For its maximal elements, first suppose $x^+ = 1$ and that $x \sqsubseteq y$. By Prop. 2.11(i), there is an index i with $x_i = x^+ = 1 \leq y^+ = y_i$, so $y_i = 1$, which means $x = y$. Hence, $x \in \max(\Delta^{n+1})$.

Conversely, if $x \in \max(\Delta^{n+1})$, then $x \sqsubseteq e_i$ by Prop. 2.11(ii), where e_i is the pure state corresponding to $x_i = x^+$. By the maximality of x, $x = e_i$, which means $x^+ = 1$. $\square$

The next result displays some fundamental properties of the order on classical states – the crucial *degeneration* lemma.

Lemma 2.13 (Degeneration) *If $x \sqsubseteq y$ in Δ^n, then*

$$(x_i = 0 \Rightarrow y_i = 0) \ \& \ (y_i = y_j > 0 \Rightarrow x_i = x_j)$$

for all $1 \leq i, j \leq n$.

Proof. Both of these are proved by induction. For $n = 2$ they are easily seen to be true; we give the arguments for $n + 1$ assuming n.

For $(x_i = 0 \Rightarrow y_i = 0)$, we can assume $x^+, y^+ < 1$: If $x^+ = 1$, then $x = y$ since x is maximal; If $y^+ = 1$, then either $y_i = 0$, which finishes the proof, or $y_i = 1$, in which case Prop. 2.11(i) gives $y_i = y^+ = 1 \geq x^+ = x_i > 0$, contradicting $x_i = 0$. Thus, since $x^+, y^+ < 1$, any $k \neq i$ yields $p_k(x) \sqsubseteq p_k(y)$, and since $x_i/(1 - x_k) = 0$, the inductive hypothesis gives $y_i/(1 - y_k) = 0$ hence $y_i = 0$.

For the other claim, suppose $y_i = y_j > 0$ with $i \neq j$. Then $y^+ < 1$. In addition, $x^+ < 1$ or else $x = y$ and we are done. Then because $n + 1 \geq 3$, there is $k \in \{1, \ldots, n + 1\} \setminus \{i, j\}$. For any such index, we have $p_k(x) \sqsubseteq p_k(y)$, so the inductive hypothesis gives $x_i/(1 - x_k) = x_j/(1 - x_k)$, i.e., $x_i = x_j$. $\square$

The standard projections $\pi_k : \Delta^n \to [0, 1]$ are $\pi_k(x) = x_k$ for $1 \leq k \leq n$. Lemma 2.10 extends to increasing sequences as follows.

Lemma 2.14 *If (x_i) is an increasing sequence in Δ^n, then*

(i) *There is an index k with $\pi_k(x_i) = x_i^-$ for all i.*

(ii) *There is an index k with $\pi_k(x_i) = x_i^+$ for all i.*

Proof. (i) Before starting, a crucial consequence of Lemma 2.13 for the present argument is that

$$\{k : y_k = y^-\} \subseteq \{k : x_k = x^-\}$$

provided that $x \sqsubseteq y$ and $y^- > 0$. Thus, *any* increasing sequence (x_i) with $x_i^- > 0$ leads to a decreasing sequence of nonempty finite sets. The intersection of such a sequence must be nonempty, and any member k in this intersection will satisfy $\pi_k(x_i) = x_i^-$ for all i.

Thus, for our sequence $(x_i)_{i\geq 1}$, we may assume that there is a *least* integer $m \geq 1$ with $x_m^- = 0$. First, the proof is finished if we find k with $\pi_k(x_i) = x_i^-$ for all $i \leq m$, since then we have $\pi_k(x_m) = 0$ and hence $\pi_k(x_i) = 0$ for all $i \geq m$, by Lemma 2.13, which means $\pi_k(x_i) = x_i^-$ for all $i \geq 1$.

The case $m = 1$ is trivial. If $m > 1$, then for the subsequence $(x_i)_{i<m}$, we have $x_i^- > 0$ for $i < m$, by the choice of m, so our opening remarks give $\pi_k(x_i) = x_i^-$, for all $i < m$, where k is *any* index in $\{k : \pi_k(x_{m-1}) = x_{m-1}^-\}$. By Lemma 2.10, there is k with $\pi_k(x_{m-1}) = x_{m-1}^-$ and $\pi_k(x_m) = x_m^-$. This value of k gives $\pi_k(x_i) = x_i^-$ for all $i \leq m$.

(ii) We simplify modify the proof of Prop. 2.11(i) using (i). $\square$

Now we take our first step toward proving that Δ^n is a domain.

Definition 2.15 A subset S of a poset is *directed* if it is nonempty and

$$(\forall x, y \in S)(\exists z \in S)\, x, y \sqsubseteq z\,.$$

A directed-complete partial order, or *dcpo*, is a poset in which every directed subset has a supremum.

A familiar example of a directed set is an *increasing sequence*: A sequence (x_i) such that $x_i \sqsubseteq x_{i+1}$ for all i. Joyfully, on classical states, one can always replace directed sets with increasing sequences, so we never have to think about the former.

Proposition 2.16 *The classical states Δ^n are a dcpo. In more detail,*

(i) *If (x_i) is an increasing sequence, then*

$$\bigsqcup_{i\geq 1} x_i = (\lim_{i\to\infty} \pi_1(x_i), \dots, \lim_{i\to\infty} \pi_n(x_i)).$$

(ii) *Every directed subset of Δ^n contains an increasing sequence with the same supremum.*

Proof. We first prove (i) by induction. It is true for $n = 2$. Assume for n. Given an increasing sequence (x_i), Lemma 2.14 yields an index k such that $\pi_k(x_i) = x_i^-$ for all i. The sequence $(p_k(x_i))$ is increasing in Δ^n, so by the inductive hypothesis, we know that

$$\lim_{i\to\infty}\left(\frac{\pi_j(x_i)}{1-x_i^-}\right)$$

exists for all $j \neq k$. The sequence (x_i^-) is decreasing and contained in $[0, 1/(n+1)]$, so it has a limit $s_k = \lim \pi_k(x_i) < 1$, which means $(1 - x_i^-)$ has a limit that is not zero. Thus,

$$s_j := \lim_{i\to\infty} \pi_j(x_i) = \lim_{i\to\infty}\left(\frac{\pi_j(x_i)}{1-x_i^-}\right)\cdot \lim_{i\to\infty}(1-x_i^-)$$

exists for $j \neq k$. Notice that

$$\sum_{j=1}^{n+1} s_j = \sum_{j=1}^{n+1}\lim_{i\to\infty}\pi_j(x_i) = \lim_{i\to\infty}\sum_{j=1}^{n+1}\pi_j(x_i) = 1,$$

which means that $s = (s_1, \dots, s_{n+1})$ is a classical state. We claim that s is the supremum of (x_i).

To avoid needless complication, we can assume $x_i^+ < 1$, since otherwise (x_i) has finitely many distinct elements, and then the claim is obvious. To prove that $x_i \sqsubseteq s$ for all i, we must show

$$(\forall i)(\forall j)\ s_j < 1 \Rightarrow p_j(x_i) \sqsubseteq p_j(s).$$

Fix an index j with $s_j < 1$. Then the sequence $(p_j(x_i))_{i\geq 1}$ is increasing in Δ^n, so by the inductive hypothesis, it has a supremum

$$\bigsqcup_{i\geq 1} p_j(x_i) = \left(\lim_{i\to\infty}\frac{\pi_1(x_i)}{1-\pi_j(x_i)}, \dots, \lim_{i\to\infty}\frac{\widehat{\pi_j(x_i)}}{1-\pi_j(x_i)}, \dots, \lim_{i\to\infty}\frac{\pi_{n+1}(x_i)}{1-\pi_j(x_i)}\right)$$

which is equal to $p_j(s)$ since $s_j = \lim_{i\to\infty}\pi_j(x_i) < 1$. Hence, $p_j(x_i) \sqsubseteq p_j(s)$ for all i and j with $s_j < 1$, which means $x_i \sqsubseteq s$ for all i.

To prove that s is the supremum of (x_i), let u be any upper bound of (x_i). We must show that $s \sqsubseteq u$, i.e.,

$$(\forall j)\ s_j < 1 \ \&\ u_j < 1 \Rightarrow p_j(s) \sqsubseteq p_j(u).$$

Let j be any index with $s_j < 1$ and $u_j < 1$. Then since $p_j(x_i) \sqsubseteq p_j(u)$ for all i, we have $p_j(s) = \bigsqcup_{i\geq 1} p_j(x_i) \sqsubseteq p_j(u)$, using the inductive hypothesis and that $s_j < 1$. Thus, $s \sqsubseteq u$, which proves $s = \bigsqcup x_i$.

The directed completeness of Δ^n and (ii) now follow from a theorem in [8] provided there is a strictly monotone map $f : \Delta^n \to [0,\infty)^*$ which preserves suprema of increasing sequences. To see that $f(x) = 1 - x^+$ is one such map, if (x_i) is an increasing sequence, Lemma 2.14(ii) yields an index k with $\pi_k(x_i) = x_i^+$ for all i, so

$$\left(\bigsqcup_{i\geq 1} x_i\right)^+ = \lim_{i\to\infty} \pi_k(x_i) = \lim_{i\to\infty} x_i^+,$$

which makes it clear that f preserves suprema of increasing sequences. That f is strictly monotone follows from Prop. 2.11(iii). $\Box$

Definition 2.17 A map $f : D \to E$ between dcpo's is *Scott continuous* if it is *monotone*

$$x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$$

and it *preserves directed suprema*:

$$f(\bigsqcup S) = \bigsqcup f(S)$$

for any directed set $S \subseteq D$.

Corollary 2.18 *A monotone map $f : \Delta^n \to E$ into a dcpo E is Scott continuous iff for each increasing sequence (x_i) in Δ^n, $f(\bigsqcup x_i) = \bigsqcup f(x_i)$.*

Proof. If $S \subseteq \Delta^n$ is directed, then $\bigsqcup f(S) \sqsubseteq f(\bigsqcup S)$ by monotonicity. For the other direction, Prop. 3.28(ii) gives an increasing sequence (x_i) in S with $\bigsqcup S = \bigsqcup x_i$, enabling

$$f(\bigsqcup S) = f(\bigsqcup x_i) = \bigsqcup f(x_i) \sqsubseteq \bigsqcup f(S),$$

confirming that f preserves suprema of *all* directed sets provided it does so for increasing sequences. $\Box$

For instance, the map $\Delta^n \to [0,1] :: x \mapsto x^+$ is Scott continuous, while $x \mapsto 1 - x^+$ is Scott continuous as a map $\Delta^n \to [0,1]^*$. An amusing example of a Scott continuous map that is not Euclidean continuous is the natural retraction from Δ^2 onto $\partial\Delta^2 = \max(\Delta^2)$.

Generally speaking, the *entropy* of an event with probability p is $-\log p$. If forced to choose a single probability representative of an entire classical state x, x^+ would be the most sensible choice, because of its qualitative significance in Prop. 2.11. Thus, one might say that $s(x) = -\log x^+$ measures the entropy of a classical state.

Corollary 2.19 *The map $s : \Delta^n \to [0,\infty)^*$ given by $s(x) = -\log x^+$ is Scott continuous. It has the following properties:*

(i) *For all $x, y \in \Delta^n$, if $x \sqsubseteq y$ and $s(x) = s(y)$, then $x = y$.*

(ii) *For all $x \in \Delta^n$, we have $s(x) = 0$ iff $x \in \max(\Delta^n)$.*

(iii) *For all $x \in \Delta^n$, we have $s(x) = \log n$ iff $x = \bot$.*

Proof. The map is well-defined because $x^+ \in [1/n, 1]$. That s is strictly monotone follows from Prop. 2.11. (ii) and (iii) follow from combinations of direct calculation and applications of (i). $\square$

By the last result, a monotone map $f : D \to \Delta^n$ from a dcpo D is Scott continuous iff $s \circ f$ is Scott continuous. We will take a closer look at entropy later on.

2.3 Symmetries for classical states

We now establish the fundamental role played by the symmetric group

$$S(n) = \{\sigma | \sigma : \{1, \dots, n\} \simeq \{1, \dots, n\}\}$$

of bijections on the set $\{1, \dots, n\}$. These we also refer to as *permutations* or *symmetries.* The composition of $x \in \Delta^n$ and $\sigma \in S(n)$ is written $x \cdot \sigma$.

Definition 2.20 A state $x \in \Delta^n$ is *monotone* if $x_i \geq x_{i+1}$ for all $i < n$.

A classical state $x \in \Delta^n$ can be completely described by a monotone state $x \cdot \sigma$ and a symmetry σ^{-1}. The order on Δ^n has an analogous representation.

Lemma 2.21 *For states $x, y \in \Delta^2$, we have $x \sqsubseteq y$ iff there is a permutation σ of $\{1, 2\}$ such that $x \cdot \sigma = (x^+, x^-)$, $y \cdot \sigma = (y^+, y^-)$ and $x^+ y^- \leq x^- y^+$.*

Theorem 2.22 *For $x, y \in \Delta^n$, we have $x \sqsubseteq y$ iff there is a permutation σ of $\{1, \dots, n\}$ such that $x \cdot \sigma$ and $y \cdot \sigma$ are monotone and*

$$(x \cdot \sigma)_i (y \cdot \sigma)_{i+1} \leq (x \cdot \sigma)_{i+1} (y \cdot \sigma)_i$$

for all i with $1 \leq i < n$.

Proof. By the last lemma, the claim is true for $n = 2$. Assume the result for $n \geq 2$. For the case $n + 1$, we prove both implications separately.

First suppose $x \sqsubseteq y$. Let k be an index with $x_k = x^- \geq y^- = y_k$. By the inductive hypothesis applied to $p_k(x) \sqsubseteq p_k(y)$, there is a permutation σ of $\{1, \dots, n\}$ such that $p_k(x) \cdot \sigma$ and $p_k(y) \cdot \sigma$ are monotone. Now compose σ with the natural bijection that maps indices of $p_k(x)$ and $p_k(y)$ to indices of x and y, and since there is no harm in doing so, call the resulting bijection $\sigma : \{1, \dots, n\} \to \{1, \dots, n+1\} \setminus \{k\}$.

We extend σ to a permutation of $\{1, \dots, n+1\}$ by setting $\sigma(n+1) := k$. It is then clear that $x \cdot \sigma$ and $y \cdot \sigma$ are monotone and that

$$(x \cdot \sigma)_i (y \cdot \sigma)_{i+1} \leq (x \cdot \sigma)_{i+1} (y \cdot \sigma)_i$$

for all $1 \leq i < n$. To finish this direction, we need to prove

$$(x \cdot \sigma)_n (y \cdot \sigma)_{n+1} \leq (x \cdot \sigma)_{n+1} (y \cdot \sigma)_n.$$

Because $(x \cdot \sigma)_{n+1} = x_k = x^- \geq y^- = y_k = (y \cdot \sigma)_{n+1}$, we need only consider the case that $(y \cdot \sigma)_n < (x \cdot \sigma)_n$.

First, $x^+ < 1$, since $(x \cdot \sigma)_n > 0 \Rightarrow (x \cdot \sigma)_1 = x^+ < 1$. Next, $y^+ < 1$, since otherwise $(y \cdot \sigma)_{n+1} = 0$, in which case the inequality is trivial. Now let j be an index with $x_j = x^+ \leq y^+ = y_j$. By the inductive hypothesis applied to $p_j(x) \sqsubseteq p_j(y)$, we obtain a permutation π of $\{1, \ldots, n\}$. Similar to the case of σ, we regard π as a bijection $\{2, \ldots, n+1\} \to \{1, \ldots, n+1\} \setminus \{j\}$ and then extend it to a permutation of $\{1, \ldots, n+1\}$ by setting $\pi(1) := j$. Again $x \cdot \pi$ and $y \cdot \pi$ are monotone, and in this case we have

$$(x \cdot \pi)_i (y \cdot \pi)_{i+1} \leq (x \cdot \pi)_{i+1} (y \cdot \pi)_i$$

for $2 \leq i < n+1$. Because $x \cdot \pi = x \cdot \sigma$ and $y \cdot \pi = y \cdot \sigma$, setting $i = n$ yields the desired inequality, finishing this direction.

For the other, let σ be a permutation of $\{1, \ldots, n+1\}$ with $x \cdot \sigma, y \cdot \sigma$ monotone and $(x \cdot \sigma)_i (y \cdot \sigma)_{i+1} \leq (x \cdot \sigma)_{i+1} (y \cdot \sigma)_i$ for all i with $1 \leq i < n+1$. First notice that slightly more is true:

$$(*) \quad (x \cdot \sigma)_i (y \cdot \sigma)_j \leq (x \cdot \sigma)_j (y \cdot \sigma)_i$$

for $1 \leq i \leq j \leq n+1$.

In the cases $(x \cdot \sigma)_i = 0$ and $(y \cdot \sigma)_j = 0$, $(*)$ is clear; if $(x \cdot \sigma)_i > 0$ and $(y \cdot \sigma)_j > 0$, then $(y \cdot \sigma)_k > 0$ for $k \leq j$ by monotonicity, which means $(x \cdot \sigma)_k > 0$ for all $i \leq k \leq j$ as well, since for $k > i$ we have

$$(x \cdot \sigma)_k \geq \frac{(x \cdot \sigma)_{k-1} (y \cdot \sigma)_k}{(y \cdot \sigma)_{k-1}} > 0$$

assuming $(x \cdot \sigma)_{k-1} > 0$. Without division by zero to worry about, $(*)$ is now clear.

To prove $x \sqsubseteq y$ we must show $p_k(x) \sqsubseteq p_k(y)$ for all k with $x_k < 1, y_k < 1$. To this end, fix one such k. We restrict σ to a bijection

$$\{1, \ldots, n+1\} \setminus \sigma^{-1}(k) \to \{1, \ldots, n+1\} \setminus \{k\}$$

which then yields a permutation σ_k of $\{1, \ldots, n\}$ such that $p_k(x) \cdot \sigma_k$ and $p_k(y) \cdot \sigma_k$ are monotone. By (*), we have

$$(p_k(x) \cdot \sigma_k)_i (p_k(y) \cdot \sigma_k)_{i+1} \leq (p_k(x) \cdot \sigma_k)_{i+1} (p_k(y) \cdot \sigma_k)_i$$

for all $1 \leq i < n$. By the inductive hypothesis, $p_k(x) \sqsubseteq p_k(y)$, finishing the proof. $\square$

The explicit nature of the representation using symmetries can be advantageous in establishing certain properties of the order.

Lemma 2.23 *The map $x \mapsto x \cdot \sigma$ is an order isomorphism of Δ^n for each $\sigma \in S(n)$.*

Proof. Let $f(x) = x \cdot \sigma$. To see that f is monotone, if $x \sqsubseteq y$, then there is $\nu \in S(n)$ with $x \cdot \nu$ and $y \cdot \nu$ monotone satisfying the inequalities of Theorem 2.22. But the same is true of $x \cdot \sigma$ and $y \cdot \sigma$ if we apply the permutation $\sigma^{-1} \cdot \nu$ to each. Thus, $f(x) = x \cdot \sigma \sqsubseteq y \cdot \sigma = f(y)$.

The same argument shows $g(x) = x \cdot \sigma^{-1}$ is monotone. Because f and g are also inverse to one another, each is an order isomorphism. $\Box$

It is now time to take a more in depth look at the order on classical states. To keep things simple initially, we start on the outside and work our way inward. The *boundary* of Δ^{n+1},

$$\partial\Delta^{n+1} = \bigcup_{1 \leq i \leq n+1} \ker \pi_i,$$

can be understood *geometrically* as $n+1$ copies of Δ^n identified at certain points. The same result holds *order theoretically.* That is, the dcpo $\partial\Delta^{n+1}$ is order isomorphic to $n+1$ copies of the dcpo Δ^n identified along their common boundaries.

Proposition 2.24 *For $n \geq 1$, we have an order isomorphism*

$$\Delta^n \simeq \{x \in \Delta^{n+1} : \pi_i(x) = 0\},$$

for any of the standard projections $\pi_i : \Delta^{n+1} \to [0,1]$ with $1 \leq i \leq n+1$.

Proof. First, $i_{n+1} : \Delta^n \to \Delta^{n+1} :: x \mapsto (x, 0)$ is an order embedding. It is order reflecting:

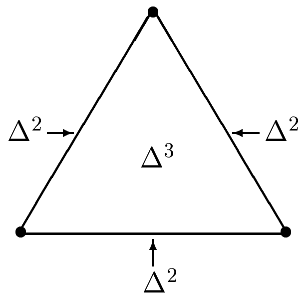
$$i_{n+1}(x) \sqsubseteq i_{n+1}(y) \implies x = p_{n+1}(i_{n+1}(x)) \sqsubseteq p_{n+1}(i_{n+1}(y)) = y.$$

For its monotonicity, let $x \sqsubseteq y$. By Theorem 2.22, there is $\sigma \in S(n)$ with $x \cdot \sigma$ and $y \cdot \sigma$ monotone such that the usual inequalities hold.

Now extend σ to a permutation in $S(n+1)$ by setting $\sigma(n+1) = n+1$. Because the value of the state $i_{n+1}(x)$ at index $n+1$ is zero, $i_{n+1}(x) \cdot \sigma$ and $i_{n+1}(y) \cdot \sigma$ are monotone and satisfy the inequalities of Theorem 2.22. Thus, $i_{n+1}(x) \sqsubseteq i_{n+1}(y)$.

The other n maps, i_k for $1 \leq k \leq n$, which produce an $n+1$ state having value zero at index k, all arise as the composition of isomorphisms (derived from right multiplication by a symmetry) followed by i_{n+1}. $\Box$

Thus, the boundary of the triangle Δ^3 is a dcpo made of three copies of Δ^2:



To get an idea of what the order is like on $\mathrm{int}(\Delta^n)$, we need to look a little closer. First, some long overdue notation.

Definition 2.25 The monotone classical states are denoted

$$\Lambda^n := \{x \in \Delta^n : (\forall i < n)\, x_i \geq x_{i+1}\}.$$

For $\sigma \in S(n)$,

$$\Delta^n_\sigma := \{x \in \Delta^n : x \cdot \sigma \in \Lambda^n\}.$$

Notice that $\Delta^n_1 = \Lambda^n$.

As we have already seen, the order on monotone states can be characterized purely algebraically. For the sake of emphasis:

Lemma 2.26 *For $x, y \in \Lambda^n$, $x \sqsubseteq y$ iff $(\forall\, 1 \leq i < n)\, x_i y_{i+1} \leq y_i x_{i+1}$.*

Just as was the case with its boundary, there is also a natural way of dividing Δ^n itself into regions: For each $n \geq 1$,

$$\Delta^n := \bigcup_{\sigma \in S(n)} \Delta^n_\sigma.$$

And just as before, these regions are identical (order-theoretically).

Proposition 2.27 *Let $n \geq 2$.*

(i) *For each $\sigma \in S(n)$, Δ^n_σ is closed under directed suprema in Δ^n.*

(ii) *For an increasing sequence (x_i) in Δ^n, there is $\sigma \in S(n)$ with $x_i \in \Delta^n_\sigma$ for all i.*

(iii) *The natural map*

$$r : \Delta^n \to \Lambda^n$$

is a Scott continuous retraction whose restriction to Δ^n_σ is an order isomorphism $\Delta^n_\sigma \simeq \Lambda^n$ for each $\sigma \in S(n)$.

Proof. (i) Since every directed set contains an increasing sequence with the same supremum, we only have to prove this result for increasing sequences (x_i) in Δ^n_σ. By Lemma 2.23 and the formula for suprema,

$$\left(\bigsqcup x_i\right) \cdot \sigma = \bigsqcup (x_i \cdot \sigma) = (\lim_{i\to\infty} \pi_1(x_i \cdot \sigma), \ldots, \lim_{i\to\infty} \pi_n(x_i \cdot \sigma)).$$

But the state on the far right is monotone because all the $x_i \cdot \sigma$ are. This proves that $\bigsqcup x_i \in \Delta^n$ also belongs to Δ^n_σ.

(ii) This is a straightforward induction using Lemma 2.14(i).

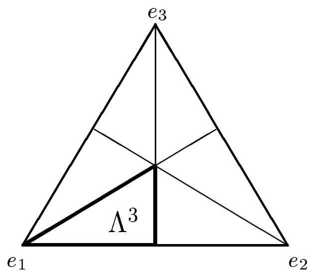
(iii) For $\sigma \in S(n)$, we denote the order isomorphism in Lemma 2.23 by $r_\sigma(x) = x \cdot \sigma$. Set $r(x) = r_\sigma(x)$ for $x \in \Delta^n_\sigma$. This map is well defined and its restriction to Δ^n_σ is an order isomorphism: $\Delta^n_\sigma = r_\sigma^{-1}(\Lambda^n) \simeq \Lambda^n$.

It is monotone: If $x \sqsubseteq y$, then by Theorem 2.22, there is $\sigma \in S(n)$ with $x, y \in \Delta^n_\sigma$ which gives $r(x) = r_\sigma(x) \sqsubseteq r_\sigma(y) = r(y)$.

It is Scott continuous: If $\mu : \Delta^n \to [0, \infty)^*$ is strictly monotone, Scott continuous and $\mu r = \mu$, then r itself is Scott continuous, since it is monotone and has continuous measure μr. Let $\mu x = -\log x^+$ (Corollary 2.19).

Finally, $r|_{\Lambda^n} = 1_{\Lambda^n}$, which proves that r is a retraction. $\Box$

Thus, we can think of Δ^n as being $n!$-many copies of the retract Λ^n identified along their common boundaries. For instance, Δ^3 splits into six different regions, all order isomorphic to Λ^3:



This, combined with an elementary analysis of Λ^3, allows us to determine the upper sets of $(\Delta^3, \sqsubseteq)$ shown in Figure 1.

We now have our first example of an *intuition* about classical states that has been *formally justified.* Consider a closed cylinder of volume V partitioned into smaller volumes V_i as follows:

V_1	V_2	V_3

The cylinder is known a priori to contain a single molecule. With no other information available to us, our knowledge of the molecule's location is (p_1, p_2, p_3) where $p_i = V_i/V$. Or is it? Well, it is if we *assume* that the volumes are labelled from left to right as 1,2,3. But if they have been labelled in the reverse order, as

3	2	1
V_1	V_2	V_3

then our knowledge is (p_3, p_2, p_1).

Naturally, we *intuitively* understand that in the grand scheme of things it makes no difference how we label things – as long as all statements made about the experiment are made with respect to the same choice of labels, we will not encounter any trouble: What is physically true for one choice of labels is also true for any other. But that's where the

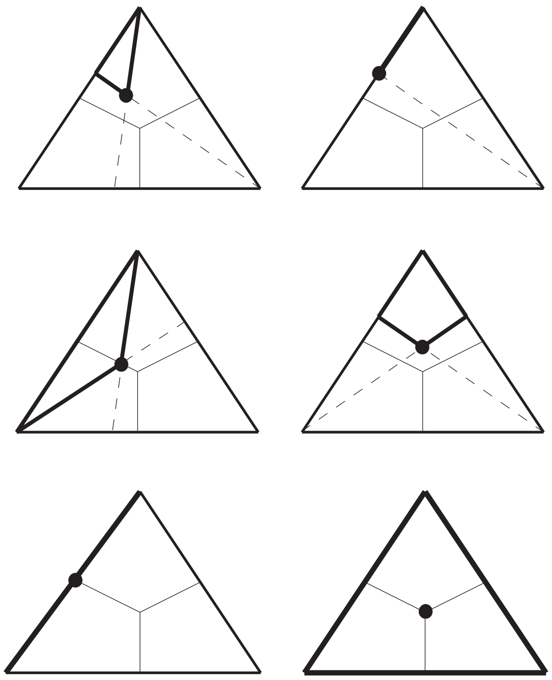


Figure 1: Pictures of $\uparrow x$ for $x \in \Delta^3$.

magic is! We have *derived* this simple truth: For each $\sigma \in S(n)$, the map $x \mapsto x \cdot \sigma$ is an order isomorphism.

In short, there is a definite physical reason why Δ^n is divided into *different* regions Δ^n_σ all of which are "identical" ($\Delta^n_\sigma \simeq \Delta^n_\nu$). For the very same reason, measures of information content in such experiments tend to be *symmetric.*

Definition 2.28 A function $f : \Delta^n \to E$ is *symmetric* if for all $\sigma \in S(n)$, we have $f(x \cdot \sigma) = f(x)$.

Lemma 2.29 *Let E be a dcpo. Then*

(i) *Every function $f : \Lambda^n \to E$ determines a unique symmetric extension $\bar{f} : \Delta^n \to E$ given by $\bar{f} = f \circ r$ where r is the natural retraction.*

(ii) *Monotonicity, strict monotonicity and Scott continuity are inherited by $\bar{f}$ whenever they are possessed by f.*

Proof. (i) For the uniqueness of $\bar{f}$, if $g : \Delta^n \to E$ is another symmetric extension of f, then for any $x \in \Delta^n_\sigma$, we can write

$$g(x) = g(x \cdot \sigma) = f(x \cdot \sigma) = \bar{f}(x \cdot \sigma) = \bar{f}(x)$$

using that g is symmetric, followed by the fact that $g = f$ on Λ^n, and then the fact that $\bar{f}$ is a symmetric extension of f.

(ii) Each property is preserved by composition and satisfied by r. $\square$

Example 2.30 *Canonical symmetric functions on Δ^n.*

(i) The maps $\Delta^n \to [0,1] :: x \mapsto x^+$ and $\Delta^n \to [0,1]^* :: x \mapsto 1 - x^+$.

(ii) Entropy $s(x) = -\log x^+$.

(iii) The natural retraction $r : \Delta^n \to \Lambda^n$.

(iv) Shannon entropy

$$\mu x = -\sum_{i=1}^{n} x_i \log x_i.$$

As the last result illustrates, the retraction $r : \Delta^n \to \Lambda^n$ provides us with a general approach for solving problems involving classical states: First solve it for Λ^n, and then for Δ^n in general.

2.4 Approximation of classical states

A decent understanding of approximation can provide insight about the nature of *partiality*. Partiality, as we will see in the next section, is imperative for a meaningful discussion on entropy.

Definition 2.31 Let D be a dcpo. For $x, y \in D$, we write $x \ll y$ iff for all directed sets $S \subseteq D$,

$$y = \bigsqcup S \Rightarrow (\exists s \in S)\, x \sqsubseteq s.$$

The *approximations* of $x \in D$ are

$$\mathop{\downarrow\!\!\downarrow} x := \{y \in D : y \ll x\},$$

and D is called *exact* if $\mathop{\downarrow\!\!\downarrow} x$ is directed with supremum x for all $x \in D$.

A *continuous* dcpo is exact, and in that case, the "way below" relation and our notion of approximation above are equivalent. In addition, the two notions also coincide on maximal elements.

Lemma 2.32 *Let D be a dcpo. For each $x \in D$, the set $\mathop{\downarrow\!\!\downarrow} x$ is directed with supremum x iff it contains a directed set with supremum x.*

The ability to approximate classical states is provided by the mixing law.

Proposition 2.33 (The mixing law) *If $x \sqsubseteq y$ in Δ^n, then*

$$x \sqsubseteq (1-p)x + py \sqsubseteq y$$

for all $p \in [0, 1]$.

Proof. Let z denote the classical state $(1-p)x+py$. Because $x \sqsubseteq y$, there is a symmetry σ with $x \cdot \sigma, y \cdot \sigma$ monotone. First,

$$\begin{aligned}(z \cdot \sigma)_i &= (1-p)(x \cdot \sigma)_i + p(y \cdot \sigma)_i \\ &\geq (1-p)(x \cdot \sigma)_{i+1} + p(y \cdot \sigma)_{i+1} \\ &= (z \cdot \sigma)_{i+1},\end{aligned}$$

for $1 \leq i < n$, which means $z \cdot \sigma$ is monotone. Thus, $x \sqsubseteq z$ follows from

$$\begin{aligned}(x \cdot \sigma)_i (z \cdot \sigma)_{i+1} &\leq (x \cdot \sigma)_{i+1} (z \cdot \sigma)_i \\ &\Leftrightarrow \\ p(x \cdot \sigma)_i (y \cdot \sigma)_{i+1} &\leq p(x \cdot \sigma)_{i+1} (y \cdot \sigma)_i,\end{aligned}$$

while $z \sqsubseteq y$ follows similarly. $\square$

A *path* from x to y in a space X is a continuous map $p : [0,1] \to X$ with $p(0) = x$ and $p(1) = y$. A *segment* of a path p is $p[a,b]$ for $b > a$. Any monotone path into Δ^n with its Euclidean topology is Scott continuous. For instance, by the mixing law (Lemma 2.3), the *straight line path* from x to y,

$$\pi_{xy}(t) = (1-t)x + ty$$

is Scott continuous iff $x \sqsubseteq y$.

Lemma 2.34 *Let $x \sqsubseteq y$ with $x \in \Delta^n$ and $y \in \Lambda^n$. Then*

(i) *If $y_i > 0$ for all i, then $x \in \Lambda^n$.*

(ii) *If $x \ll y$, then $x \in \Lambda^n$.*

Proof. (i) The proof is by induction. For the $n+1$ case, Lemma 2.10 gives an index i with $x_i = x^- \geq y_i = y^-$, while the monotonicity of y yields $y_{n+1} = y^- = y_i > 0$. By degeneration (Lemma 2.13), $x_{n+1} = x_i = x^- > 0$.

Now we can apply the inductive hypothesis to $p_{n+1}(x) \sqsubseteq p_{n+1}(y)$, since $p_{n+1}(y) \in \Lambda^n$ and all its values are positive, to deduce that $p_{n+1}(x) \in \Lambda^n$. But since $x_{n+1} = x^-$, we have $x \in \Lambda^{n+1}$.

(ii) We apply (i). By the Scott continuity of $\pi_{\perp y}$,

$$y = \bigsqcup_{t<1} \pi_{\perp y}(t),$$

and since $x \ll y$, we have $x \sqsubseteq \pi_{\perp y}(t)$ for some $t < 1$. Because $y \in \Lambda^n$, $\pi_{\perp y}(t) \in \Lambda^n$, and $\pi_{\perp y}(t)_i > 0$ for all i since $t < 1$. By (i), $x \in \Lambda^n$. $\Box$

Proposition 2.35 *Let $r : \Delta^n \to \Lambda^n$ be the natural retraction.*

(i) *If $x, y \in \Delta^n_\sigma$ and $x \sqsubseteq y$, then $\pi_{xy}(t) \in \Delta^n_\sigma$ for all $t \in [0,1]$.*

(ii) *For $x, y \in \Delta^n$, we have $x \ll y$ iff*

$$(\forall \sigma \in S(n))(y \in \Delta^n_\sigma \Rightarrow x \in \Delta^n_\sigma) \quad \text{and} \quad (r(x) \ll r(y) \text{ in } \Lambda^n)\,.$$

Proof. (i) This was shown in the proof of the mixing law. (ii) First recall that right multiplication by $\sigma \in S(n)$, $r_\sigma(x) = x \cdot \sigma$, is an order isomorphism of Δ^n. If $x \ll y$, then $x \sqsubseteq y$, which means $x, y \in \Delta^n_\sigma$ for some $\sigma \in S(n)$. Because r_σ is an order isomorphism,

$$x \ll y \Rightarrow r_\sigma(x) \ll r_\sigma(y) \text{ in } \Delta^n.$$

But $r(x) = r_\sigma(x)$ and $r(y) = r_\sigma(y)$, which means $r(x) \ll r(y)$ in Δ^n. However, $r(x), r(y) \in \Lambda^n$ *and in addition* Λ^n is closed under directed suprema in Δ^n by Prop. 2.27(i), which means that the supremum in Λ^n of a directed set $S \subseteq \Lambda^n$ is *equal* to the supremum it has as a subset of Δ^n. Thus, $r(x) \ll r(y)$ in Λ^n.

To finish this direction, suppose $y \in \Delta^n_\sigma$. Then $x \cdot \sigma = r_\sigma(x) \ll r_\sigma(y) = y \cdot \sigma$ in Δ^n, since r_σ is an order isomorphism. But $y \cdot \sigma$ is monotone, so Lemma 2.34 implies that $x \cdot \sigma$ is too, i.e., $x \in \Delta^n_\sigma$.

For the other direction, if we choose any $\sigma \in S(n)$ with $y \in \Delta^n_\sigma$, then $x \in \Delta^n_\sigma$. By assumption, we have $r_\sigma(x) = r(x) \ll r(y) = r_\sigma(y)$ in Λ^n. If we show that $r_\sigma(x) \ll r_\sigma(y)$ in Δ^n, then because r_σ is an order isomorphism, we may conclude $x \ll y$ in Δ^n.

Let (y_i) be an increasing sequence in Δ^n with $r_\sigma(y) = \bigsqcup y_i$. By Prop. 2.27, there is $\nu \in S(n)$ with $y_i \in \Delta^n_\nu$ for all i, and hence $y \cdot \sigma \in \Delta^n_\nu$. Then because $y \in \Delta^n_{\sigma\cdot\nu}$, we have $x \in \Delta^n_{\sigma\cdot\nu}$ by assumption, so the following relation involves only states in Λ^n:

$$x \cdot (\sigma \cdot \nu) = x \cdot \sigma \ll y \cdot \sigma = y \cdot (\sigma \cdot \nu) = \bigsqcup (y_i \cdot \nu).$$

Because $x \cdot \sigma \ll y \cdot \sigma$ in Λ^n, we must have $x \cdot (\sigma \cdot \nu) \sqsubseteq y_i \cdot \nu$ for some i, i.e., $r_\nu(r_\sigma(x)) \sqsubseteq r_\nu(y_i)$ which gives $r_\sigma(x) \sqsubseteq y_i$. Then $r_\sigma(x) \ll r_\sigma(y)$ in Δ^n, and now the proof is finished. □

Theorem 2.36 *The classical states Δ^n are exact.*

(i) *For every $x \in \Delta^n$, $\pi_{\perp x}(t) \ll x$ for all $t < 1$.*

(ii) *The approximation relation $\ll$ is interpolative: If $x \ll y$ in Δ^n, then there is $z \in \Delta^n$ with $x \ll z \ll y$.*

Proof. The exactness of Δ^n follows from (i), the Scott continuity of $\pi_{\perp x}$, and Lemma 2.32. To prove (i), we first show that $\pi_{\perp x}(t) \ll x$ in Λ^n for any $x \in \Lambda^n$ and $t < 1$. Notice that $\pi_{\perp x}(t) \in \Lambda^n$ for all $t \in [0,1]$ by Prop. 2.35(i). Let $x = \bigsqcup y_k \in \Lambda^n$ for an increasing sequence $(y_k)_{k\geq 1}$ in Λ^n.

For $i < n$ fixed, we will show that there is an integer k_i such that

$$\left(\frac{1-t}{n} + tx_i\right) \pi_{i+1}(y_k) \leq \pi_i(y_k) \left(\frac{1-t}{n} + tx_{i+1}\right)$$

for all $k \geq k_i$. If $x_i = 0$, then $x_{i+1} = 0$ by the monotonicity of x, and then we can take $k_i = 1$, by the monotonicity of each y_k. Thus, we can assume $x_i > 0$.

If $x_{i+1} = 0$, then we can write

$$\left(\frac{1-t}{n} + tx_i\right) \lim_{k\to\infty} \pi_{i+1}(y_k) = 0 < \delta < x_i \left(\frac{1-t}{n}\right) = \lim_{k\to\infty} \pi_i(y_k) \left(\frac{1-t}{n}\right),$$

where $\delta > 0$ is some constant, and we use $x = \bigsqcup y_k$. This makes it clear that such a k_i exists in this case. Thus, we can also assume $x_{i+1} > 0$.

If $x_i = x_{i+1} > 0$, then because $y_k \sqsubseteq x$, degeneration (Lemma 2.13) gives $\pi_i(y_k) = \pi_{i+1}(y_k) > 0$ for each k. In this case, we can again take $k_i = 1$. Thus, we assume $x_i > x_{i+1} > 0$. By the degeneration lemma, this also implies $\pi_i(y_k) > 0$ and $\pi_{i+1}(y_k) > 0$ for all k. But then we get

$$\frac{(1-t)/n + tx_i}{(1-t)/n + tx_{i+1}} < \frac{x_i}{x_{i+1}} = \lim_{k\to\infty} \frac{\pi_i(y_k)}{\pi_{i+1}(y_k)},$$

using $x_i > x_{i+1} > 0$, $t < 1$ and $\bigsqcup y_k = x$. Thus, in this case there is also a large enough k_i such that the desired inequality holds for all $k \geq k_i$.

Then $\pi_{\perp x}(t) \sqsubseteq y_k$ where $k \geq \max\{k_i : 1 \leq i < n\}$, which proves $\pi_{\perp x}(t) \ll x$ in Λ^n for all $t < 1$. To finish the proof, let x be *any* classical state and $r : \Delta^n \to \Lambda^n$ the natural retract. We know

- $x \in \Delta^n_\sigma \Rightarrow \pi_{\perp x}(t) \in \Delta^n_\sigma$ for all $t \in [0, 1]$, and
- $r(\pi_{\perp x}(t)) = \pi_{\perp r(x)}(t) \ll r(x)$ in Λ^n, for all $t < 1$,

where the first follows from Prop. 2.35(i), and the second from what we proved above. By Prop. 2.35(ii), these two give $\pi_{\perp x}(t) \ll x$ for all $t < 1$.

(ii) First, for any $x \in \Delta^n$, we have $\pi_{\perp x}(s) \ll \pi_{\perp x}(t)$ whenever $s < t$. This easily follows from (i): For $p := \pi_{\perp x}(t)$ we have

$$\pi_{\perp x}(s) = \pi_{\perp p}(s/t) \ll p = \pi_{\perp x}(t).$$

If $x \ll y$, then $x \sqsubseteq \pi_{\perp y}(t) \ll y$ for some $t < 1$. Thus,

$$x \sqsubseteq \pi_{\perp y}(t) \ll \pi_{\perp y}((t + 1)/2) \ll y,$$

so taking $z := \pi_{\perp y}((t + 1)/2)$ finishes the proof. $\square$

The last result demonstrates the existence of a natural approximative structure on classical states: The dcpo Δ^n can rightfully be called a *domain*. As we said at the start, domains normally have *partial* elements, and *total* or ideal elements. We now explain the relationship between the qualitative notion of approximation $\ll$ and the natural intuitive notions of 'partiality' and 'totality' for classical states.

Intuitively, a classical state x is *partial* iff it offers no certainty about any outcome iff $(\forall i)\, 0 < x_i < 1$ iff $(\forall i)\, x_i > 0$. One may object that $x = (1/2, 1/2, 0) \in \Delta^3$ seems partial but is excluded from the above. However, only "some" of x is partial, the element $p_3(x) = \perp \in \Delta^2$. As a state in Δ^3, though, x is not genuinely partial because it imparts certainty about the third outcome.

On the other hand, if we *assume* that the order theoretic structure of Δ^n has captured our *intuitive* understanding of classical states, we easily arrive at an alternative formulation of partiality: An object is partial when it approximates something. The latter of course is *purely qualitative* and provides exactly what one hopes for: A formalization of intuition.

Lemma 2.37 (Partiality) *For each $x \in \Delta^n$, the set $\twoheaduparrow x$ is nonempty iff $x_i > 0$ for all i.*

Proof. If $x \ll y$, there there is $t < 1$ with $x \sqsubseteq \pi_{\perp y}(t)$. Because $t < 1$, $\pi_{\perp y}(t)_i > 0$, so degeneration (Lemma 2.13) gives $x_i > 0$. For the other direction, let $x_i > 0$ for all i.

Intuitively, because x is in the interior of Δ^n, the line segment from $\perp$ to x can be extended *nontrivially* to a point y on the boundary of Δ^n, for which we then have $x \ll y$. Formally now, we can assume $x \neq \perp$. Then

$$0 < x^- < 1/n \;\Rightarrow\; \lambda := \frac{1}{1 - nx^-} > 1.$$

Let y be the classical state defined pointwise by

$$y_i = \frac{1}{n} \cdot (1-\lambda) + \lambda \cdot x_i$$

for each $1 \leq i \leq n$. To see that y is in fact a classical state, notice that

$$0 = \frac{1}{n} \cdot (1-\lambda) + \lambda x^- \leq y_i \leq \sum_{i=1}^{n} y_i = 1.$$

Since $0 \leq 1/\lambda < 1$, $\pi_{\perp y}(1/\lambda) = x \ll y$, which proves $\twoheaduparrow x \neq \emptyset$. $\Box$

The 'opposite' of partiality is totality: A classical state is *total* when it imparts certainty about *all* of its outcomes. Thus, the total or *ideal* classical states are exactly the pure states e_i, which we have already characterized qualitatively as being precisely $\max(\Delta^n)$. But the approximation relation can offer additional insight about the sense in which pure states are total.

To understand the connection between the two, let's begin by thinking about $x \ll y$, which we could say means that

- All paths (y_i) to y must *qualitatively* exceed x after some finite stage,

which can be read as

- All paths to y essentially begin with x,

and finally

- A process (y_i) can only end up in state $y = \bigsqcup y_i$ provided that it has the information represented by x: x is *necessary* for having y, i.e., the only way to know y is to first know x.

In each version of $\ll$ above, some reference to a process is made (a path is assumed to be generated by some process), providing us with a crucial distinction between $\ll$ and $\sqsubseteq$: $x \ll y$ is a statement about *processes,* $x \sqsubseteq y$ is a statement about information. The difference between these two becomes clear by considering states x, y, z with $x \ll y \sqsubseteq z$ but *not* $x \ll z$.

Example 2.38 Let $\perp \neq x \ll y := (1/2, 1/2, 0) \sqsubseteq z := e_1$. Then $x \not\ll z$. Here are two equivalent perspectives:

(i) In terms of knowledge: We are not *required* to know that an object is not in box 3 before we can know that it is in box 1.

(ii) In terms of processes: From an initial state of $\perp$, one way to conclude the object is in box 1 is to begin by ruling out box 3 as a possibility, and then look in one of the others – but this does not describe *all* ways. We could just look in box 1.

Thus, $\sqsubseteq$ makes statements about *potential* evolutions of state; $\ll$ is concerned with what we must know in order to obtain information using the process of observation.

This example suggests that $\ll$ is capable of expressing a characteristic of totality: The only time we expect the implication

$$(\forall y, z)\, y \in \twoheaduparrow x \text{ and } y \sqsubseteq z \Rightarrow z \in \twoheaduparrow x$$

to hold nontrivially is when x is a state from which a *unique* outcome is likely, i.e., x approximates a unique pure state. When $\twoheaduparrow x$ satisfies the implication above, it is called an *upper set*.

Proposition 2.39 (Approximation of pure states) *Let* $n \geq 2$.

(i) *For all* $x \in \Delta^n$, $x \ll e_i$ *iff* $x = \pi_{\perp e_i}(t)$ *for some* $t < 1$.

(ii) *For all* $x \in \Delta^n$, $\twoheaduparrow x$ *is an upper set iff it is empty, all of* Δ^n, *or contains a unique pure state.*

Proof. (i) For $n = 2$ this is clear. Assume $n \geq 3$. Because $x \ll e_i$, there is $s < 1$ with $x \sqsubseteq \pi_{\perp e_i}(s)$. By degeneration, $(\exists a > 0)(\forall k \neq i)(x_k = a)$, which now makes the claim obvious.

(ii) For $(\Rightarrow)$, every nonempty upper set contains at least one maximal element. By (i), either $x = \perp$, or $\twoheaduparrow x$ contains a unique pure state.

For the other direction, we need to prove that $\twoheaduparrow x$ is an upper set when it contains a unique pure state e_i. Suppose $x \ll y \sqsubseteq z$. First, because $x \ll e_i$, it is routine to show that $x_i = x^+$ and $x_k = x^-$ for all $k \neq i$. Because $\twoheaduparrow x$ contains a unique pure state, $x \neq \perp$, which means $x^+ > x^- > 0$. To apply Prop. 2.35(ii), we first show $z \in \Delta^n_\sigma \Rightarrow x \in \Delta^n_\sigma$.

Let $z \cdot \sigma$ be monotone. Because $x \sqsubseteq z$, $x \cdot \sigma \sqsubseteq z \cdot \sigma$, which means there is an index k with $(x \cdot \sigma)_k = x^+ \leq (z \cdot \sigma)_k = z^+$. By the monotonicity of $z \cdot \sigma$ and degeneration,

$$(z \cdot \sigma)_k = (z \cdot \sigma)_1 = z^+ \geq x^+ > 0 \implies (x \cdot \sigma)_k = (x \cdot \sigma)_1 = x^+ > 0,$$

which means $x \cdot \sigma$ is monotone, since the only other value it assumes is x^-.

To finish, we need to show $r(x) \ll r(z)$ in Λ^n. First, $r(z)_2 > 0$, since otherwise $r(z) = e_1$, for which we already know $r(x) \ll r(e_i) = e_1 = r(z)$. By degeneration, this also means $r(y)_2 > 0$. Because $r(x) \ll r(y)$ in Λ^n, there is $t < 1$ with $r(x) \sqsubseteq \pi_{\perp r(y)}(t)$. Thus,

$$\frac{r(x)_1}{r(x)_2} \leq \frac{(1/n)(1-t) + tr(y)_1}{(1/n)(1-t) + tr(y)_2} < \frac{r(y)_1}{r(y)_2} \leq \frac{r(z)_1}{r(z)_2},$$

where the strict inequality follows from $r(y)_1 > r(y)_2 > 0$ (which is a consequence of degeneration using $r(x)_1 > r(x)_2 > 0$ and $r(x) \sqsubseteq r(y)$). Because $r(x)_i / r(x)_{i+1} = 1$ for $1 < i < n$, it is clear that $r(x) \ll r(z)$ in Λ^n. $\square$

An approximation a of a pure state x defines a region $\twoheaduparrow a$ of Δ^n known in domain theory as a *Scott open set*.

Definition 2.40 A subset U of a dcpo D is *Scott open* if

- U is an *upper set*: $(\forall x \in U)(\forall y \in D)\, x \sqsubseteq y \Rightarrow y \in U$, and
- U is *inaccessible by directed suprema*: For any directed set $S \subseteq D$,

$$\bigsqcup S \in U \Rightarrow S \cap U \neq \emptyset\,.$$

The collection of all Scott open subsets of D is σ_D.

Notice that a map $f : D \to E$ between dcpo's is Scott continuous in the sense defined earlier iff $f^{-1}(U)$ is Scott open in D whenever U is Scott open in E.

Lemma 2.41 *For all $x \in \Delta^n$, $\mathrel{\uparrow\!\!\!\!\uparrow} x$ is an upper set iff it is Scott open.*

Proof. If $z = \bigsqcup S \in \mathrel{\uparrow\!\!\!\!\uparrow} x$, then by interpolation (Theorem 2.36), there is $y \in \Delta^n$ with $x \ll y \ll z$. Thus, by $y \ll z$, there is $s \in S$ with $y \sqsubseteq s$, and since $\mathrel{\uparrow\!\!\!\!\uparrow} x$ is an upper set, $s \in \mathrel{\uparrow\!\!\!\!\uparrow} x$. Interestingly, one can also show that $\mathrel{\uparrow\!\!\!\!\uparrow} x$ is Scott open iff $\uparrow(\mathrel{\uparrow\!\!\!\!\uparrow} x)$ is Scott open. $\square$

The relation between approximation, partiality and purity can now be summarized as follows:

(i) The partial elements are those $x \in \Delta^n$ with $\mathrel{\uparrow\!\!\!\!\uparrow} x \neq \emptyset$.

(ii) For a partial element $x \in \Delta^n$, $\mathrel{\uparrow\!\!\!\!\uparrow} x$ is Scott open iff $x = \pi_{\perp e_i}(t)$ for some i and some $t < 1$ iff ($x = \perp$ or x approximates a unique pure state).

Thus, the 'totality' of a pure state x is largely explained by the fact that $\mathrel{\uparrow\!\!\!\!\uparrow} a$ is Scott open whenever $a \ll x$. To complete the picture,

Lemma 2.42 *A subset $U \subseteq \Delta^n$ is Scott open iff*

- *Any monotone path from $x \in U$ to a pure state lies in U, and*
- *The line from $\perp$ to $x \in U$ has a segment contained in U,*

and for pure states x, there is an equivalence between 'approximation of x' and 'Scott open set containing x': Given any $a \ll x$, the set $\mathrel{\uparrow\!\!\!\!\uparrow} a$ is Scott open, while given any Scott open U with $x \in U$, we can (by exactness) find an approximation $a \in U$ of x with $x \in \mathrel{\uparrow\!\!\!\!\uparrow} a \subseteq U$.

Approximation can also describe things of a more concrete nature. Because of its close connection to the mixing law, which is especially evident in the case of pure states (Prop. 2.39(i)), we can sometimes reinterpret mixing as approximation. This, for instance, can be useful when one seeks to explain the sense in which certain forms of noise work "against" the state σ of a system.

Example 2.43 *The depolarization channel.* The map $d_p : \Delta^n \to \Delta^n$ by

$$d_p(\sigma) = p\bot + (1-p)\sigma$$

describes the process by which a state $\sigma \in \Delta^n$ is *depolarized* with probability $p > 0$ (has all bias and hence all information removed from it) and is otherwise unaltered. But notice:

$$d_p(\sigma) = \pi_{\bot\sigma}(1-p),$$

which means $d_p(\sigma) \ll \sigma$ for $p > 0$. In particular, the effect of depolarization on a state is *qualitative.*

To say that the effect of noise is qualitative essentially means that while the state of the system has suffered, it has not been "degraded beyond recognition." This is not always the case: Some forms of noise are more destructive than others and the order on classical states can at times capture this.

Example 2.44 *Classical bit flipping.* A state $\sigma \in \Delta^2$ suffering the effect of a magnetic field is "flipped" with probability p and otherwise left alone

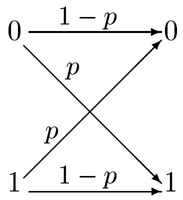
$$f_p(\sigma) = p\sigma^* + (1-p)\sigma,$$

where $*$ is the *involution* $(x,y)^* = (y,x)$. In this case, we have

$$(\forall\sigma. f_p(\sigma) \ll \sigma) \Leftrightarrow 0 < p \leq 1/2,$$

i.e., the effect of the noise is qualitative iff the field is weak enough.

Those familiar with classical information theory may know what we call classical bit flipping by another name, the *binary symmetric channel.* In this important example, a bit (a '0' or a '1') is transmitted correctly through a channel with probability $1-p$ and reversed with probability p:



Given that information is sent through the binary symmetric channel, we want to determine the information that is actually received. The information sent is modelled by $\sigma = (x,y) \in \Delta^2$, where x is the probability that 0 is sent and y is the probability that 1 is sent. The effect that the channel has on information passing through it (σ) is captured by its *channel matrix*

$$\begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}$$

To determine the information received when (x, y) is sent, we calculate a distribution for the output using the channel matrix as follows:

$$\begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (1-p)x + py \\ px + (1-p)y \end{bmatrix}$$

All of this is implicit in the operator $f_p(\sigma) = p\sigma^* + (1-p)\sigma$ of Example 2.44: The distribution for the output is $f_p(\sigma)$, the 0 bit is $e_1 = (1, 0)$, the 1 bit is $e_2 = (0, 1)$, and reversing σ means applying the involution $*$ to obtain σ^*.

2.5 Entropy, content and partiality

We have already seen how the use of $\sqsubseteq$ on Δ^n enables a precise formulation of what it means to say that a classical state is "information." One of the advantages in taking this approach to *defining* information is that the structure of a domain can then be used to define the notion "information content," i.e., we can say what it *means* to measure the *content* of information.

The idea introduced in [8] is this: Assuming that information is formally specified as a domain, measuring content *means* measuring partiality, i.e., the amount of partiality in an object. The importance of this conceptually is that partiality, as we have already seen, is intimately connected to the order theoretic structure of a domain.

To slightly motivate the formal definition we are about to see, suppose that $\mu : \Delta^n \to [0, \infty)$ is a measure of content on classical states. Then μx is the amount of uncertainty (or partiality) in x. As we move up in the order $\sqsubseteq$ on Δ^n, states become more informative, so uncertainty *decreases*:

$$x \sqsubseteq y \Rightarrow \mu x \geq \mu y.$$

That is, as a map from Δ^n to $[0, \infty)^*$, μ is monotone. If μ is defined in terms of the usual formulae from physics (arithmetic, logarithms, other elementary functions), then it is continuous in the sense of analysis, and hence *Scott continuous* from Δ^n to $[0, \infty)^*$.

The essence of the distinction between content and a random continuous map on a domain is subtle. Consider a pure state $x \in \max(\Delta^n)$ and one of its approximations $a \ll x$, so that a is information any process must have before it can evolve to x. Then we also expect $a \ll y$ *provided* that

(i) y is a state from which it is possible to evolve to x, and

(ii) y is "close enough" to x in content.

The first translates as "$y \sqsubseteq x$"; the second translates as "$|\mu x - \mu y| < \varepsilon$," on the *assumption* that μ measures information content. Putting everything together now, *if* μ is a measure of content, *then* we expect that

$$x \in \Uparrow a \Rightarrow (\exists \varepsilon > 0)(y \sqsubseteq x \;\&\; |\mu x - \mu y| < \varepsilon \Rightarrow y \in \Uparrow a).$$

Because x is pure, we can replace $\Uparrow a$ with a Scott open set $U \subseteq \Delta^n$, as we saw in the last section.

Definition 2.45 A Scott continuous map $\mu : D \to [0, \infty)^*$ on a dcpo is said to *measure the content of* $x \in D$ if

$$x \in U \Rightarrow (\exists \varepsilon > 0)\, x \in \mu_\varepsilon(x) \subseteq U,$$

whenever $U \in \sigma_D$ is Scott open and

$$\mu_\varepsilon(x) := \{y \in D : y \sqsubseteq x \ \&\ |\mu x - \mu y| < \varepsilon\}$$

are the elements ε close to x in content. The map μ *measures* X if it measures the content of each $x \in X$.

In order for a map μ to be regarded "a measure of content," it must *minimally* be capable of distinguishing those elements which *it claims* are maximally informative. That is, μ must measure all of the objects which it regards as possessing no uncertainty $\ker \mu := \{x : \mu x = 0\}$.

Definition 2.46 A *measurement* is a Scott continuous map $\mu : D \to [0, \infty)^*$ on a dcpo that measures $\ker \mu := \{x \in D : \mu x = 0\}$.

The measurement formalism [8] teaches that the ability to measure content is indicative of a purely structural relationship that exists between two classes of informative objects. Neither class need consist of numbers. This relationship is formally expressed by a map $\mu : D \to E$ whose general nature is to *reflect* properties of simpler objects E onto more complex objects D.

The motivation for the idea stems from the empirical fact that it is often easier to reason about D in terms of E rather than deal with D directly [8]. Hence the reflective nature of μ: It confirms that we *actually can* learn about $x \in D$ by studying the properties of its simplification $\mu x \in E$.

Definition 2.47 A Scott continuous map $\mu : D \to E$ between dcpo's is said to *measure the content of* $x \in D$ if

$$x \in U \Rightarrow (\exists \varepsilon \in \sigma_E)\, x \in \mu_\varepsilon(x) \subseteq U,$$

whenever $U \in \sigma_D$ is Scott open and

$$\mu_\varepsilon(x) := \mu^{-1}(\varepsilon) \cap {\downarrow} x$$

are the elements ε close to x in content. The map μ *measures* X if it measures the content of each $x \in X$.

Definition 2.48 A *measurement* is a Scott continuous map $\mu : D \to E$ between dcpo's that measures $\ker \mu := \{x \in D : \mu x \in \max(E)\}$.

These definitions are easily seen to be equivalent to the quantitative formulations we saw earlier by setting $E = [0, \infty)^*$. To establish the reflective nature of content, we use the following relationship between the order $\sqsubseteq$ on a dcpo D and its Scott open sets σ_D:

$$x \sqsubseteq y \Leftrightarrow (\forall U \in \sigma_D)(x \in U \Rightarrow y \in U).$$

Proposition 2.49 *Let $\mu : D \to E$ be a measurement and x an object that it measures.*

(i) *If $\mu x \in \max(E)$, then $x \in \max(D)$.*

(ii) *If $\mu x = \bot$, then $x = \bot$, provided $\bot \in D$ exists.*

(iii) *If $y \sqsubseteq x$ and $\mu x = \mu y$, then $x = y$.*

(iv) *If $x_n \sqsubseteq x$ and (μx_n) is directed with supremum μx, then $\bigsqcup x_n = x$.*

In addition, the composition of measurements is again a measurement.

Proof. The proofs here are essentially taken verbatim from [8], where other properties of content can be found.

(iii) Let $x \in U$. Then $y \in \mu_\varepsilon(x) \subseteq U$, for some $\varepsilon \in \sigma_E$. Since U was arbitrary, $x \sqsubseteq y$. By antisymmetry, $x = y$.

(i) If $x \sqsubseteq y$, then $\mu x = \mu y$, since $\mu x \in \max(E)$, which gives $y \in \ker \mu$. Since μ is a measurement, it measures y, so $x = y$ by (iii).

(ii) First, $\mu(\bot) \sqsubseteq \mu x = \bot$, so $\mu(\bot) = \bot = \mu x$. Since $\bot \sqsubseteq x$, we can apply (iii) to obtain $x = \bot$.

(iv) Let $x_n \sqsubseteq u$ for all n. If $x \in U$, then $x \in \mu_\varepsilon(x) \subseteq U$, which means

$$\mu x = \bigsqcup \mu x_n \in \varepsilon,$$

and so $\mu x_n \in \varepsilon$ for some n, which gives $x_n \in U$ and hence $u \in U$. Since U was arbitrary, $x \sqsubseteq u$. Thus, $\bigsqcup x_n = x$.

Finally, if we have measurements $D \xrightarrow{\mu} E \xrightarrow{\lambda} F$, then $\lambda\mu$ measures $\ker \lambda\mu$ as follows. First, if $x \in \ker \lambda\mu$ and $x \in U \in \sigma_D$, then $x \in \ker \mu$ so there is $\varepsilon \in \sigma_E$ with $x \in \mu_\varepsilon(x) \subseteq U$. Then, since $\mu x \in \varepsilon$ and $\mu x \in \ker \lambda$, there is $\delta \in \sigma_F$ with $\mu x \in \lambda_\delta(\mu x) \subseteq \varepsilon$. We have

$$x \in (\lambda\mu)^{-1}(\delta) \cap \downarrow\!x \subseteq \mu_\varepsilon(x) \subseteq U,$$

which finishes the proof. $\square$

With the benefit of the abstract formulation of content, let us take a second look at uncertainty ($E = [0, \infty)^*$). By Prop. 2.49(i), we know that

$$\mu x = 0 \Rightarrow x \in \max(D),$$

for any measurement $\mu : D \to [0,\infty)^*$. That is, quantitative certainty implies qualitative certainty. As a case in point, if $D = \Delta^n$, then, as we will see shortly, Shannon entropy $\mu : D \to [0,\infty)^*$ given by

$$\mu x = -\sum_{i=1}^{n} x_i \log x_i$$

is a measurement. Thus, any classical state x with entropy $\mu x = 0$ is *pure.* But now we have an explanation for why such properties hold:

(i) In the sense of the measurement formalism, μ is a measure of content between the domains Δ^n and $[0,\infty)^*$, and

(ii) Measures of content between domains *always* reflect maximality.

The same is true of the von Neumann entropy on quantum states (that we will see later). But the moral of the last result is what is most important: *Subject to moderate hypotheses, information behaves in the same manner as its content.*

Proposition 2.50 *The natural retraction $r : \Delta^n \to \Lambda^n$ is a measurement.*

Proof. To start, notice that $\ker r = \max(\Delta^n)$. Let $U \subseteq \Delta^n$ be a Scott open set that contains the pure state x. By exactness, there is $\perp \neq a \ll x$ with $a \in U$. By Prop. 2.35(ii), $r(a) \ll r(x)$ in Λ^n. Because x is pure, $\varepsilon := \twoheaduparrow r(a)$ is a Scott open subset of Λ^n (a corollary of Theorem 2.36 and Prop. 2.39). We claim $x \in r_\varepsilon(x) \subseteq \twoheaduparrow a \subseteq U$ as follows.

First, $x \in r_\varepsilon(x)$ by $r(a) \ll r(x)$. Then, if $y \in r_\varepsilon(x)$, we have $r(a) \ll r(y)$ in Λ^n and $y \sqsubseteq x$. To prove that $a \ll y$ in Δ^n and finish the proof, we must show $y \in \Delta^n_\sigma \Rightarrow a \in \Delta^n_\sigma$.

For this subtle point, a takes its maximum at a unique index, because $a \neq \perp$ and it approximates a pure state (Prop. 2.39(i)). Then $r(a)$ does as well. Since $r(a) \sqsubseteq r(y)$, degeneration implies the same is true of $r(y)$ and hence of y. Thus, because y takes its maximum at a unique index, and because $y \sqsubseteq x \in \max(\Delta^n)$, we have

$$y \in \Delta^n_\sigma \Rightarrow x \in \Delta^n_\sigma,$$

while $a \ll x$ then implies $a \in \Delta^n_\sigma$. $\square$

We have made intuitive use of this fact numerous times: Whenever we prove a statement about classical states by first proving it for monotone states, we are *implicitly* appealing to the fact that $r(x)$ provides a decent measure of the content of x.

Example 2.51 The standard variable $v : \Delta^n \to [0,\infty)^*$ given by

$$v(x) = 1 - x^+$$

is a measurement with $\ker v = \max(\Delta^n)$. To prove as much, we need only show that its restriction to Λ^n, $\lambda := v|_{\Lambda^n}$, is a measurement, since then $v = \lambda \circ r$ must be another.

To this end, let $U \subseteq \Lambda^n$ be a Scott open set and $x \in \ker \lambda$. Because U is Scott open, there is $t < 1$ with $a := \pi_{\perp x}(t) \in U$. We then have

$$x \in \lambda_\varepsilon(x) \subseteq \uparrow\! a \subseteq U,$$

where

$$\varepsilon := \frac{1}{2} \cdot \frac{a_2}{a_1 + a_2} > 0\,.$$

Example 2.52 The entropy $s : \Delta^n \to [0, \infty)^*$ given by

$$s(x) = -\log x^+$$

is a measurement with $\ker s = \max(\Delta^n)$. First, $s(x) \geq v(x)$, using the classic inequality $\log t \leq t - 1$ for $t > 0$. Thus,

$$x \in s_\varepsilon(x) \subseteq v_\varepsilon(x),$$

for any pure state x and $\varepsilon > 0$. Because v is a measurement, so is s.

Now for Shannon entropy.

Lemma 2.53 *Let $x \sqsubseteq y$ be monotone classical states in Δ^n. Then there is $k \in \{1, \ldots, n\}$ such that*

(i) $(\forall i < k)\, x_i \leq y_i$, *and*

(ii) $(\forall i \geq k)\, x_i \geq y_i$.

Proof. First, since $x \sqsubseteq y$, we have by induction that $x_i y_{i+j} \leq y_i x_{i+j}$, for each $j \in \{0, \ldots, n - i\}$. Thus, if $x_i \geq y_i$, then $x_{i+j} \geq y_{i+j}$ for each $j \in \{0, \ldots, n - i\}$. Now let k be the least integer $1 \leq k \leq n$ with $x_k \geq y_k$. Notice that such a k exists since $x_n \geq y_n$. This finishes the proof. $\square$

The *relative Shannon entropy* of y given x is

$$\mu(y \| x) := \sum_{i=1}^{n} y_i \log(y_i / x_i)$$

where $x, y \in \Delta^n$. This quantity is always nonnegative and is zero iff $x = y$.

Theorem 2.54 *Let $\mu : \Delta^n \to [0,\infty)^*$ be the Shannon entropy on classical states*

$$\mu x = -\sum_{i=1}^{n} x_i \log x_i$$

where the logarithm is natural. Then μ is a measurement. In addition,

(i) *For all $x, y \in \Delta^n$, if $x \sqsubseteq y$ and $\mu(x) = \mu(y)$, then $x = y$.*

(ii) *For all $x \in \Delta^n$, we have $\mu(x) = 0$ iff $x \in \max(\Delta^n)$.*

(iii) *For all $x \in \Delta^n$, we have $\mu(x) = \log n$ iff $x = \perp$.*

Proof. Because μ is symmetric, its Scott continuity follows if we show that its restriction to the dcpo Λ^n is Scott continuous. First we prove its monotonicity into $[0,\infty)^*$.

Let $x \sqsubseteq y$ be monotone classical states. By Lemma 2.53, there is an integer $k \in \{1, \ldots, n\}$ such that $x_i \le y_i$ for $i < k$ and $x_i \ge y_i$ for $i \ge k$. Then

$$\sum_{i=1}^{n}(y_i - x_i)\log x_i = \sum_{i<k}(y_i - x_i)\log(x_i/x_k) + \sum_{i>k}(y_i - x_i)\log(x_i/x_k) \ge 0.$$

Notice that if $x_k = 0$ then the sum of the $i > k$ vanishes, while the sum of the $i < k$ blows up, but is nevertheless nonnegative. From the nonnegativity of this sum, we have

$$\mu x \ge -\sum_{i=1}^{n} y_i \log x_i \ge \mu y,$$

where the second inequality follows from $\mu(y\|x) \ge 0$. This proves that μ is monotone into $[0,\infty)^*$.

If in addition to $x \sqsubseteq y$ we also have $\mu x = \mu y$, then the inequality above immediately gives $\mu(y\|x) = 0$, which implies $x = y$. This establishes that μ is strictly monotone. For its Scott continuity, if (x_i) is increasing, then

$$\begin{aligned}
\mu(\bigsqcup x_i) &= \mu(\lim_{i\to\infty} \pi_1(x_i), \ldots, \lim_{i\to\infty} \pi_n(x_i)) \\
&= \lim_{i\to\infty} \mu(\pi_1(x_i), \ldots, \pi_n(x_i)) \\
&= \lim_{i\to\infty} \mu x_i,
\end{aligned}$$

where the first equality uses Prop. 3.28 and the second uses the continuity of μ with respect to the Euclidean topology. By Lemma 2.18, μ is Scott continuous. Finally, μ is a measurement: For $x \in \Delta^n$, we have

$$\mu x \ge -x^+ \log x^+ \ge \frac{1}{n} \cdot v(x),$$

using $\log t \le t - 1$ for $t > 0$ and $x^+ \ge 1/n$, where v is the variable from Example 2.51. Since v is a measurement, $(1/n) \cdot v$ is a measurement, which means that μ is as well. $\square$

It is important to realize that the minimal account of content given here is more substantial than it may seem: There are natural mappings which do not measure content.

Example 2.55 *Numbers are not enough.* For $n \geq 3$, consider

$$f : \Delta^n \to [0, \infty)^* :: x \mapsto x^- .$$

It is Scott continuous, symmetric and assumes its order theoretic minimum at $\perp$. Furthermore, even though $f(x) = 0$ for all $x \in \max(\Delta^n)$, f *does not* measure the content of a single pure state.

For instance, suppose f measured the content of $e_1 \in \Delta^3$. Then given any open $U \subseteq \Delta^n$ with $e_1 \in U$, there would exist $\varepsilon > 0$ with $e_1 \in f_\varepsilon(e_1) \subseteq U$. Then $(1/2, 1/2, 0) \in U$. But because this applies to *any* open set U, we now have a proof that $e_1 \sqsubseteq (1/2, 1/2, 0)$.

More intuitively: Many states are assigned maximal measure by f which are not pure. For instance $f(x, y, 0) = 0$ on Δ^3, even though the only time $(x, y, 0)$ is pure is when $x = 1$ or $y = 1$.

Here is a summary.

Example 2.56 *Canonical measures of content on Δ^n.*

(i) The maps $\Delta^n \to [0, 1] :: x \mapsto x^+$ and $\Delta^n \to [0, 1]^* :: x \mapsto 1 - x^+$.

(ii) Entropy $s(x) = -\log x^+$.

(iii) The natural retraction $r : \Delta^n \to \Lambda^n$.

(iv) Shannon entropy

$$\mu x = -\sum_{i=1}^{n} x_i \log x_i .$$

3 Quantum states

We now pursue the idea which motivated our study of the Bayesian order on classical states: The *spectral order* on quantum states. Later we will see that the spectral order can be characterized in a manner completely analogous to the order on classical states:

- The *inductive formulation*, in terms of quantum projections, and
- The *symmetric formulation*, in terms of unitary transformations.

These two accounts of the quantum order, when restricted to a class of states exhibiting classical behavior, are equivalent to the inductive and symmetric characterizations of the Bayesian order on classical states studied in the last section.

3.1 Essentials

An n-dimensional complex Hilbert space $\mathcal{H}^n$ is an n-dimensional vector space over $\mathbb{C}$ with specified inner product $\langle \cdot \mid \cdot \rangle$.

Definition 3.1 A *base* of $\mathcal{H}^n$ is a sequence $(\psi_i)_{i=1}^n$ of *unit vectors,*

$$\langle \psi_i \mid \psi_i \rangle = 1,$$

which are *mutually orthogonal*:

$$i \neq j \Rightarrow \langle \psi_i \mid \psi_j \rangle = 0.$$

We write $x \perp y$ to express the orthogonality of two vectors $x, y \in \mathcal{H}^n$, and as is customary, extend this notation to subspaces of $\mathcal{H}^n$ as follows:

$$\Psi \perp \Phi \;\Leftrightarrow\; \forall \psi \in \Psi \setminus \{o\}, \forall \phi \in \Phi \setminus \{o\} : \psi \perp \phi$$

where o is the zero of $\mathcal{H}^n$.

Definition 3.2 A linear operator $\rho : \mathcal{H}^n \to \mathcal{H}^n$ is *self-adjoint* if

$$\langle \phi \mid \rho\psi \rangle = \langle \rho\phi \mid \psi \rangle,$$

for all $\phi, \psi \in \mathcal{H}^n$, *positive* when

$$\langle \psi \mid \rho\psi \rangle \geq 0$$

for all $\psi \in \mathcal{H}^n$, and *idempotent* when $\rho^2 := \rho \circ \rho = \rho$.

The *spectral theorem* of von Neumann [12], roughly speaking, states that each self-adjoint operator on a Hilbert space decomposes into a sum of simple operators called *projections.*

Definition 3.3 A *projection* or *projector* is a self-adjoint, linear, idempotent operator. The set of projections is denoted $\mathbb{P}^n$. A projection $P \in \mathbb{P}^n$ is fully characterized by its subspace of fixed points $\mathsf{fix}(P) \subseteq \mathcal{H}^n$.

All we need here is the finite dimensional case of the spectral theorem.

Theorem 3.4 *A self-adjoint linear operator $\rho : \mathcal{H}^n \to \mathcal{H}^n$ decomposes uniquely into a linear combination of mutually orthogonal projections*

$$\rho = \sum_{\lambda \in \mathsf{spec}(\rho)} \lambda \cdot P_\rho^\lambda \quad \text{with} \quad \sum_{\lambda \in \mathsf{spec}(\rho)} P_\rho^\lambda = I$$

whose images span $\mathcal{H}^n$. The set $\mathsf{spec}(\rho) \subseteq \mathbb{R}$ is called the spectrum of ρ.

We write the fact that the images of the projections span $\mathcal{H}^n$ as

$$\mathsf{span}\left(\bigcup_{\lambda \in \mathsf{spec}(\rho)} \mathsf{fix}(P_\rho^\lambda)\right) = \mathcal{H}^n , \tag{1}$$

where by idempotence we have $\mathsf{fix}(P) = \mathsf{Im}(P) = P(\mathcal{H}^n)$.

Definition 3.5 The *trace* of a linear operator ρ on $\mathcal{H}^n$ is

$$\mathsf{tr}(\rho) := \sum_i \langle \psi_i \mid \rho \psi_i \rangle,$$

where $\{\psi_i\}$ is any base of $\mathcal{H}^n$. If A is *any* matrix representation of ρ, then $\mathsf{tr}(\rho) = \sum A_{ii}$ is the sum of the elements on the diagonal of A.

The standard kinematical account of a quantum system includes both a description of the *states* a system can take, and of its *observables,* i.e., the *measurements* that can be performed on the system.

Definition 3.6 A *density operator* ρ on $\mathcal{H}^n$ is a self-adjoint, positive, linear operator with $\mathsf{tr}(\rho) = 1$. A *quantum n-state* is a density operator. The class of quantum n-states is denoted Ω^n.

Definition 3.7 A quantum state ρ is *pure* if $\mathsf{spec}(\rho) \subseteq \{0, 1\}$. The set of pure states is written Σ^n.

A classical state is a distribution on the set of pure states $\max(\Delta^n)$. Similarly, Gleason's theorem [5] establishes that density operators encode precisely the measures on the closed subspaces of $\mathcal{H}^n$, i.e., density operators are distributions on the set of pure states.

Definition 3.8 A *quantum n-measurement* is a self-adjoint linear operator $e : \mathcal{H}^n \to \mathcal{H}^n$.

For instance, if e is the energy observable, then its spectrum $\mathsf{spec}(e)$ contains the actual energy values a system can assume. According to quantum mechanics, if the density operator of a system is ρ, then a measurement of the observable e yields $\lambda \in \mathsf{spec}(e)$ as the result with probability

$$\mathsf{prob}_e^\lambda(\rho) := \mathsf{tr}(P_e^\lambda \cdot \rho).$$

Now what we want to do is rewrite all of this in a form more amenable to the task at hand.

Definition 3.9 $\mathbb{L}^n$ is the set of closed subspaces of $\mathcal{H}^n$.

By the spectral theorem, we can write a self-adjoint operator e as

$$e\psi = \sum_{\lambda \in \mathsf{spec}(e)} (\lambda \cdot P_e^\lambda)\psi.$$

By mutual orthogonality, $e\psi = \lambda\psi \ \Leftrightarrow\ P_e^\lambda \psi = \psi$, so the eigenspaces

$$e_\lambda := \{\psi \in \mathcal{H}^n \mid e\psi = \lambda\psi\} = \mathsf{fix}(P_e^\lambda)$$

give rise to a labeled collection of mutually orthogonal subspaces

$$\mathcal{D}_e := \{e_\lambda \mid \lambda \in \mathsf{spec}(e)\}$$

which span $\mathcal{H}^n$.

Definition 3.10 A *decomposition* of $\mathcal{H}^n$ is a family of mutually orthogonal subspaces of $\mathcal{H}^n$ of dimension at least one which span $\mathcal{H}^n$. The decompositions of $\mathcal{H}^n$ are denoted $\mathbb{D}^n$.

We will also refer to the union $\bigcup \mathcal{D}$ of a decomposition $\mathcal{D}$ as being the decomposition itself since the first characterizes the latter.

Definition 3.11 A *spectral decomposition* of $\mathcal{H}^n$ is an injective function $f : X \to \mathbb{L}^n$ defined on a nonempty set $X \subseteq \mathbb{R}$ with $f(X) \in \mathbb{D}^n$. The domain of f is written $\mathsf{spec}(f) = X$ and called the *spectrum* of f.

Equivalently, a spectral decomposition is a partial injection $f : \mathbb{R} \rightharpoonup \mathbb{L}^n$ with $\mathsf{Im}(f) \in \mathbb{D}^n$ and $\mathsf{spec}(f) := \mathsf{dom}(f)$.

Lemma 3.12 *There is a one to one correspondence between self-adjoint operators on $\mathcal{H}^n$ and spectral decompositions of $\mathcal{H}^n$.*

Thus, we frequently use the operator and decomposition language interchangeably. For example, here is an alternate formulation of quantum states:

Definition 3.13 A *density operator* is a spectral decomposition r with

$$\sum_{\lambda \in \mathsf{spec}(r)} \lambda \cdot \mathsf{dim}(r_\lambda) = 1$$

and $\mathsf{spec}(r) \subseteq [0, \infty)$.

In particular, a *pure state* $r \in \Sigma^n$ is a decomposition $r : \{0,1\} \to \mathbb{L}^n$ with

$$\sum_{\lambda \in \{0,1\}} \lambda \cdot \mathsf{dim}(r_\lambda) = 1.$$

From this equation we see that the subspace r_1 is one-dimensional. In fact, r_1 serves to characterize r, since r_0 must then be a certain $n-1$ dimensional subspace known as the *orthocomplement* of r_1,

$$r_0 = r_1^\perp := \{\psi \in \mathcal{H}^n \mid \psi \perp r_1\}\,.$$

We have proven the following.

Lemma 3.14 *The pure states on $\mathcal{H}^n$ are in bijective correspondence with the one dimensional subspaces of $\mathcal{H}^n$.*

So much for states. For observables, we will consider only those e on $\mathcal{H}^n$ with the maximum number of distinguishable outcomes n. By simple renaming then, we can take $\mathsf{spec}(e) = \{1, \ldots, n\}$. This convention highlights the role played by measurements: They are *labelings,* i.e., to each outcome $1 \leq i \leq n$, a measurement assigns those states e_i of the system for which observable e has value i with certainty (probability one).

Definition 3.15 A *labeling* is a spectral decomposition

$$e : \{1, \ldots, n\} \to \mathbb{L}^n\,.$$

Notice that non-degeneration of $\mathsf{spec}(e)$ implies that the decomposition $\mathcal{D}_e$ consists only of one-dimensional subspaces (pure states), i.e., $\mathcal{D}_e$ cannot be refined any further:

Definition 3.16 A decomposition $\mathcal{D}$ is a *refinement* of decomposition $\mathcal{D}'$ iff

$$\bigcup \mathcal{D} \subseteq \bigcup \mathcal{D}'\,.$$

Finally, the probabilities. Recall that the probability of obtaining outcome i in a measurement of observable e on a system with density operator r is given by

$$\mathsf{prob}_e^i(r) := \mathsf{tr}(P_e^i \cdot r)\,.$$

For a state r and a labeling e, $\langle r|e_i\rangle$ denotes the ith diagonal element of the matrix representation of r when expressed in a base B in which all P_e^i diagonalize, and thus, by the spectral decomposition theorem, in which e itself diagonalizes. Writing $P_e^i \cdot r$ in base B then yields

$$\begin{pmatrix} 0 & & & & & & \\ & \ddots & & & 0 & & \\ & & 0 & & & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & 0 & & & & \ddots & \\ & & & & & & 0 \end{pmatrix} \cdot \begin{pmatrix} \langle r|e_1\rangle & & & \\ & \ddots & & ? \\ & & \langle r|e_i\rangle & \\ & ? & & \ddots \\ & & & \langle r|e_n\rangle \end{pmatrix} = \begin{pmatrix} 0 & & & & ? & & & \\ & \ddots & & & \vdots & & 0 & \\ & & & 0 & ? & & & \\ ? & \cdots & & ? & \langle r|e_i\rangle & ? & \cdots & ? \\ & & & & ? & 0 & & \\ & 0 & & & \vdots & & \ddots & \\ & & & & ? & & & 0 \end{pmatrix}$$

and thus

$$\mathsf{tr}(P_e^i \cdot r) = \langle r|e_i\rangle\,,$$

from which we conclude

$$\mathsf{prob}_e^i(r) = \langle r|e_i\rangle\,. \tag{2}$$

In particular, $\langle r|e_i\rangle$ does *not* depend on B, but only on r and e_i, since it is equal to $\mathsf{tr}(P_e^i \cdot r)$. Further, since $\mathsf{tr}(r) = 1$,

$$\sum_{i=1}^{n} \langle r|e_i\rangle = 1,$$

which simply says that a measurement for observable e yields some outcome $i \in \mathsf{spec}(e)$ with probability one.

Definition 3.17 For a state r and labeling e, we define

$$\mathsf{spec}(r|e) := (\langle r|e_1\rangle, \ldots, \langle r|e_n\rangle) \in \Delta^n\,.$$

Notice that $\mathsf{spec}(r|e)$ is a *list,* while $\mathsf{spec}(r)$ is a *set.*

In general, $\mathsf{spec}(r|e)$ may not consist of eigenvalues of r, i.e., elements of $\mathsf{spec}(r)$. However, if r also diagonalizes in base B, then its diagonal consists of eigenvalues of r. And this is the case we are most interested in.

Definition 3.18 A state r *admits* a labeling e if

$$\mathsf{Im}(\mathsf{spec}(r|e)) = \mathsf{spec}(r).$$

Any state admits at least $n!$ labelings, corresponding with different permutations of $\mathsf{spec}(r|e)$. More generally, the following result tells us exactly when a labeling yields the spectrum of a state.

Proposition 3.19 *The following are equivalent for state r and labeling e:*

- *r admits labeling e.*
- *$\mathcal{D}_e$ is a refinement of $\mathcal{D}_r$.*
- *r diagonalizes in a base B in which e is diagonal.*
- *r and e commute, that is, $[r, e] = r \cdot e - e \cdot r = 0$.*

The following are equivalent for states r and s:

- *They admit a joint labeling e.*
- *They admit joint refinement $\mathcal{D}$.*
- *They diagonalize in a common base B.*
- *They commute, that is, $[r, s] = 0$.*

Additionally, states r and s admit labeling e iff

$$[r, s] = [r, e] = [s, e] = 0\,.$$

In particular we have in all the above cases that

$$B \subset \bigcup \mathcal{D}_e \subseteq \bigcup \mathcal{D} \subseteq \bigcup \mathcal{D}_r \cap \bigcup \mathcal{D}_s \subseteq \bigcup \mathcal{D}_r$$

whenever one of the inclusions applies.

Proof. Given a base B, for all $\psi \in B$ we have $\psi \in \bigcup \mathcal{D}_e$ iff all $\psi \in B$ are eigenvectors of e iff e diagonalizes in the base B. Thus, any self-adjoint operator e diagonalizes in a base B iff $B \subseteq \bigcup \mathcal{D}_e$.

We already showed above that r admits labeling e when there exists a base B in which both r and e diagnalize, that is, whenever B is included both in $\bigcup \mathcal{D}_r$ and $\bigcup \mathcal{D}_e$ and as such

$$B \subseteq \bigcup \{\mathsf{span}(\psi) \mid \psi \in B\} \subseteq \bigcup \mathcal{D}_r$$

where since e is non-degenerated we have

$$\{\mathsf{span}(\psi) \mid \psi \in B\} = \mathcal{D}_e$$

and thus $\bigcup \mathcal{D}_e \subseteq \bigcup \mathcal{D}_r$. Two states r and s then admit a joint labeling e whenever

$$\bigcup \mathcal{D}_e \subseteq \bigcup \mathcal{D}_r \cap \bigcup \mathcal{D}_s\,.$$

The converses of these derivations is obvious. Whenever r and s admit diagonalization in a common base B, when representing them in B commutation reduces to commutation of reals. For the other results with respect to commutation, in particular the fact that self-adjoint operators diagonalize in a common base if they commute, we refer to relevant literature. □

Lemma 3.20 *Let r be a state and e be a labeling with $[r,e]=0$. Then*

$$\langle r|e_i\rangle = \lambda \Leftrightarrow \psi_i \in r_\lambda \Leftrightarrow e_i \subseteq r_\lambda \tag{3}$$

and

$$\mathsf{dim}(r_\lambda) = \mathsf{card}(\{1 \le i \le n \mid \langle r|e_i\rangle = \lambda\})\,. \tag{4}$$

In particular, $\mathsf{dim}(r_\lambda)$ does not depend on the choice of e, so neither do the multiplicities of eigenvalues.

Finally, the following result is indispensable and we will appeal to it time and time again (often implicitly).

Lemma 3.21 (Definability) *For any labeling e and classical state x, there is a unique quantum state $r \in \Omega^n$ with $[r,e]=0$ and $\mathsf{spec}(r|e) = x$.*

Although the notions decomposition, refinement and labeling as well as the representation of states and measurements as maps that label subspaces in terms of spectra are not standard in orthodox quantum theory [12], they prove to be useful in our setting since they highlight degeneration of spectra, a fundamental ingredient in the ordering of both classical and quantum states.

3.2 A partial order on quantum states

Here is the *spectral order* on quantum states Ω^n.

Definition 3.22 *For states $r, s \in \Omega^n$, we write $r \sqsubseteq s$ iff there exists a labeling e such that*

- *e is admitted both by r and s,*
- *$\mathsf{spec}(r|e) \sqsubseteq \mathsf{spec}(s|e)$ in Δ^n.*

Though the order on quantum states only requires that *there exist* a single joint labeling, it nevertheless applies to *all* labels shared by r and s. This is like the way that $x \sqsubseteq y$ for classical states implies $x \cdot \sigma \sqsubseteq y \cdot \sigma$, for any $\sigma \in S(n)$ with $x, y \in \Delta^n_\sigma$.

Proposition 3.23 *If $r \sqsubseteq s$ in Ω^n, then $\mathsf{spec}(r|e) \sqsubseteq \mathsf{spec}(s|e)$ in Δ^n, for any labeling e with $[r,e]=[s,e]=0$.*

Proof. We prove the equivalent statement that

$$\mathsf{spec}(r|e) \sqsubseteq \mathsf{spec}(s|e) \Leftrightarrow \mathsf{spec}(r|e') \sqsubseteq \mathsf{spec}(s|e') \tag{5}$$

whenever $[r,e]=[s,e]=[r,e']=[s,e']=0$. Since

$$\begin{aligned} \bigcup \mathcal{D}_r \cap \bigcup \mathcal{D}_s &= \bigcup\{r_\lambda \mid \lambda \in \mathsf{spec}(r)\} \cap \bigcup\{s_{\lambda'} \mid \lambda' \in \mathsf{spec}(s)\} \\ &= \bigcup\{r_\lambda \cap s_{\lambda'} \mid \lambda \in \mathsf{spec}(r)\,, \lambda' \in \mathsf{spec}(s)\}\,, \end{aligned}$$

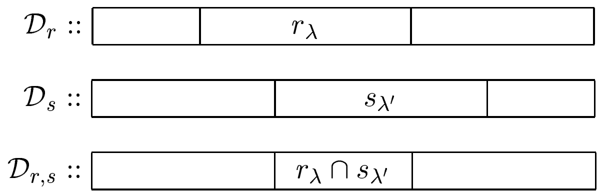
and, since whenever $[r,e]=[s,e]=0$ we have $\bigcup \mathcal{D}_e \subseteq \bigcup \mathcal{D}_r \cap \bigcup \mathcal{D}_s$ by Proposition 3.19, it follows that

$$\bigcup \mathcal{D}_e \quad \subseteq \quad \bigcup \{ r_\lambda \cap s_{\lambda'} \mid \lambda \in \mathsf{spec}(r)\,, \lambda' \in \mathsf{spec}(s) \}\,,$$

where, since $r_\lambda \perp r_{\lambda'}$ and $s_\lambda \perp s_{\lambda'}$ for $\lambda \neq \lambda'$, the subspaces $r_\lambda \cap s_{\lambda'}$ are mutually orthogonal for non-coincideng labels (λ, λ') and thus mutually exclusive. Since their union includes $\bigcup \mathcal{D}_e$ they span $\mathcal{H}^n$, so they constitute a decomposition

$$\mathcal{D}_{r,s} := \{ r_\lambda \cap s_{\lambda'} \mid \lambda \in \mathsf{spec}(r)\,, \lambda' \in \mathsf{spec}(s) \}$$

with $\mathcal{D}_e$ as a refinement.



Since $\mathcal{D}_e$ is a refinement of $\mathcal{D}_{r,s}$ it also follows that

$$\mathsf{dim}(r_\lambda \cap s_{\lambda'}) = \mathsf{card}\left(\{ i \in \{1, \ldots, n\} \bigm| e_i \subseteq r_\lambda \cap s_{\lambda'} \}\right)$$

where the quantity on the left does not depend on e. Since,

$$\begin{aligned} e_i \subseteq r_\lambda \cap s_{\lambda'} \quad &\Leftrightarrow \quad e_i \subseteq r_\lambda, e_i \subseteq s_{\lambda'} \\ &\Leftrightarrow \quad \lambda = \langle r|e_i\rangle, \lambda' = \langle s|e_i\rangle \\ &\Leftrightarrow \quad (\langle r|e_i\rangle, \langle s|e_i\rangle) = (\lambda, \lambda') \end{aligned}$$

for $(\lambda, \lambda') \in \mathsf{spec}(r) \times \mathsf{spec}(s)$, we have

$$\mathsf{card}\Big(\{ i \in \{1, \ldots, n\} \bigm| (\langle r|e_i\rangle, \langle s|e_i\rangle) = (\lambda, \lambda') \}\Big) = \mathsf{dim}(r_\lambda \cap s_{\lambda'})\,.$$

Thus, writing

$$\mathsf{spec}(r, s|e) = \Big((\langle r|e_1\rangle, \langle s|e_1\rangle)\,, \ldots\,, (\langle r|e_n\rangle, \langle s|e_n\rangle) \Big)$$

it follows that the list $\mathsf{spec}(r, s|e)$ contains a fixed collection of elements

$$\Big(\ldots\,, \underbrace{(\lambda, \lambda')\,, \ldots\,, (\lambda, \lambda')}_{\mathsf{dim}(r_\lambda \cap s_{\lambda'})}\,, \ldots \Big)\,,$$

where all $(\lambda, \lambda') \in \mathsf{spec}(r) \times \mathsf{spec}(s)$, independent on the choice of e except for the order of the elements in this list, that is, given e and e' such that $[r,e]=[s,e]=[r,e']=[s,e']=0$ we have

$$\mathsf{spec}(r, s|e') = \mathsf{spec}(r, s|e) \cdot \sigma$$

for some permutation $\sigma : \{1, \ldots , n\} \to \{1, \ldots , n\}$ and thus

$$\mathsf{spec}(r|e') = \mathsf{spec}(r|e) \cdot \sigma \quad \text{and} \quad \mathsf{spec}(s|e') = \mathsf{spec}(s|e) \cdot \sigma \,.$$

But these are classical states, so

$$\mathsf{spec}(r|e) \sqsubseteq \mathsf{spec}(s|e) \;\Leftrightarrow\; \mathsf{spec}(r|e) \cdot \sigma \sqsubseteq \mathsf{spec}(s|e) \cdot \sigma,$$

from which implication (5) follows. $\square$

The last result uses one of the two fundamental properties possessed by the Bayesian order on Δ^n: It is *symmetric*, i.e., the map

$$\Delta^n \to \Delta^n :: x \mapsto x \cdot \sigma$$

is an order isomorphism, for any $\sigma \in S(n)$. This *label independence* of the Bayesian order is a simple case of a more general notion satisfied by the spectral order which we will study in the section on symmetries. To hint at the connection: The equation

$$\mathsf{spec}(r|e) \cdot \sigma = \mathsf{spec}(r|e \cdot \sigma)$$

indicates that permuting the classical state $\mathsf{spec}(r|e)$ is the same as permuting the subspaces $(e_i)_{i=1}^{i=n}$ of the labeling e.

The second crucial property of the Bayesian order on Δ^n is that it is *degenerative*:

$$x \sqsubseteq y \Rightarrow (y_i = y_j > 0 \Rightarrow x_i = x_j > 0).$$

Here is the quantum version of the *degeneration lemma* for classical states.

Lemma 3.24 *If $r \sqsubseteq s$ in Ω^n then*

$$r_0 \subseteq s_0 \tag{6}$$

and

$$\bigcup s_{>0} \subseteq \bigcup r_{>0} \,, \tag{7}$$

where

$$r_{>0} := \mathcal{D}_r \setminus \{r_0\} \qquad \text{and} \qquad s_{>0} := \mathcal{D}_s \setminus \{s_0\} \,.$$

Proof. Since $r \sqsubseteq s$ they admit a labeling e such that $\mathsf{spec}(r|e) \sqsubseteq \mathsf{spec}(s|e)$ and thus by degeneration for classical states (Lemma 2.13), we have

$$\{1 \leq i \leq n \mid \langle r|e_i\rangle = 0\} \subseteq \{1 \leq i \leq n \mid \langle s|e_i\rangle = 0\}$$

so eq.(6) follows. Analogously, for

$$\langle s|e_i\rangle \in \mathsf{spec}_0(s) := \mathsf{spec}(s) \setminus \{0\}$$

classical degeneration again yields

$$\{1 \le j \le n \mid \langle s|e_j\rangle = \langle s|e_i\rangle\} \subseteq \{1 \le j \le n \mid \langle r|e_j\rangle = \langle r|e_i\rangle\}$$

so eq.(7) follows. $\square$

Recall that an increasing sequence (x_i) of classical states must be confined to some region Δ^n_σ. Here is the analogous result for the spectral order.

Lemma 3.25 *Let $(r_i)_{i\ge1}$ be a sequence such that for all $i \ge 1$ we have that $r_i \sqsubseteq r_{i+1}$. Then there exists a joint refinement $\mathcal{D}_{(r_i)}$ of $(\mathcal{D}_{r_i})_{i\ge1}$ and thus the states $(r_i)_{i\ge1}$ admit joint labeling.*

Proof. We agree that the first index for states refers to the sequence index and that the second refers to eigenvalues. First note that by Lemma 3.24, since $r_i \sqsubseteq r_{i+1}$ we have

$$r_{i,0} \subseteq r_{i+1,0} \tag{8}$$

$$\bigcup r_{i+1,>0} \subseteq \bigcup r_{i,>0}\,. \tag{9}$$

We now proceed by induction.

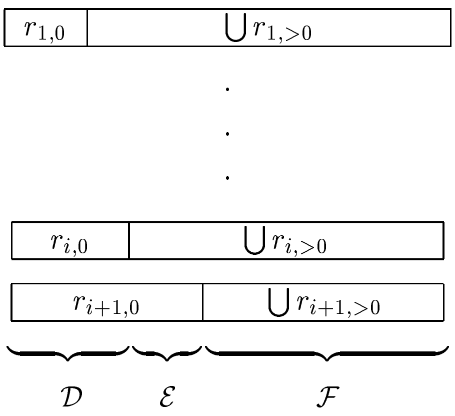
As base case we take r_1 as its own refinement. Note that the spectrum of a state r decomposes in a zero and a non-zero part to which we refer as $\mathsf{spec}_0(r)$. Let $\mathcal{D}_i$ be the constructed joint refinement for $(r_1, \ldots, r_i)$. Set

$$\mathcal{D}_{i+1} = (\mathcal{D} \cup \mathcal{E} \cup \mathcal{F}) \setminus \{o\}\,,$$

where

$$\mathcal{D} = \{a \cap r_{i,0} \mid a \in \mathcal{D}_i\}$$
$$\mathcal{E} = \{a \cap r_{i+1,0} \mid a \in r_{i,>0}\}$$
$$\mathcal{F} = r_{i+1,>0}\,.$$

Graphically, in terms of decompositions of $\mathcal{H}^n$ in subspaces,



We now prove that $\mathcal{D}_{i+1}$ is a joint refinement for $(r_1, \ldots, r_{i+1})$. Since $r_i \sqsubseteq r_{i+1}$ they admit a joint refinement $\mathcal{G}$ so we have by Proposition 3.19 that

$$\begin{aligned}
\bigcup \mathcal{G} &\subseteq \bigcup \mathcal{D}_{r_i} \cap \bigcup \mathcal{D}_{r_{i+1}} \\
&= \left(r_{i,0} \cup \bigcup r_{i,>0}\right) \cap \left(r_{i+1,0} \cup \bigcup r_{i+1,>0}\right) \\
&= \left(r_{i,0} \cap r_{i+1,0}\right) \cup \left(\bigcup r_{i,>0} \cap r_{i+1,0}\right) \cup \left(\bigcup r_{i,>0} \cap \bigcup r_{i+1,>0}\right) \\
&= r_{i,0} \cup \left(\bigcup r_{i,>0} \cap r_{i+1,0}\right) \cup \bigcup r_{i+1,>0}
\end{aligned}$$

by eq.(8) and eq.(9) and since

$$r_{i,0} \cap \bigcup r_{i+1,>0} = \emptyset\,.$$

We moreover have

$$\begin{aligned}
\mathsf{span}\left(\bigcup r_{i,>0} \cap r_{i+1,0}\right) &= \mathsf{span}\left(\bigcup \{a \cap r_{i+1,0} \mid a \in r_{i,>0}\}\right) \\
&= \mathsf{span}\left(\bigcup \mathcal{E}\right)
\end{aligned}$$

Since $\mathcal{D}_i$ is a refinement for $\mathcal{D}_{r_i}$ it also follows that

$$\begin{aligned}
\mathsf{span}(r_{i,0}) &= \mathsf{span}\left(\bigcup \{a \cap r_{i,0} \mid a \in \mathcal{D}_i\}\right) \\
&= \mathsf{span}\left(\bigcup \mathcal{D}\right)\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathcal{H}^n &= \mathsf{span}(\bigcup \mathcal{G}) \\
&= \mathsf{span}\left(\bigcup \mathcal{D}_{r_i} \cap \bigcup \mathcal{D}_{r_{i+1}}\right) \\
&= \mathsf{span}\left(\bigcup \mathcal{D} \cup \bigcup \mathcal{E} \cup \bigcup \mathcal{F}\right) \\
&= \mathsf{span}\left(\bigcup \mathcal{D}_{i+1}\right)\,.
\end{aligned}$$

The elements in $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ are mutually orthogonal since $\mathcal{D}_i$, $r_{i,>0}$ and $r_{i+1,>0}$ consist of mutually orthogonal elements. Moreover, the sets $\bigcup \mathcal{D}$, $\bigcup \mathcal{E}$ and $\bigcup \mathcal{F}$ are themselves mutually orthogonal since

- $\bigcup \mathcal{F} = \bigcup r_{i+1,>0} \perp r_{i+1,0} \supseteq \bigcup \mathcal{E}\,,$
- $\bigcup \mathcal{D} \subseteq r_{i,0} \perp \bigcup r_{i,>0} \supseteq \bigcup \mathcal{E}$, and,
- $\bigcup \mathcal{D} \subseteq r_{i,0} \perp \bigcup r_{n,>0} \supseteq \bigcup r_{i+1,>0} = \bigcup \mathcal{F}\,,$

where the last inclusion follows from eq.(9). Thus, $\mathcal{D}_{i+1}$ is a decomposition of $\mathcal{H}^n$. Since

- $\bigcup \mathcal{F} = \bigcup r_{i+1,>0}$,
- $\bigcup \mathcal{E} = r_{i+1,0}$, and,
- $\bigcup \mathcal{D} \subseteq r_{i,0} \subseteq r_{i+1,0}$,

by eq.(8), it follows that

$$\bigcup \mathcal{D}_{i+1} \subseteq r_{i+1,0} \cup \bigcup r_{i+1,>0} = \bigcup \mathcal{D}_{r_{i+1}}$$

so $\mathcal{D}_{i+1}$ is a refinement of $\mathcal{D}_{r_{i+1}}$. Since

- $\bigcup \mathcal{E} = \bigcup r_{i+1,>0} \subseteq \bigcup r_{i,>0} \subseteq \bigcup \mathcal{D}_n$, by eq.(9) and the inductive assumption,
- $\bigcup \mathcal{F} \subseteq \bigcup r_{i,>0} \subseteq \bigcup \mathcal{D}_i$, and,
- $\bigcup \mathcal{D} \subseteq \bigcup \mathcal{D}_i$,

it follows that $\mathcal{D}_{i+1}$ is a refinement of $\mathcal{D}_i$ and thus of all $\mathcal{D}_{r_j}$ for $1 \leq j \leq i$.

Finally, consider an infinite sequence $(r_i)_{i\geq 1}$ such that for all $i \geq 1$ we have that $r_i \sqsubseteq r_{i+1}$ and let $(\mathcal{D}_i)_{i\geq 1}$ be the corresponding series of refinements, each member being the above constructed common refinement of $(\mathcal{D}_{r_1}, \ldots, \mathcal{D}_{r_i})$. Note that $(\bigcup \mathcal{D}_i)_{i\geq 1}$ is decreasing with respect to intersection. Then, since $\mathcal{H}^n$ is n-dimensional, there can only be n distinct decompositions contained in $(\mathcal{D}_i)_{i\geq 1}$, that is $n-1$ non-trivial refinements steps $\mathcal{D}_i \mapsto \mathcal{D}_{i+1}$. Thus,

$$\bigcup \mathcal{D}_{(r_i)} := \bigcap_{i\geq 1} \bigcup \mathcal{D}_i$$

is equal to the intersection of a finitely many decreasing sets and thus must be equal to its smallest member, which is a common refinement for $(\mathcal{D}_{r_i})_{i\geq 1}$. $\square$

Theorem 3.26 *Ω^n is a partially ordered set for each $n \geq 2$. Its maximal elements are the pure states,*

$$\max(\Omega^n) = \Sigma^n,$$

while its least element is the completely mixed state

$$\perp := \begin{pmatrix} 1/n & & 0 \\ & \ddots & \\ 0 & & 1/n \end{pmatrix}.$$

Proof. For reflexivity, consider any labeling e admitted by state r. Then, due to reflexivity in Δ^n (Theorem 2.12), reflexivity in Ω^n follows.

For anti-symmetry assume that $r \sqsubseteq s$ and $s \sqsubseteq r$. By Lemma 3.25 there exists a joint labeling e and thus by definition 3.22 we have

$$\mathsf{spec}(r|e) \sqsubseteq \mathsf{spec}(s|e) \ \text{ and } \ \mathsf{spec}(s|e) \sqsubseteq \mathsf{spec}(r|e)\,.$$

Due to anti-symmetry in Δ^n we obtain $\mathsf{spec}(r|e) = \mathsf{spec}(s|e)$ so $r = s$ by Lemma 3.21.

For transitivity assume that $r \sqsubseteq s$ and $s \sqsubseteq t$. By Lemma 3.25 there exists a joint labeling e and thus we have

$$\mathsf{spec}(r|e) \sqsubseteq \mathsf{spec}(s|e) \quad \text{and} \quad \mathsf{spec}(s|e) \sqsubseteq \mathsf{spec}(t|e)\,.$$

Thus, due to transitivity in Δ^n we obtain $\mathsf{spec}(r|e) \sqsubseteq \mathsf{spec}(t|e)$ and thus by definition 3.22 we have $r \sqsubseteq t$.

Since $\mathsf{spec}(r) = \{0, 1\}$ for any $r \in \Sigma^n$, when $s \in \Omega^n$ satisfies $r \sqsubseteq s$ it follows for any labeling e admitted by r and s that we have

$$\mathsf{spec}(r|e) = (1, 0, \dots, 0) \cdot \sigma \sqsubseteq \mathsf{spec}(s|e)$$

in Δ^n for some permutation σ, so $\mathsf{spec}(r|e) = \mathsf{spec}(s|e)$ since $\mathsf{spec}(r|e) \in \max(\Delta^n)$, and thus $r = s$.

Conversely, for any state $r \in \Omega^n$ expressed in a base $B \in \mathcal{D}_e$ in which it diagonalizes, we have

$$\mathsf{spec}(r|e) \sqsubseteq (1, 0, \dots, 0) \cdot \sigma$$

in Δ^n for some permutation σ, so r has a pure state above it, and thus the pure quantum states are the only maximal elements of Ω^n.

Since $\mathsf{spec}(\perp) = \{1/n\}$ we have $\mathcal{D}_\perp = \mathcal{H}^n$ and thus $\perp$ admits any labeling. Given $r \in \Omega^n$ and labeling e admitted by e we then have

$$\mathsf{spec}(\perp|e) = (1/n, \dots, 1/n) \sqsubseteq \mathsf{spec}(r|e)\,,$$

so $\perp \sqsubseteq r$ and thus $\perp$ is the least element of Ω^n. $\square$

Examining the proofs given so far reveals that the technique used in defining the spectral order serves to distinguish an interesting class of partial orders on classical states for which the Bayesian order is the canonical member.

Corollary 3.27 *If $\sqsubseteq$ is a symmetric and degenerative partial order on Δ^n, then the relation in Definition 3.22 is a partial order on Ω^n. Moreover,*

- $\max(\Omega^n) = \Sigma^n$ *whenever* $\max(\Delta^n) = \{e_i : 1 \leq i \leq n\}$*, and*
- *The completely mixed state is the bottom of Ω^n whenever $(1/n, \dots, 1/n)$ is the bottom of Δ^n.*

By Lemma 3.21, we can define a quantum state r by specifying two pieces of information: (i) a labeling e which it admits, that is $[r, e] = 0$, and (ii) a classical state x for which $\mathsf{spec}(r|e) := x$. We use this idea in what follows.

Proposition 3.28 *The quantum states Ω^n are a dcpo. In more detail,*

(i) *If $(r_i)_{i\geq 1}$ is an increasing sequence, then its supremum $\bigsqcup_{i\geq 1} r_i$ exists and is implicitly defined by*

$$\mathsf{spec}\Big(\bigsqcup_{i\geq 1} r_i \;\Big|\; e\Big) = \Big(\lim_{i\to\infty}\langle r_i|e_1\rangle, \dots , \lim_{i\to\infty}\langle r_i|e_n\rangle\Big) \tag{10}$$

for some and thus any joint labeling e of $(r_i)_{i\geq 1}$.

(ii) *Every directed subset of Ω^n contains an increasing sequence with the same supremum.*

Proof. By Lemma 3.25 there exists a joint labeling e for $(r_i)_{i\geq 1}$ and thus by Definition 3.22 it follows that $(\mathsf{spec}(r_i|e))_{i\geq 1}$ is an increasing sequence in Δ^n. Then by Proposition 3.28 we know that the pointwise limit

$$\lim_{i\to\infty} \mathsf{spec}(r_i|e) := \Big(\lim_{i\to\infty}\langle r_i|e_1\rangle, \dots , \lim_{i\to\infty}\langle r_i|e_n\rangle\Big)$$

exists. We define a state r implicitly via

$$\mathsf{spec}(r|e) = \lim_{i\to\infty} \mathsf{spec}(r_i|e)\,.$$

We first show that this state r is independent on the choice of e. Since

$$\begin{aligned}\bigcup \mathcal{D}_{(r_i)} &= \bigcap_i \bigcup \mathcal{D}_{r_i} = \bigcap_i \bigcup \{r_{i,\lambda_i} | \lambda_i \in \mathsf{spec}(r_i)\} \\ &= \bigcup \Big\{\bigcap_i r_{i,\lambda_i} \;\Big|\; \forall i : \lambda_i \in \mathsf{spec}(r_i)\Big\}\,,\end{aligned}$$

where we leave the proof of the first equality to the reader (straightforward verification via the inductive definition of $\mathcal{D}_{(r_i)}$), so

$$\mathcal{D}_{(r_i)} = \Big\{\bigcap_i r_{i,\lambda_i} \;\Big|\; \forall i : \lambda_i \in \mathsf{spec}(r_i)\Big\}\,.$$

Note that by $\bigcup \mathcal{D}_{(r_i)} = \bigcap_i \bigcup \mathcal{D}_{r_i}$ it also follows that for any joint labeling e of $(r_i)_{i\geq 1}$, since

$$\forall i \geq 1 : \bigcup \mathcal{D}_e \subseteq \bigcup \mathcal{D}_{r_i} \;\Rightarrow\; \bigcup \mathcal{D}_e \subseteq \bigcap_i \bigcup \mathcal{D}_{r_i}\,,$$

we have $\bigcup \mathcal{D}_e \subseteq \bigcup \mathcal{D}_{(r_i)}$, that is, $\bigcup \mathcal{D}_{(r_i)}$ contains all joint labelings of $(r_i)_{i\geq 1}$ (and only those, so it is maximal with respect to this property).

If e is a joint labeling of $(r_i)_{i\geq 1}$ and $\bigcap_i r_{i,\lambda_i} \neq \emptyset$, where

$$(\lambda_i)_{i\leq 1} \in \prod_{i\leq 1} \mathsf{spec}(r_i)\,,$$

then there exists some $e_j \in \mathcal{D}_e$ such that $e_j \subseteq \bigcap_i r_{i,\lambda_i}$ for which we have

$$\begin{aligned} e_j \subseteq \bigcap_i r_{i,\lambda_i} \quad &\Leftrightarrow \quad \forall i : e_j \subseteq r_{i,\lambda_i} \\ &\Leftrightarrow \quad \forall i : \langle r_i | e_j \rangle = \lambda_i \\ &\Leftrightarrow \quad \big(\langle r_i | e_j \rangle\big)_{i\geq 1} = (\lambda_i)_{i\geq 1} \,. \end{aligned}$$

Since $\lim_{i\to\infty}\langle r_i|e_j\rangle$ exists, $\lim_{i\to\infty}\lambda_i$ exists and is equal to it. However, $(\lambda_i)_{i\geq 1}$ does not depend on any labeling so neither does its limit $\lim_{i\to\infty}\lambda_i$. can define r now as follows without any reference to a labeling:

$$\mathsf{spec}(r) := \left\{ \lim_{i\to\infty} \lambda_i \;\middle|\; (\lambda_i)_{i\leq 1} \in \prod_{i\leq 1} \mathsf{spec}(r_i) \,, \bigcap_i r_{i,\lambda_i} \neq \emptyset \right\},$$

$$r : \mathsf{spec}(r) \to \mathbb{L}^n :: \lim_{i\to\infty} \lambda_i \mapsto \bigcap_i r_{i,\lambda_i} \,.$$

Next we prove that r is an upper bound of $(r_i)_{i\geq 1}$. By Proposition 3.28 we have

$$\begin{aligned} \bigsqcup_{i\geq 1} \mathsf{spec}(r_i|e) \quad &= \quad \bigsqcup_{i\geq 1} \Big(\langle r_i|e_1\rangle, \ldots, \langle r_i|e_n\rangle \Big) \\ &= \quad \lim_{i\to\infty} \mathsf{spec}(r_i|e) \\ &= \quad \mathsf{spec}(r|e) \,. \end{aligned}$$

so for all $i \geq 1$ we have $\mathsf{spec}(r_i|e) \sqsubseteq \mathsf{spec}(r|e)$ and thus by definition of the order on quantum states it follows that $r_i \sqsubseteq r$ for all $i \geq 1$.

We now show that r is the least upper bound of $(r_i)_{i\geq 1}$. Let s be any upper bound of the sequence $(r_i)_{i\geq 1}$, i.e., for all $i \geq 1$, $r_i \sqsubseteq s$. We now prove that $r \sqsubseteq s$. By the proof of Lemma 3.25 we know that there exists a finite subsequence of $(r_i)_{i\geq 1}$ (of which we can assume that it has n members) which yields the same common refinement of $(r_i)_{i\geq 1}$ for the given construction — since there are only $n-1$ refinement steps possible. Denote this finite subsequence by $(r_{i_j})_{j=1}^{j=n}$. Then, since

$$r_{i_1} \sqsubseteq \ldots \sqsubseteq r_{i_n} \sqsubseteq s$$

they admit a common refinement and thus a common labeling e, which is also a common labeling for the whole sequence (r_i), and which we can assume to be the one by means of which we defined r since the definition of r does not depend on the choice of labeling. In this labeling we then have for each $i \geq 1$ that

$$\mathsf{spec}(r|e) = \bigsqcup_{i\geq 1} \mathsf{spec}(r_i|e) \sqsubseteq \mathsf{spec}(s|e)$$

in Δ^n since for all $i \geq 1$ we have $\mathsf{spec}(r_i|e) \sqsubseteq \mathsf{spec}(s|e)$. Thus $r \sqsubseteq s$ by definition of the order on quantum states.

We conclude $r = \bigsqcup_{i\geq 1} r_i$ from which eq.(10) then follows.

(ii) The map $\Omega^n \to [0,1] :: r \mapsto \max(\mathsf{spec}(r))$ preserves suprema of increasing sequences and is strictly monotone. $\square$

Thus, we can think of $\mathsf{spec}(\cdot|\cdot)$ as being Scott continuous in its first argument: For any observable e,

$$\mathsf{spec}(\bigsqcup r_i|e) = \bigsqcup \mathsf{spec}(r_i|e)$$

whenever (r_i) is an increasing sequence in Ω^n.

3.3 Symmetries for quantum states

We introduce symmetries, the quantum analogue of permutations for classical states.

Definition 3.29 A *unitary transformation* is a surjective linear operator $U : \mathcal{H}^n \to \mathcal{H}^n$ which preserves angles:

$$\langle U\phi \mid U\psi\rangle = \langle \phi \mid \psi\rangle\,,$$

for all $\psi, \phi \in \mathcal{H}^n$. U is called a *quantum n-symmetry.*

In particular, the inverse U^{-1} of a unitary operator U is unitary.

Lemma 3.30 *Let U be a quantum symmetry on $\mathcal{H}^n$. For any labeling e,*

$$U \cdot e : \{1, \ldots, n\} \to \mathbb{L}^n :: i \mapsto \{U(\psi) \in \mathcal{H}^n \mid \psi \in e_i\}$$

is a labeling, while for any state r,

$$U \cdot r : \mathsf{spec}(r) \to \mathbb{L}^n :: \lambda \mapsto \{U(\psi) \in \mathcal{H}^n \mid \psi \in r_\lambda\}$$

is a state with $\mathsf{spec}(U \cdot r) = \mathsf{spec}(r)$.

In both cases only the action of U on subspaces comes into play. Thus, two unitary operators U and U' related by $U = re^{i\theta}\cdot U'$ with $r > 0$ and $\theta \in [0, 2\pi)$ should be thought of as equivalent. The linearity of the maps, in conjuction with the coincidence of the action of U and U' on subspaces does force them to essentially be the same [4], e.g. $\mathsf{span}(e^{i\theta}\psi) = \mathsf{span}(\psi)$, though for $\psi \neq \phi$ and both nonzero we find $\mathsf{span}(\phi + e^{i\theta}\psi) \neq \mathsf{span}(\phi + \psi)$ for $\theta \neq 0$. Thus, a *quantum n-symmetry* should be conceived of as a class of unitary operators on $\mathcal{H}^n$ with equivalent action on subspaces. We will freely represent such a class by one of its representatives.

Lemma 3.31 *For a state r and a labeling e with $[r, e] = 0$,*

$$\langle r|(U \cdot e)_i\rangle = \langle U^{-1} \cdot r|e_i\rangle\,. \tag{11}$$

Proof. First note that

$$(U \cdot e)_i = \{U(\psi) \in \mathcal{H}^n \mid \psi \in e_i\} = \{\psi \in \mathcal{H}^n \mid U^{-1}(\psi) \in e_i\}$$

and

$$(U^{-1} \cdot r)_\lambda = \{U^{-1}(\psi) \in \mathcal{H}^n \mid \psi \in r_\lambda\} = \{\psi \in \mathcal{H}^n \mid U(\psi) \in r_\lambda\}.$$

Next, following eq.(3) we have

$$\begin{aligned}
\langle r|(U \cdot e)_i\rangle = \lambda \quad &\Leftrightarrow \quad (U \cdot e)_i \subseteq r_\lambda \\
&\Leftrightarrow \quad \forall \psi \in \mathcal{H}^n : U^{-1}(\psi) \in e_i \Rightarrow \psi \in r_\lambda \\
&\Leftrightarrow \quad \forall \psi \in \mathcal{H}^n : \psi \in e_i \Rightarrow U(\psi) \in r_\lambda \\
&\Leftrightarrow \quad e_i \subseteq (U^{-1} \cdot r)_\lambda \\
&\Leftrightarrow \quad \langle U^{-1} \cdot r|e_i\rangle = \lambda,
\end{aligned}$$

which completes the proof. □

Now we give a symmetric characterization of the spectral order analogous to the symmetric characterization of the Bayesian order on classical states. Eq.(11) leads us to the following dual formulations, which we call the *active* and *passive* (cfr. active and passive transformations in classical mechanics are those acting on the system and the reference frame, respectively). We assume for both theorems that *a labeling e has been fixed in advance.*

Theorem 3.32 (Active) *For $r, s \in \Omega^n$, we have $r \sqsubseteq s$ iff there exists a quantum symmetry $U : \mathcal{H}^n \to \mathcal{H}^n$ such that*

- $\mathsf{spec}(U \cdot r|e)$ *and* $\mathsf{spec}(U \cdot s|e)$ *are monotone*
- $[r, e] = [s, e] = 0$

and

$$\langle U \cdot r|e_i\rangle\langle U \cdot s|e_{i+1}\rangle \leq \langle U \cdot r|e_{i+1}\rangle\langle U \cdot s|e_i\rangle$$

for all i with $1 \leq i < n$.

Theorem 3.33 (Passive) *For $r, s \in \Omega^n$, we have $r \sqsubseteq s$ iff there exists a quantum symmetry $U : \mathcal{H}^n \to \mathcal{H}^n$ such that*

- $\mathsf{spec}(r|\, U \cdot e)$ *and* $\mathsf{spec}(s|\, U \cdot e)$ *are monotone*
- $[r, U \cdot e] = [s, U \cdot e] = 0$

and

$$\langle r|(U \cdot e)_i\rangle\langle s|(U \cdot e)_{i+1}\rangle \leq \langle r|(U \cdot e)_{i+1}\rangle\langle s|(U \cdot e)_i\rangle$$

for all i with $1 \leq i < n$.

Proof. Any labeling e' can be obtained from a given one e as $U \cdot e$ for some unitary transformation U. Indeed, in terms of linear operators this correspondence translates as $e' = U \circ e \circ U^{-1}$ so $e \cdot \psi = i\,\psi$ iff $e' \cdot U(\psi) = i\,U(\psi)$, that is, $\psi \in e_i \Leftrightarrow U(\psi) \in e'_i$ yielding the definition of $U \cdot e$ in terms of labelings. The result then straightforwardly follows from Theorem 2.22 and Lemma 3.31. $\square$

The following is now merely an observation.

Proposition 3.34 *The map* $(U \cdot -) : \Omega^n \to \Omega^n$ *is an order isomorphism for any quantum symmetry* $U : \mathcal{H}^n \to \mathcal{H}^n$.

Theorem 3.32 is the quantum counterpart of Theorem 2.22 for classical states: The action on states $(U \cdot -) : \Omega^n \to \Omega^n$ in terms of a unitary transformation U corresponds to the action on states $(- \cdot \sigma) : \Delta^n \to \Delta^n$ in terms of a permutation σ. But what is the classical analogue of the passive formulation of the spectral order?

Definition 3.35 A *classical labeling* is an injective function

$$e : \{1, \ldots, n\} \to \max(\Delta^n).$$

The *standard labeling* is 1 defined by $1(i) = e_i$.

Like the quantum case, we can write a classical state x from the point of view of a classical labeling e as

$$\mathsf{spec}(x|e) := (\langle x|e_1\rangle, \ldots, \langle x|e_n\rangle),$$

where $\langle\cdot|\cdot\rangle$ is the standard inner product on $\mathbb{R}^n$. For $e = 1$, $\mathsf{spec}(x|1) = x$. Notice too that $\langle e_i|e_j\rangle = 0$ for $i \neq j$, so the image of a classical labeling e is by definition a mutually orthogonal collection of pure states.

A classical labeling e induces a permutation $1^{-1} \circ e \in S(n)$. Thus, a *classical labeling* is merely a way of *rearranging* a *fixed* set of n orthogonal pure states $\max(\Delta^n)$. By contrast, a quantum labeling corresponds to selecting n orthogonal pure states from an infinite set of potential pure states *and* arranging the n pure states chosen.

Because classical labelings and symmetries are essentially the same, Theorem 2.22 is the passive formulation of the Bayesian order when we fix the standard classical label 1 as our reference frame. All other classical labels e can be written as $e = 1 \circ \sigma$ for some $\sigma \in S(n)$, analogous to the quantum case. To summarize:

	Classically	Quantum
Labeling	permutation e	self-adjoint operator e with spectrum $\{1, \ldots, n\}$
Symmetry	permutation σ	unitary transformation U

The equivalence of 'symmetry' and 'labeling' for classical states suggests the following analogy: Symmetries are to classical states as labelings are to quantum states. Though this is not entirely conceptually satisfying, it is a useful mathematical view of things. To illustrate, notice the strong resemblance between the following characterization of the spectral order, in terms of labels, and the symmetric characterization of the Bayesian order.

Theorem 3.36 *For $r, s \in \Omega^n$, we have $r \sqsubseteq s$ iff there is a quantum labeling e such that*

- $\mathsf{spec}(r|e)$ *and* $\mathsf{spec}(s|e)$ *are monotone*
- $[r, e] = [s, e] = 0$

and

$$\langle r|e_i\rangle\langle s|e_{i+1}\rangle \leq \langle r|e_{i+1}\rangle\langle s|e_i\rangle$$

for all i with $1 \leq i < n$.

Compared to Theorem 3.32 and Theorem 3.33, in this result it is the *act* of labeling itself that transforms a state into a monotone classical state. In the classical case, it is obviously a permuation (classical label) which converts a state to monotone form.

As a second example, first recall that the symmetric group $S(n)$ divides Δ^n into order isomorphic regions,

$$\Delta^n := \bigcup_{\sigma \in S(n)} \Delta^n_\sigma,$$

where $\Delta^n_\sigma \simeq \Lambda^n$. Similarly, quantum states are divided into order isomorphic regions by the class of measurement operators:

$$\Omega^n := \bigcup_e \Omega^n|e,$$

where $\Omega^n|e := \{r \in \Omega^n : [r, e] = 0\}$, i.e., the set of quantum states admitted by measurement e. Here is the quantum version of Prop. 2.27.

Proposition 3.37 *Let $n \geq 2$. Then*

(i) *For each labeling e, $\Omega^n|e$ is closed under directed suprema.*

(ii) *For an increasing sequence (r_i), there is a labeling e with $r_i \in \Omega^n|e$ for all i.*

(iii) *The natural map*

$$q : \Omega^n \to \Lambda^n$$

is Scott continuous, strictly monotone and restricts to a retraction

$$r_e : \Omega^n|e \simeq \Delta^n \to \Lambda^n$$

for each e.

Proof. The precise definition of q is as follows: For $s \in \Omega|e$, we define $q(s) := r(\mathsf{spec}(s|e))$, where $r : \Delta^n \to \Lambda^n$ is the natural retraction. $\square$

In particular, the Bayesian order on classical states is an instance of the spectral order on quantum states, which is realized whenever we specify a labeling e. It may interest the reader to know that both authors claim that $q : \Omega^n \to \Lambda^n$ *cannot* be factored into a composition of monotone maps

$$\Omega^n \xrightarrow{?} \Delta^n \xrightarrow{r} \Lambda^n,$$

where $? : \Omega^n \to \Delta^n$ denotes a monotone map that probably doesn't exist.

3.4 Approximation of quantum states

Like classical states, the ability to approximate quantum states order theoretically is a consequence of the mixing law.

Proposition 3.38 *If $r \sqsubseteq s$ in Ω^n, then*

$$r \sqsubseteq (1-p)r + ps \sqsubseteq s$$

for all $p \in [0,1]$.

Proof. First, $(1-p)r + ps$ is a density operator. Because $r \sqsubseteq s$, there is a labeling e with $[r,e] = [s,e] = 0$. Then

$$\begin{aligned} [(1-p)r + ps, e] &= ((1-p)r + ps)e - e((1-p)r + ps) \\ &= (1-p)[r,e] + p[s,e] \\ &= 0. \end{aligned}$$

Next,

$$\mathsf{spec}((1-p)r + ps|e) = (1-p)\mathsf{spec}(r|e) + p \cdot \mathsf{spec}(s|e),$$

because $(1-p)r + ps$, r and s are *diagonal* when written in the base e. The result now follows from the mixing law for classical states. $\square$

Like the classical case, the mixing law is equivalent to saying that the path $\pi_{rs} : [0,1] \to \Omega^n$ from r to s given by

$$\pi_{rs}(t) = (1-t)r + ts$$

is Scott continuous iff $r \sqsubseteq s$.

Lemma 3.39 *If $r \sqsubseteq s$ in Ω^n and $\mathsf{spec}(s) \subseteq (0,\infty)$, then*

$$[s,e] = 0 \Rightarrow [r,e] = 0,$$

for any labeling e.

Proof. First recall that $[s, e] = 0$ means that $\mathcal{D}_e$ is a refinement of $\mathcal{D}_s$. But the spectrum of s is *positive,* so Lemma 3.24 implies that $\mathcal{D}_s$ is a refinement of $\mathcal{D}_r$. Thus, $\mathcal{D}_e$ is a refinement of $\mathcal{D}_r$, which means $[r, e] = 0$. $\square$

The last result is the quantum analogue of Lemma 2.34(i) for classical states. The next few results further demonstrate the parallel between Δ^n_σ for classical states and $\Omega^n|e$ for quantum states.

Proposition 3.40 *Let* $n \geq 2$.

(i) *If* $r, s \in \Omega^n|e$, *then* $\pi_{rs}(t) \in \Omega^n|e$ *for all* $t \in [0, 1]$.

(ii) *For* $r, s \in \Omega^n$, *we have* $r \ll s$ *iff for any labeling* e, *if* $s \in \Omega^n|e$, *then* $r \in \Omega^n|e$ *and* $\mathsf{spec}(r|e) \ll \mathsf{spec}(s|e)$ *in* Δ^n.

Proof. (i) This was established in the proof of the mixing law.

(ii)($\Rightarrow$) Let $r \ll s$. Then for some $t < 1$, $r \sqsubseteq \pi_{\perp s}(t)$. If $[s, e] = 0$, then by (i), $[\pi_{\perp s}(t), e] = 0$ for all t, since we always have $[\perp, e] = 0$. However, because $t < 1$, the spectrum of $\pi_{\perp s}(t)$ is *positive*, which is clear since

$$\mathsf{spec}(\pi_{\perp s}(t)|e) = (1 - t)\perp + t \cdot \mathsf{spec}(s|e).$$

By Lemma 3.39, $[r, e] = 0$. The other part is obvious.

(ii)($\Leftarrow$) Suppose $s = \bigsqcup s_i$ for an increasing sequence (s_i). Then there is a labeling e with $[s_i, e] = 0$ for all i and $[s, e] = 0$. By assumption, $[r, e] = 0$, and since

$$\mathsf{spec}(r|e) \ll \mathsf{spec}(s|e) = \bigsqcup_{i \geq 1} \mathsf{spec}(s_i|e) \text{ in } \Delta^n,$$

we have $\mathsf{spec}(r|e) \sqsubseteq \mathsf{spec}(s_i|e)$ for some i, and hence $r \sqsubseteq s_i$. Thus, $r \ll s$. $\square$

Ω^n is a domain: A dcpo with an intrinsic notion of approximation.

Theorem 3.41 *The quantum states* Ω^n *are exact. In addition,*

(i) *For all* $r \in \Omega^n$, $\pi_{\perp r}(t) \ll r$ *for all* $t < 1$.

(ii) *The approximation relation* $\ll$ *is interpolative: If* $r \ll s$ *in* Ω^n, *then there is* $q \in \Omega^n$ *with* $r \ll q \ll s$.

Proof. (i) By Prop. 3.40(i), $[\pi_{\perp r}(t), e] = 0$ whenever $[r, e] = 0$. Since

$$\mathsf{spec}(\pi_{\perp r}(t)|e) = (1 - t)\perp + t \cdot \mathsf{spec}(r|e) \ll \mathsf{spec}(r|e) \text{ in } \Delta^n,$$

Prop. 3.40(ii) gives $\pi_{\perp r}(t) \ll r$ for all $t < 1$.

The map $\pi_{\perp r}$ is Scott continuous, so r is the supremum of an increasing sequence of approximations. This implies that $\mathord{\downarrow\!\!\downarrow} r$ is directed with supremum r, proving the exactness of Ω^n.

(ii) Mimic the argument for classical states to show $\pi_{\perp r}(t_1) \ll \pi_{\perp r}(t_2)$ whenever $t_1 < t_2$. $\square$

The notion of partiality derivable from $\ll$ on Ω^n is worth taking a brief look at. As before, we call $r \in \Omega^n$ partial iff $\Uparrow r \neq \emptyset$.

Lemma 3.42 (Partiality) *For $r \in \Omega^n$, the set $\Uparrow r \neq \emptyset$ iff* $\mathsf{spec(r)} \subseteq (0, \infty)$.

Proof. The only direction which requires proof is ($\Leftarrow$). Let e be a labeling with $[r, e] = 0$. Let $x := \mathsf{spec}(r|e) \in \Delta^n$. From the proof of Lemma 2.37 for classical states, there is $y \in \Delta^n$ such that $\pi_{\perp y}(t) = x$ for some $t < 1$.

Let $s \in \Omega^n$ with $[s, e] = 0$ and $\mathsf{spec}(s|e) = y$. First, $\pi_{\perp s}(t) = r$, since $\pi_{\perp s}(t), r \in \Omega^n|e$ and

$$\mathsf{spec}(\pi_{\perp s}(t)|e) = \pi_{\perp y}(t) = x = \mathsf{spec}(r|e).$$

Because $t < 1$, $r = \pi_{\perp s}(t) \ll s$ in Ω^n. $\square$

Thus, a quantum state which is partial cannot be pure. In addition, by exactness, all quantum states r arise as the supremum of an increasing sequence $(\pi_{\perp r}(1 - 1/n))_{n \geq 1}$ of partial states which approximate r.

Lemma 3.43 (Approximation of pure states) *Let $n \geq 2$ and $\psi \in \max(\Omega^n)$ be a pure state. For all $r \in \Omega^n$, $r \ll \psi \Leftrightarrow r = \pi_{\perp\psi}(t)$ for some $t < 1$.*

Proof. Let $r \ll \psi$. Let e be any labeling with $[\psi, e] = 0$. Then $[r, e] = 0$ and

$$x := \mathsf{spec}(r|e) \ll y := \mathsf{spec}(\psi|e) \in \max(\Delta^n)\,.$$

Thus, by Prop. 2.39,

$$(\exists t < 1)\, x = \pi_{\perp y}(t).$$

But $\pi_{\perp\psi}(t) \in \Omega^n|e$ and $\mathsf{spec}(\pi_{\perp\psi}(t)|e) = \pi_{\perp y}(t) = x = \mathsf{spec}(r|e)$, so $r = \pi_{\perp\psi}(t)$, since each is diagonal in e and their spectra are equal. $\square$

Thus, the order theoretic approximations of pure states ψ are precisely the mixtures of ψ with the completely mixed ensemble $\perp$.

Example 3.44 *The depolarization channel $d_p : \Omega^n \to \Omega^n$* describes the process by which the density operator of a system has all bias removed from it with probability p

$$d_p(r) = p \cdot I/n + (1 - p)r.$$

It can be rewritten as

$$d_p(r) = p\perp + (1 - p)r,$$

very similar to the classical case we considered earlier. Just as in the classical case, we also have $d_p(r) \ll r$ for $p > 0$.

3.5 Entropy

The word 'measurement' is used in domain theory and in quantum mechanics. They are related as follows: Domain theoretically, to measure the content of an object x, we must do something to x that will convert the information it represents into a simpler form μx that can be *understood.* Physically, as it turns out, the content of a quantum state r can be measured by selecting an appropriate quantum measurement e that converts r into a monotone classical state $\mathsf{spec}(r|e)$. (We might think of e as a way of extracting classical information from r.) This defines the map

$$q : \Omega^n \to \Lambda^n,$$

completely analogous to $r : \Delta^n \to \Lambda^n$ for classical states, which is a measurement in the sense of domain theory.

Proposition 3.45 *The map $q : \Omega^n \to \Lambda^n$ is a measurement.*

Proof. First, q is Scott continuous, strictly monotone, and preserves and reflects maximal elements. To show that it measures $\ker q = \max(\Omega^n)$, let $\psi \in \Omega^n$ and $U \subseteq \Omega^n$ be Scott open with $\psi \in U$.

Then there is $0 < t < 1$ with $a := \pi_{\perp\psi}(t) \in U$. Because $a \ll \psi$ in Ω^n, $q(a) \ll q(\psi)$ in Λ^n, which means $\varepsilon := \twoheaduparrow q(a)$ is a Scott open subset of Λ^n. We claim that $\psi \in q_\varepsilon(\psi) \subseteq \uparrow a \subseteq U$. That $\psi \in q_\varepsilon(\psi)$ is clear.

Now let $s \in q_\varepsilon(\psi)$. Then there is a labeling e with $[s, e] = [\psi, e] = 0$. Because $a \ll \psi$, we must have $[a, e] = 0$. But we also know

$$\perp \neq r(\mathsf{spec}(a|e)) = q(a) \ll q(s) = r(\mathsf{spec}(s|e)),$$

where $r : \Delta^n \to \Lambda^n$ is the natural retraction. Now the proof that r is a measurement gives

$$\mathsf{spec}(a|e) \ll \mathsf{spec}(s|e) \text{ in } \Delta^n,$$

which implies that $a \sqsubseteq s$, and thus $s \in U$, finishing the proof. $\square$

In particular, we can measure the content of a quantum state with a classical state.

Example 3.46 The content of a density operator ρ can also be measured with its largest eigenvalue,

$$\rho \mapsto \max(\mathsf{spec}(\rho)).$$

This is a measurement into $[0, 1]$ since it factors as $q(\rho)^+$. Similarly, $\rho \mapsto 1 - q(\rho)^+$ and $\rho \mapsto -\log q(\rho)^+$ are measurements into $[0, \infty)^*$.

The measures of content in the last example are the quantum versions of the maps $x \mapsto x^+$, $x \mapsto 1 - x^+$ and $x \mapsto -\log x^+$ on classical states. The extension of Shannon entropy to quantum states is called *von Neumann entropy.*

Theorem 3.47 *Let $\sigma : \Omega^n \to [0,\infty)^*$ be the von Neumann entropy on quantum states*

$$\sigma(r) = -\mathsf{tr}(r \cdot \log r)$$

where the logarithm is natural. Then σ is a measurement in the sense of domain theory. In addition,

(i) *For all $r, s \in \Omega^n$, if $r \sqsubseteq s$ and $\sigma(r) = \sigma(s)$, then $r = s$.*

(ii) *For all $r \in \Omega^n$, we have $\sigma(r) = 0$ iff $r \in \max(\Omega^n) = \Sigma^n$.*

(iii) *For all $r \in \Omega^n$, we have $\sigma(r) = \log n$ iff $r = \perp$.*

Proof. The von Neumann entropy σ factors as

$$\sigma = \mu \circ q$$

where $q : \Omega^n \to \Lambda^n$ assigns to a quantum state its *monotone* spectrum, and $\mu : \Lambda^n \to [0,\infty)^*$ is Shannon entropy. Since q and μ have all the properties mentioned in this result, so does σ. $\square$

By now it is clear that quantum information is more intricate than classical information, if for no other reason than the superficial observation that a density operator is "more complicated" than a classical state. What we now want is a precise formulation of the intuitive idea that there is more information in the quantum than in the classical.

One hint is provided by Prop. 3.37: We can associate each classical state to a quantum state in such a way that *information is conserved*:

$$\text{conservation of information}$$
$$=$$
$$(\text{qualitative conservation}) + (\text{quantitative conservation})$$
$$=$$
$$(\text{order embedding}) + (\text{preservation of entropy}).$$

And this is what we now prove: While each classical state can be associated to a quantum state in such a way that information is conserved, the converse is *never true.*

Theorem 3.48 *Let $n \geq 2$. Then*

- *There is an order embedding $\phi : \Delta^n \to \Omega^n$ such that $\sigma \circ \phi = \mu$.*
- *For $m \geq 2$, there is no order embedding $\phi : \Omega^n \to \Delta^m$ with $\mu \circ \phi = \sigma$.*

Proof. For the first, Prop. 3.37 gives an embedding which preserves entropy. For the second, if there is an embedding of Ω^n into Δ^m which preserves entropy, it yields an injection of $\max(\Omega^n)$ into $\max(\Delta^m)$, which is impossible since the first of these sets is infinite, while the latter is finite. $\square$

The reader may be interested to know that the authors both claim that the above result holds independent of entropic considerations.

4 Synthesis

We now obtain a unified perspective on classical and quantum which leads to a methodology applicable in any setting where one has (i) a notion of state and (ii) a notion of state update as the result of observation.

4.1 Classical projections

We first turn back to classical states to show that they admit a more general class of projectors and that the inductive definition of the Bayesian order extends to this larger class. First note that a classical projection

$$p_i : \Delta^{n+1} \rightharpoonup \Delta^n$$

is undefined in the singleton

$$\mathsf{fix}(p_i)^\perp := \{x \in \Delta^{n+1} \mid x_i = 1\}$$

and has as "fixed points"

$$\mathsf{fix}(p_i) := \{x \in \Delta^{n+1} \mid x_i = 0\}\,.$$

Any projection p_i moreover has a complementary projector, namely

$$p_i^\perp : \Delta^{n+1} \rightharpoonup \Delta^1 : x \mapsto (1)$$

which is undefined in

$$\mathsf{fix}(p_i)^\perp := \{x \in \Delta^{n+1} \mid x_i = 0\}\,.$$

This projector expresses the update that the observer experiences when he looks in box i and the object of his desire is actually there. Equivalently, this corresponds to looking in all boxes except box i and not finding the object. The condition

$$x \sqsubseteq y \Rightarrow p_i^\perp(x) \sqsubseteq p_i^\perp(y)$$

is however trivially satisfied whenever $p_i^\perp$ is defined both in x and y. As such, one could have included it in definition 2.7 providing an interpretation "whatever outcome we obtain when looking in box i, the corresponding collapse of knowledge preserves the partial order".

We define general projectors, encoding knowledge update when looking in several boxes at once. Let $n \geq 2$ and $1 \leq k \leq n$. The map which collapses all $i_1, \ldots, i_k^{th}$ outcomes is

$$p_{i_1,\ldots,i_k} : \Delta^n \rightharpoonup \Delta^{n-k}$$

$$p_{i_1,\ldots,i_k}(x) = \frac{1}{1-\sum_j x_{i_j}}(x_1, \ldots, \widehat{x_{i_1}}, \ldots, \widehat{x_{i_j}}, \ldots, \widehat{x_{i_k}}, \ldots, x_n)$$

for $1 \leq i_1, \dots, i_k \leq n$ and $0 \leq x_{i_1}, \dots, x_{i_j}, \dots, x_{i_k} < 1$. The projector corresponding to "looking in all boxes except" is

$$p^{\perp}_{i_1,\dots,i_k} : \Delta^n \rightharpoonup \Delta^{n-k}$$

$$p^{\perp}_{i_1,\dots,i_k}(x) = \frac{1}{\sum_j x_{i_j}} (\widehat{x_1}, \dots, x_{i_1}, \dots, x_{i_j}, \dots, x_{i_k}, \dots, \widehat{x_n})$$

for $1 \leq i_1, \dots, i_k \leq n$.

The set of projectors as defined constitute a Boolen algebra isomorphic to the powerset $\mathcal{P}(\{1, \dots, n\})$ when we adjoin the empty map

$$p_{1,\dots,n} : \Delta^n \rightharpoonup \emptyset$$

and the identity

$$p : \Delta^n \to \Delta^n .$$

In particular we have

$$p^{\perp}_{1,\dots\widehat{i_1},\dots,\widehat{i_j},\dots,\widehat{i_k},\dots,k} = p_{i_1,\dots,i_k} ,$$

that is, projections inherit orthogonality from the complementation of the Boolean algebra $\mathcal{P}(\{1, \dots, n\})$.

Proposition 4.1 *Let $x, y \in \Delta^{n+1}$. Then*

$$x \sqsubseteq y \;\Leftrightarrow\; (\forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}) \; p_{i_1,\dots,i_k}(x) \sqsubseteq p_{i_1,\dots,i_k}(y).$$

Proof. Given $p_i : \Delta^n \rightharpoonup \Delta^{n-1}$ define $\tilde{p}_i : \Delta^n \rightharpoonup \Delta^n$ via

$$\tilde{p}_i : \Delta^n \overset{p_i}{\rightharpoonup} \Delta^{n-1} \overset{\iota_i}{\to} \Delta^n$$

where

$$\begin{array}{ll} \pi_j(\iota_i(x)) = x_j & \text{for} \quad j < i \\ \pi_j(\iota_i(x)) = 0 & \text{for} \quad j = i \\ \pi_j(\iota_i(x)) = x_{j+1} & \text{for} \quad j > i \,. \end{array}$$

Analogously we introduce the map

$$\tilde{p}_{i_1,\dots,i_k} = \iota_{i_1,\dots,i_k} \cdot p_{i_1,\dots,i_k} : \Delta^n \rightharpoonup \Delta^n$$

where

$$\begin{array}{ll} \pi_j(\iota_i(x)) = x_{j+l-1} & \text{for} \quad i_{l-1} < j < i_l \\ \pi_j(\iota_i(x)) = 0 & \text{for} \quad j = i_l \\ \pi_j(\iota_i(x)) = x_{j+l} & \text{for} \quad i_l < j < i_{l+1} \end{array}$$

when assuming that $i_1, \dots, i_k$ is monotone and formally setting $i_0 = 0$ and $i_{k+1} = n + 1$. We then have

$$\tilde{p}_{i_1,\dots,i_k} = \tilde{p}_{i_1} \cdot \dots \cdot \tilde{p}_{i_k}$$

from which the result follows by induction. $\square$

Clearly, we rely on the following.

Proposition 4.2 *Projections commute with respect to composition.*

In particular, the Boolean algebra of projections is defined from concatenated action of projections

$$\tilde{p} \leq \tilde{q} \Leftrightarrow \tilde{p} \cdot \tilde{q} = \tilde{p}$$

where $\tilde{p}$ and $\tilde{q}$ are defined as in the proof of Proposition 4.1.

4.2 Quantum projections

We now show that the projective structure of classical states and its corresponding inductive definition of the Bayesian order are preserved by the natural embeddings of Δ^n into Ω^n. In particular, the classical projections become instances of Hilbert space projectors.

The Hilbert space projectors $\mathbb{P}^n$ also constitute an orthocomplemented lattice for the partial order

$$P \leq Q \Leftrightarrow P \cdot Q = P\,.$$

This lattice is however no longer distributive, e.g. [2, 3], and as such is not a Boolean algebra. Related to this, commutativity for projections as we have in Proposition 4.2 is not valid anymore for Hilbert space projectors.

Analogous to the introduction of $\tilde{p}$ given p in order to be able to compose projections, we now will have to do the converse for Hilbert space projectors in order to state an inductive definition of the partial ordering ofthe quantum states.

Any projector $P \in \mathbb{P}^n$ can be equivalently represented as a partial surjective map

$$P : \mathcal{H}^n \rightharpoonup \mathsf{fix}(P)$$

which is undefined in $\mathsf{fix}(P)^\perp$. When an isomorphism

$$h : \mathsf{fix}(P) \to \mathcal{H}^k$$

is specified, with $0 \leq k = \mathsf{dim}(\mathsf{fix}(P)) \leq n$, we can define

$$P^\downarrow : \mathcal{H}^n \overset{P}{\rightharpoonup} \mathsf{fix}(P) \overset{h}{\to} \mathcal{H}^k$$

of which the codomain does not depend on P anymore.

Note that such a map $P^\downarrow$ as well as P itself is fully characterized by its kernel, thus these maps are in bijective correspondence with the subspaces $\mathbb{L}^n$ via

$$\mathbb{P}^n \to \mathbb{L}^n : P \mapsto \mathsf{fix}(P)$$

and also with $\{0,1\}$-labeled decompositions, or equivalently, two element ordered decompositions, via

$$P \mapsto \left(\mathsf{fix}(P), \mathsf{fix}(P)^\perp\right).$$

Abstracting over the $\{0,1\}$-labeling we set

$$\mathcal{D}_P := \left\{\mathsf{fix}(P), \mathsf{fix}(P)^\perp\right\}.$$

Proposition 4.3 *The following are equivalent for state r and projector P:*

- *They admit joint labeling e.*
- *$\mathcal{D}_r$ and $\mathcal{D}_P$ admit a joint refinement $\mathcal{D}$.*
- *They diagonalize in a common base B.*
- $[r, P] = 0$.

The following are equivalent for states r and s and projector P:

- *They admit joint labeling e, i.e., $\mathcal{D}_r$, $\mathcal{D}_s$ and $\mathcal{D}_P$ admit a joint refinement $\mathcal{D}$, i.e., they diagonalize in a common base B.*
- *They pairwise admit joint labeling, i.e., $\mathcal{D}_r$, $\mathcal{D}_s$ and $\mathcal{D}_P$ pairwise admit joint refinement, i.e., they pairwise diagonalize in a common base.*
- $[r, P] = [s, P] = [r, s] = 0$.

Proof. Equivalence of the first four conditions follows from Proposition 3.19 since P is a state up to normaliztion, that is,

$$\frac{1}{\mathsf{dim}\,(\mathsf{fix}(P))} P \in \Omega^n\,.$$

Given a joint refinement $\mathcal{D}$ for r, s and P any labeling e such that $\bigcup \mathcal{D}_e = \bigcup \mathcal{D}$ is a joint labeling, and a base $B \in \bigcup \mathcal{D}_e$ yields joint diagonalization. At its turn, given a base B in which r, s and P diagonalize then

$$\bigcup \{\mathsf{span}(\psi) \mid \psi \in B\}$$

is a joint refinement.

Whenever we have a joint refinement, a joint labeling or a joint base for s, t and P then we have pairwise existence of one too. For the converse statement we provide a proof. We are going to prove a more general statement however, namely, that whenever we have a set of decompositions $\{\mathcal{D}_i \mid i \in I\}$, for technical simplicity envisioned as being finite, and such that for all $i, j \in I$ we have that $\mathcal{D}_i$ and $\mathcal{D}_j$ admits a joint refinement, then $\{\mathcal{D}_i \mid i \in I\}$ as a whole admits one. (This fact is implied by well-known results in the study of quantum structures [2, 3, 7, 13], though the terminology there is different from ours. For the sake of a self-contained discussion, we provide a complete proof.)

We call $a, b \in \mathbb{L}^n$ compatible, denoted $a \leftrightarrow b$, iff $\{a, a^\perp\}$ and $\{b, b^\perp\}$ admit joint refinement — in lattice terms this means that they generate a subalgebra of $\mathbb{L}^n$ which is Boolean [3]. Then we have that

$$\mathsf{span}\left(a \cap b\,,\, a \cap b^\perp\right) = a\,. \tag{12}$$

Indeed, existence of a joint refinement for $\{a, a^\perp\}$ and $\{b, b^\perp\}$ implies

$$\begin{aligned}
\mathcal{H}^n &= \mathsf{span}\Big((a \cup a^\perp) \cap (b \cup b^\perp)\Big) \\
&= \mathsf{span}\Big((a \cap (b \cup b^\perp)) \cup (a^\perp \cap (b \cup b^\perp))\Big) \\
&= \mathsf{span}\Big(\mathsf{span}\Big(a \cap (b \cup b^\perp)\Big)\,,\, \mathsf{span}\Big(a^\perp \cap (b \cup b^\perp)\Big)\Big)
\end{aligned}$$

and since

$$\mathsf{span}\Big(a \cap (b \cup b^\perp)\Big) \subseteq a \quad \text{and} \quad \mathsf{span}\Big(a^\perp \cap (b \cup b^\perp)\Big) \subseteq a^\perp$$

are subspaces of $\mathcal{H}^n$ this forces eq.(12).

The fact that each $\mathcal{D}_i$ is a decomposition, implying mutual orthogonality of its members, and that we have pairwise existence of a joint refinement for all decompositions in $\{\mathcal{D}_i \mid i \in I\}$, implies that

$$\forall a, b \in \bigcup \{\mathcal{D}_i \mid i \in I\} : a \leftrightarrow b\,.$$

We will now construct a joint refinement inductively, that is, we build a series (d_i) containing all elements of $\bigcup\{\mathcal{D}_i \mid i \in I\}$ and construct a joint refinement $\mathcal{E}_{k+1}$for $(d_1, \ldots, d_{k+1})$ given a joint refinement $\mathcal{E}_k$ for $(d_1, \ldots, d_k)$, taking as base case $\mathcal{E}_1 := \{d_1, d_1^\perp\}$. Set

$$\mathcal{E}_{k+1} := \{a \cap d_{k+1}, a \cap d_{k+1}^\perp \mid a \in \mathcal{E}_k\} \setminus \{o\}\,.$$

It clearly follows that $\bigcup \mathcal{E}_{k+1} \subseteq \bigcup \mathcal{E}_k$ so we obtain a decreasing sequence.

We then also have that

- $\mathsf{span}(\mathcal{E}_1) = \mathcal{H}^n$, and,
- $\mathsf{span}(\mathcal{E}_{k+1}) = \mathsf{span}\left(\{a \cap d_{k+1}, a \cap d_{k+1}^\perp \mid a \in \mathcal{E}_k\}\right) = \mathsf{span}\left(\mathcal{E}_k\right)$

due to eq.(12), what proves that the inductive procedure preserves spanning $\mathcal{H}^n$. Mutual orthogonality of the elements in $\mathcal{E}_{k+1}$ also follows construction.

It remains to be proven that

$$\bigcap_{j \in I} \bigcup \mathcal{E}_j \subseteq \bigcup \mathcal{D}_i$$

for all $i \in I$. Let $\#(d_i)$ be the length of (d_i). From the construction it follows that the elements of $\mathcal{E}_{\#(d_i)}$ are of the form $a_1 \cap \ldots \cap a_{\#(d_i)}$ where $a_j \in \{d_j, d_j^\perp\}$. For every $\mathcal{D}_i$ there is a subsequence of elements a_{i_j} such that $d_{i_j} \in \mathcal{D}_i$. Let $a_{i_1} \cap \ldots \cap a_{i_{\#\mathcal{D}_i}}$ be the corresponding subterm. We claim that the only non-empty such terms are those for which there is exactly one $1 \leq j \leq \#\mathcal{D}_i$ such that $a_{i_j} = d_{i_j}$ and for all others $k \neq j$ we have $a_{i_j} = d_{i_k}^\perp$. That there is at most one follows from the fact that the elements in $\mathcal{D}_i$ are mutually orthogonal. That there is necessarily one follows from the fact that we otherwise have $d_{i_1}^\perp \cap \ldots \cap d_{i_{\#\mathcal{D}_i}}^\perp$ as this subterm what implies that any vector contained in it should be orthogonal to all $d_{i_j} \in \mathcal{D}_i$, what is impossible since $\mathcal{D}_i$ spans $\mathcal{H}^n$. So every subterm $a_{i_1} \cap \ldots \cap a_{i_{\#\mathcal{D}_i}}$ and thus also every term $a_1 \cap \ldots \cap a_{\#(d_i)}$ contains $d_{i_j} \in \mathcal{D}_i$ and thus

$$a_1 \cap \ldots \cap a_{\#(d_i)} \subseteq \bigcup \mathcal{D}_i$$

so

$$\bigcap_{j \in I} \bigcup \mathcal{E}_j = \bigcup \mathcal{E}_{\#(d_i)} \subseteq \bigcup \mathcal{D}_i$$

for all $i \in I$ what completes the proof. □

It is well-known that projectors on $\mathcal{H}^n$ induce maps on Ω^n in terms of Luders' rule [11], that is

$$P[-] : \Omega^n \rightharpoonup \Omega^n : r \mapsto \frac{P \cdot r \cdot P}{\mathsf{tr}(P \cdot r)}$$

for $\mathsf{tr}(P \cdot r) > 0$. Note that $P : \Omega^n \rightharpoonup \Omega^n$ is still idempotent since

$$P \cdot (P \cdot r \cdot P) \cdot P = P \cdot r \cdot P\,,$$

so we can set

$$\mathsf{fix}(P[-]) := \{P \cdot r \cdot P \mid r \in \Omega^n\}\,.$$

Given an isomorphism

$$g : \mathsf{fix}(P[-]) \to \Omega^k,$$

with $0 \leq k = \mathsf{dim}(\mathsf{fix}(P)) \leq n$, and possibly induced by an isomorphism h on the underlying Hilbert spaces, we can define a map

$$\Omega^n \overset{P[-]}{\rightharpoonup} \mathsf{fix}(P[-]) \overset{g}{\to} \Omega^k\,. \tag{13}$$

This will be of our view of projectors in this section, except for an additional extension of the kernel to those density matrices that do not commute with P.

- By a projector $P^{\downarrow}[-] : \Omega^n \rightharpoonup \Omega^k$ we refer to the partial map induced by $P \in \mathbb{P}^n$ which has as kernel those states $x \in \Omega^n$ which are such that either
  - $\mathsf{tr}(P \cdot x) = 0$
  - $[x, P] \neq 0$

  and with the images defined by eq.(13).

When writing down $P^{\downarrow}[-]$ we as such assume that an isomorphism h, or equivalently, g has been specified.

We introduce some dialectics analogous to that of labelings.

- A state r *admits* a projector $P^{\downarrow}[-]$ iff $P^{\downarrow}[\,r\,]$ is defined.

Then by Proposition 4.3 P and r admit joint labeling.

Let $\mathbb{I}(k,n)$ be the collection of monotone maps of the form

$$\iota : \{1, \ldots, k\} \to \{1, \ldots, n\}$$

for $0 \leq k \leq n$, where the monotonicity is with respect to the usual order on natural numbers, and let

$$\mathbb{I}^n = \bigcup \{\mathbb{I}(k,n) \mid 0 \leq k \leq n\}\,.$$

Let

$$\iota^* : \{1, \ldots, n\} \rightharpoonup \{1, \ldots, k\}$$

be the partial inverse for any given ι and let $\mathbb{P}^n|e$ be the projectors $P \in \mathbb{P}^n$ that admit a given labeling e, that is,

$$\mathbb{P}^n|e := \left\{ P \in \mathbb{P}^n \;\middle|\; \bigcup \mathcal{D}_e \subseteq \bigcup \mathcal{D}_P \right\}.$$

Lemma 4.4 *Given a labeling e of $\mathcal{H}^n$ then $\mathbb{P}^n|e \cong \mathbb{I}^n$.*

Proof. Given $I \subseteq \{1, \ldots, n\}$ define $\iota \in \mathbb{I}^n$ such that I is its range. It then follows that

$$\mathcal{P}(\{1, \ldots, n\}) \cong \mathbb{I}^n$$

via $I \mapsto \iota$ due to monotonicity of ι.

We moreover have that $P \in \mathbb{P}^n|e$ iff $\bigcup \mathcal{D}_e \subseteq \bigcup \{\mathsf{fix}(P), \mathsf{fix}(P)^\perp\}$ iff there exists $I_P \subseteq \{1, \ldots, n\}$ such that $e_i \in \mathsf{fix}(P) \Leftrightarrow i \in I_P$, and thus we have $\mathbb{P}^n|e \cong \mathcal{P}(\{1, \ldots, n\})$ via $P \mapsto I_P$. $\square$

In Prop. 4.1, we characterized the Bayesian order in terms of projections. Here is the formulation for the spectral order in terms of quantum projections.

Theorem 4.5 *Let $n \geq 2$. For $r, s \in \Omega^n$, we have*

$$r \sqsubseteq s \Leftrightarrow P^{\downarrow}[\,r\,] \sqsubseteq P^{\downarrow}[\,s\,] \tag{14}$$

- *for all projectors $P^{\downarrow}[-]$ admitting both r and s, and,*
- *provided there are enough projectors admitting both r and s,*

where we adopt the base cases

- $\Omega^0 := \emptyset$;
- $\Omega^1 := \{(1)\}$ *with* $(1) \sqsubseteq (1)$;
- *For $r, s \in \Omega^2$ we have $r \sqsubseteq s$ iff there exist $p, q \in [0,1]$ with $p \leq q$ and a pure state $t \in \Sigma^n$ such that*

  $$r = (1-p)\bot + pt \qquad s = (1-q)\bot + qt\,,$$

  with

  $$\bot := \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Proof. Any self-adjoint operator on $\mathcal{H}^2$ either has a non-degenerated spectrum or is $\perp$. Excluding the latter case, given $r \in \Omega^2$ there exists a unique labeling e (up to permutation of the labels) such that r admits e. This labeling is obtained by setting $e_1 = r_+$ and $e_2 = r_+^\perp$, where r_+ are the eigenvectors for eigenvalue $\mathsf{max}(\mathsf{spec}(r))$. In view of Definition 3.22, it then follows that if e is admitted both by r and s and $\mathsf{spec}(r|e) \sqsubseteq \mathsf{spec}(s|e)$ then

$$\langle r|e_1\rangle = \mathsf{max}(\mathsf{spec}(r)) \leq \mathsf{max}(\mathsf{spec}(s)) = \langle s|e_1\rangle$$

with $s_+ = r_+$ is necessary and sufficient for $r \sqsubseteq s$. Defining $p, q \in [0,1]$ by

$$p = 2\langle r|e_1\rangle - 1 \quad \text{and} \quad q = 2\langle s|e_1\rangle - 1\,,$$

and the pure state $t \in \Omega^n$ such that $t_1 = e_1$ we obtain

$$(1-p)\perp + pt = (1-p)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} + p\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \langle r|e_1\rangle & 0 \\ 0 & 1-\langle r|e_1\rangle \end{pmatrix} = r$$

$$\text{and} \quad (1-q)\perp + qt = \begin{pmatrix} \langle s|e_1\rangle & 0 \\ 0 & 1-\langle s|e_1\rangle \end{pmatrix} = s$$

what encodes $s_+ = r_+$ and $\langle r|e_1\rangle \leq \langle s|e_1\rangle$ provided that $p \leq q$.

Let $r \sqsubseteq s$ according to Definition 3.22. We need to prove that r and s admit enough projectors and that they satisfy eq.(14) with respect to those admitted. So let us first define what we mean by enough projectors.

We mean by this having at least enough to consitute a family of mutually orthogonal projectors $\{P_e^i \mid 1 \leq i \leq n\}$ that support the spectral decomposition of a labeling

$$e = \sum_i i\, P_e^i\,.$$

One verifies that this is equivalent to saying that there exists a family of mutually orthogonal projectors $\{P_i \mid 1 \leq i \leq n\}$ such that $\bigcap_i \bigcup \mathcal{D}_{P_i}$ is the decomposition of some labeling e. It is moreover not restrictive to assume that for all i we have $\mathsf{dim}(\mathsf{fix}(P_i)) = n-1$.

Since $r \sqsubseteq s$ they admit a joint labeling e that admits projectors $\mathbb{P}^n|e$, among which we have those defined by $\mathsf{fix}(P_i) = e_i$. Then $\mathcal{D}_e = \bigcap_i \bigcup \mathcal{D}_{P_i}$ so r and s admit enough projectors.

We now show that $r \sqsubseteq s$ implies $P^\downarrow[\,r\,] \sqsubseteq P^\downarrow[\,s\,]$ provided $P^\downarrow[-]$ is admitted both by r and s. By Proposition 4.3 we have, since there exists a pairwise common refinement for r, s and P, that they all together admit a joint labeling e for which we then moreover have $\mathsf{spec}(r|e) \sqsubseteq \mathsf{spec}(s|e) \in \Delta^n$. Let

$$I_P := \{1 \leq i \leq n \mid e_i \in \mathsf{fix}(P)\}$$

and set $I_P^\perp := \{0, \ldots, 1\} \setminus I_P$. By Proposition 4.1 we then have that

$$p_{I_P^\perp}(\mathsf{spec}(r|e)) \sqsubseteq p_{I_P^\perp}(\mathsf{spec}(s|e))\ .$$

Let $\iota \in \mathbb{I}(k,n)$ with $k = \mathsf{dim}(\mathsf{fix}(P))$ such that it range coincides with I_P. Next, given any isomorphism $h : \mathsf{fix}(P) \to \mathcal{H}^k$, and thus also a corresponding one on states

$$g : \mathsf{fix}(P[-]) \to \Omega^k$$

choose a labeling (e'_i) of $\mathcal{H}^k$ such that $h(e_{\iota(i)}) = e'_i$ — we slightly abusively refer to a base vector by the subspace of the labeling in which it is contained. We obtain commutation of the following maps

$$\begin{tikzcd}
\{1,\ldots,n\} \arrow[r, "e"] & \mathbb{L}^n \arrow[d, "\tilde{h}"] \\
\{1,\ldots,k\} \arrow[u, "\iota"] \arrow[r, "e'"] & \mathbb{L}^k
\end{tikzcd}$$

where $\tilde{h} : \mathbb{L}^n \rightharpoonup \mathbb{L}^k$ is here the partial surjective map arising when $h : \mathsf{fix}(P) \to \mathcal{H}^k$ is applied pointwisely to those one-dimensional $a \in \mathbb{L}^n$ which are such that $a \subseteq \mathsf{fix}(P)$. Indeed, we have

$$i \overset{\iota}{\mapsto} \iota(i) \overset{e}{\mapsto} e_{\iota(i)} \overset{\tilde{h}}{\mapsto} e'_i\,.$$

Since due to $\bigcup \mathcal{D}_e \subseteq \bigcup \mathcal{D}_P$ we have

$$\begin{aligned}
P^{\downarrow}(r) &= g\left(\frac{1}{\mathsf{tr}(P\cdot r)} P\cdot r\cdot P\right) \\
&= \frac{1}{\sum_{i\in I_P}\langle r|e_i\rangle}\begin{pmatrix} \langle r|e_{\iota(1)}\rangle & & 0 \\ & \ddots & \\ 0 & & \langle r|e_{\iota(k)}\rangle \end{pmatrix} \quad \text{in } (e'_i)
\end{aligned}$$

It then follows that

$$\begin{aligned}
\pi_j\left(p_{I_P^{\perp}}(\mathsf{spec}(r|e))\right) &= \frac{1}{\sum_{i\in I_P}\langle r|e_i\rangle}\langle r|e_{\iota(j)}\rangle \\
&= \left\langle P^{\downarrow}(r) \,\middle|\, e'_j\right\rangle \\
&= \pi_j\left(\mathsf{spec}(P^{\downarrow}(r)|e')\right)
\end{aligned}$$

so

$$p_{I_P^{\perp}}(\mathsf{spec}(r|e)) = \mathsf{spec}(P^{\downarrow}(r)|e')$$

and thus

$$\mathsf{spec}(P^{\downarrow}(r)|e') \sqsubseteq \mathsf{spec}(P^{\downarrow}(s)|e')\,.$$

We then conclude $P^{\downarrow}(r) \sqsubseteq P^{\downarrow}(s)$.

Conversely, assume that there exists mutually orthogonal projectors $\{P_i \mid 1 \le i \le n\}$ such that $\bigcap_i \bigcup \mathcal{D}_{P_i}$ is the decomposition of some labeling e and for which we have

$P^{\downarrow}[\,r\,]_i \sqsubseteq P^{\downarrow}[\,s\,]_i$. Since r and s are admitted by all P_i we have by the proof of Proposition 4.3 that there exists a joint labeling for r and s and all P_i, which as such can only be e itself due to $\mathcal{D}_e = \bigcap_i \bigcup \mathcal{D}_{P_i}$. By constructing the isomorphisms

$$h_e : \Omega^n|e \to \Delta^n \quad \text{and} \quad h_{e'} : \Omega^{n-1}|e' \to \Delta^{n-1}$$

for each projector P_i such that we have commutation of

$$\begin{tikzcd}
\Omega^n|e \arrow[r, "h_e"] \arrow[d, "P_i^{\downarrow}"'] & \Delta^n \arrow[d, "p_i"] \\
\Omega^{n-1}|e' \arrow[r, "h_{e'}"] & \Delta^{n-1}
\end{tikzcd}$$

taking into account the isomorphisms $g_i : \mathsf{fix}(P_i[-]) \to \Omega^k$ that are different for each P_i, we can embed the quantum case for projectors P_i and states r and s that admit a fixed labeling e in the classical one with projectors p_i. Then $r \sqsubseteq s$ follows from $\mathsf{spec}(r|e) \sqsubseteq \mathsf{spec}(s|e) \in \Delta^n$ which itself results from the inductive definition for classical states. $\square$

Denote by $\mathbb{P}^n_\bullet$ projectors on $n-1$ dimensional subspaces of $\mathcal{H}^n$. We have the following analogy between classical and quantum states:

	Classical	Quantum
States	Δ^n	Ω^n
Pure states	$\max(\Delta^n)$	Σ^n
Primitive projections	$\{p_i \mid 0 \leq i \leq n\}$	$\mathbb{P}^n_\bullet$
General projections	$\{p_{i_1,\ldots,i_k} \mid 0 \leq i_1,\ldots,i_k \leq n\}$	$\mathbb{P}^n$

4.3 The lattices of Birkhoff and von Neumann

In the spectral order, quantum states are ordered by requiring of a labeling e that it *commute* with the states r and s under consideration. In the corresponding inductive formulation, a condition of commutation with states is imposed on projections. Knowing the structural importance of non-commutativity of observables in quantum mechanics, it may surprise the reader to learn that the lattices of Birkhoff and von Neumann [2], the powerset $\mathcal{P}\{1,\ldots,n\}$ and the collection $\mathbb{L}^n$ of subspaces of $\mathcal{H}^n$ ordered by inclusion, can be recovered from Δ^n and Ω^n in a *purely order theoretic manner.*

Recall here that the fundamentally different nature of quantum versus classical observables can also be explained in order theoretic terms, roughly, by the distributivity of $\mathcal{P}\{1,\ldots,n\}$ versus the non-distributivity of $\mathbb{L}^n$ [3, 7, 13]. The relation between observables and these lattices is as follows.

A (real-valued) observable of a classical system, with pure states $\Sigma^n_{\rm cl} \cong \{1,\ldots,n\}$, is a map $a : \Sigma^n_{\rm cl} \to \mathbb{R}$ with range $\mathsf{spec}(a)$ that assigns to each pure state the value of that observable. As a consequence, any proposition about the system of the form

"The value of classical observable a is contained in $E \subseteq \mathsf{spec}(a)$"

encodes as the set $a^{-1}(E)$. By considering all observables, that is, all such maps a, we obtain $\mathcal{P}\{1, \ldots, n\} \cong \mathcal{P}(\Sigma^n_{\mathrm{cl}})$ as the algebra of propositions about the system. Inclusion of sets encodes implication of propositions.

Envision the pure states Σ^n_{qm} of a quantum system as the one-dimensional subspaces of $\mathcal{H}^n$, that is, a pure state is the set of fixed points r_1 of a density operator r with $\mathsf{spec}(r) \subseteq \{0, 1\}$. For an observable on a quantum system, i.e., a self-adjoint operator A with spectrum $\mathsf{spec}(A)$, any proposition about the system of the form

"The value of quantum observable A is contained in $E \subseteq \mathsf{spec}(A)$"

encodes as the fixed points of the projector P^E_A, since the states included in it will yield an outcome in E in a measurement of the observable A (the probability of all other outcomes is zero). This set of fixed points is a subspace of $\mathcal{H}^n$. By considering all observables, that is, all self-adjoint operators A, we obtain $\mathbb{L}^n \subset \mathcal{P}(\Sigma^n_{\mathrm{qm}})$ as the algebra of propositions about the system. And once again, inclusion of subspaces encodes implication of propositions.

And so now, to briefly state things, in this section we establish the following: Though the domain of quantum states is grounded in commutativity, it is nevertheless a genuine *quantum structure* in the sense of present day theoretical physics.

Definition 4.6 Let D be a dcpo. An element $x \in D$ is *irreducible* if

$$\bigwedge\big(\uparrow x \cap \max(D)\big) = x.$$

The set of irreducible elements in D is $\mathrm{Ir}(D)$.

Our first result establishes that the irreducible states in Δ^n have an unmistakable operational significance: They are precisely the states one derives by applying all possible combinations of projections p_i to the initial state $\perp \in \Delta^n$.

Lemma 4.7 *For $x \in \Delta^n$, the following are equivalent:*

(i) *The state x is irreducible.*

(ii) *For all $i \in \{1, \ldots, n\}$, either $x_i = x^+$ or $x_i = 0$.*

(iii) *There is a nonempty subset $X \subseteq \max(\Delta^n)$ with $x = \bigwedge X$.*

Proof. Recall from Prop. 2.11(ii) that for any classical state x and any index i, $x \sqsubseteq e_i \Leftrightarrow x_i = x^+$. This fact is used implicitly in what follows.

(i) $\Rightarrow$ (ii): Let x be irreducible. Let $y \in \Delta^n$ be the classical state with $y_i = y^+ \Leftrightarrow e_i \in \uparrow x \cap \max(\Delta^n)$ and $y_i = 0$ otherwise. Then y is a lower bound for $\uparrow x \cap \max(\Delta^n)$, so $y \sqsubseteq x$. We claim that $y = x$. If $y^+ < x^+$, then

$$\sum_{i=1}^{n} x_i \geq \sum_{x_i = x^+} x_i > \sum_{y_i = y^+} y_i = 1,$$

which contradicts the fact that x is a classical state. Then $y^+ \geq x^+$. But since $y \sqsubseteq x$, we know $y^+ \leq x^+$. Thus, $x^+ = y^+$, which gives $x = y$. This proves (ii).

(ii) $\Rightarrow$ (i): The proof is by induction. It is true for $n = 2$. Assume it for Δ^n, and let $x \in \Delta^{n+1}$ be a state of the desired form. We can assume x is not pure, since otherwise x is clearly irreducible.

Let $y \in \Delta^{n+1}$ be a lower bound for the set $\uparrow x \cap \Delta^{n+1}$. Because x is not pure, y cannot be pure either. Let i be any index with $1 \leq i \leq n+1$. Then $p_i(x)$ has the form mentioned in (ii), so it is irreducible by the inductive hypothesis. The state $p_i(y)$ is a lower bound of $\uparrow p_i(x) \cap \max(\Delta^n)$, so the irreduciblity of $p_i(x)$ gives $p_i(y) \sqsubseteq p_i(x)$. Then $y \sqsubseteq x$. This puts $x \in \mathrm{Ir}(\Delta^{n+1})$.

(iii) $\Rightarrow$ (i): Let y be the state with $y_i = y^+$ iff $e_i \in X$ and $y_i = 0$ otherwise. By (i)=(ii), y is irreducible, while $X = \uparrow y \cap \max(\Delta^n)$ gives

$$x = \bigwedge X = \bigwedge \big(\uparrow y \cap \max(\Delta^n)\big) = y,$$

which shows that x is irreducible.

(i) $\Rightarrow$ (iii): Obvious. $\square$

Now we prove that $\mathcal{P}\{1, \ldots, n\}$ is recoverable from the irreducible elements of Δ^n. Specifically, $\mathrm{Ir}(D)$ in the order it inherits from D is order isomorphic to a subset of $\mathcal{P}\big(\max(D)\big)$ ordered by *reverse inclusion,* so we must consider $\mathrm{Ir}(D)$ in its dual order, $\mathrm{Ir}(D)^*$. Second, since every $x \in D$ has a maximal element above it, the empty set is not represented in $\mathrm{Ir}(D)^*$, so we adjoin a least element 0 to obtain the poset

$$\mathrm{Ir}(D)^*_\perp := \mathrm{Ir}(D)^* \cup \{0\}\,.$$

Proposition 4.8 *For any $n \geq 2$,*

$$\mathrm{Ir}(\Delta^n)^*_\perp \simeq \mathcal{P}\{1, \ldots, n\}.$$

Proof. Let $e : \{1, \ldots, n\} \to \max(\Delta^n)$ be the natural bijection that takes an outcome i to its associated pure state $e(i) = e_i$. An order isomorphism

$$\varphi : \mathrm{Ir}(\Delta^n)^* \to \mathcal{P}\{1, \cdots, n\} \setminus \{\emptyset\}$$

is then given by

$$\varphi(x) = e^{-1}\big(\uparrow x \cap \max(\Delta^n)\big).$$

First, φ is surjective: Given $\emptyset \neq X \in \mathcal{P}\{1, \ldots, n\}$,

$$\varphi\Big(\bigwedge e(X)\Big) = X,$$

using Lemma 4.7(iii). Next, it is an order embedding: For $x, y \in \mathrm{Ir}(\Delta^n)^*$,

$$\begin{aligned} x \sqsubseteq y \quad &\Leftrightarrow \quad \uparrow x \cap \max(\Delta^n) \subseteq \uparrow y \cap \max(\Delta^n) \\ &\Leftrightarrow \quad e^{-1}\big(\uparrow x \cap \max(\Delta^n)\big) \subseteq e^{-1}\big(\uparrow y \cap \max(\Delta^n)\big) \\ &\Leftrightarrow \quad \varphi(x) \subseteq \varphi(y). \end{aligned}$$

Now we simply extend φ to an order isomorphism from $\mathrm{Ir}(\Delta^n)^*_\perp$ to $\mathcal{P}\{1, \cdots, n\}$ by setting $\varphi(0) = \emptyset$, and the proof is finished. $\square$

We turn now to the analogous result for quantum states. To stress the analogy with classical states we denote pure states now as $\max(\Omega^n)$ rather than as Σ^n.

First we prove the analogue of Prop. 2.11(ii) for quantum states. Denote by r^+ the subspace of eigenvectors for the largest eigenvalue, that is $r^+ = r_\lambda$ for $\lambda = \max\big(\mathsf{spec}(r)\big)$.

Lemma 4.9 *For $r \in \Omega^n$ and $t \in \max(\Omega^n)$, we have*

$$r \sqsubseteq t \Leftrightarrow t_1 \subseteq r^+$$

and thus

$$r^+ = \bigcup \{t_1 \mid t \in \uparrow r \cap \max(\Omega^n)\}\,.$$

Proof. Let $t \in \max(\Omega^n)$ be such that $t_1 \subseteq r^+$. Define a labeling e that satisfies

- $e_1 = t_1$,
- $e_2, \ldots, e_{\mathsf{dim}(r^+)} \in r^+ \cap t_1^\perp$, and,
- $e_{\mathsf{dim}(r^+)+1}, \ldots, e_n \in \bigcup \mathcal{D}_r \cap (r^+)^\perp$.

By Prop. 2.11(ii) we then have $\mathsf{spec}(r|e) \sqsubseteq \mathsf{spec}(t|e)$ and thus $r \sqsubseteq t$.

Conversely, let $t \in \max(\Omega^n)$ be such that $r \sqsubseteq t$. Then there exists labeling e such that $[r, e] = [t, e] = 0$, that is $\bigcup \mathcal{D}_e \subseteq \bigcup \mathcal{D}_r \cap \bigcup \mathcal{D}_t$, which implies since $\mathcal{D}_t = \{t_1^\perp, t_1\}$ with t_1 one-dimensional that there exists $i \in \{1, \ldots, n\}$ such that $t_1 = e_i$, say $i = 1$. Then

$$\mathsf{spec}(r|e) \sqsubseteq \mathsf{spec}(t|e) = (1, 0, \ldots, 0)$$

in Δ^n. By Prop. 2.11(ii) it then follows that $\langle r|e_i\rangle = \mathsf{spec}(r|e)^+$ so $e_i \subseteq r^+$ and thus $t_1 \subseteq r^+$. $\square$

A quantum state is irreducible iff its spectrum can be viewed as an irreducible classical state.

Lemma 4.10 *For $r \in \Omega^n$, the following are equivalent:*

(i) *The state r is irreducible.*

(ii) *There is a labeling e with $[r, e] = 0$ and $\mathsf{spec}(r|e) \in \mathrm{Ir}(\Delta^n)$.*

(iii) *Either there exists $\lambda \in (0, 1]$ such that $\mathsf{spec}(r) = \{0, \lambda\}$ or $r = \perp$.*

In either case, $\mathsf{spec}(r|e) \in \mathrm{Ir}(\Delta^n)$ for any labeling e with $[r, e] = 0$.

Proof. By Lemma 4.7, (ii) $\Leftrightarrow$ (iii) is obvious. The rest of the proof essentially relies on the analogous result for classical states.

(i) $\Rightarrow$ (iii) Let r be irreducible and e be a labeling with $[r,e]=0$. Let

$$X = \{\mathsf{spec}(t|e) : t \in \uparrow r \cap \max(\Omega^n) \cap (\Omega^n|e)\}.$$

By Lemma 4.7(iii), the infimum of X is an irreducible classical state, and we use this to implicitly define a quantum state $s \in \Omega^n$ by

$$\mathsf{spec}(s|e) := \bigwedge X \in \mathrm{Ir}(\Delta^n).$$

By the definition of $\mathsf{spec}(s|e)$, we immediately have $\mathsf{spec}(r|e) \sqsubseteq \mathsf{spec}(s|e)$, which implies $r \sqsubseteq s$ in Ω^n.

We claim that $r = s$. To prove this, we need only show that

$$\uparrow r \cap \max(\Omega^n) \subseteq \uparrow s \cap \max(\Omega^n),$$

for then we have

$$\begin{aligned} r \sqsubseteq s \quad &\Rightarrow \quad \uparrow r \cap \max(\Omega^n) = \uparrow s \cap \max(\Omega^n) \\ &\Rightarrow \quad r = \bigwedge \uparrow r \cap \max(\Omega^n) = \bigwedge \uparrow s \cap \max(\Omega^n) \\ &\Rightarrow \quad s \sqsubseteq r \end{aligned}$$

using the irreducibility of r.

Let $t \in \uparrow r \cap \max(\Omega^n)$. By Lemma 4.9, $t_1 \subseteq r^+$, and since $r^+ \subseteq s^+$, $t_1 \subseteq s^+$, which again by Lemma 4.9 gives $t \in \uparrow s \cap \max(\Omega^n)$. But why do we have $r^+ \subseteq s^+$?

This is the crucial part of the argument: Since $[r,e]=0$, $\bigcup \mathcal{D}_e \subseteq \bigcup \mathcal{D}_r$, so $\mathcal{D}_e$ contains a subset S of cardinality $\mathsf{dim}(r^+)$ whose union is contained in r^+. Each element of S is a one dimensional subspace of $\mathcal{H}^n$, so the usual bijection allows us to treat S as a collection of pure states.

For each pure state $t \in S$, we have $[t,e]=0$, since $t_1 \in \mathcal{D}_e$, and $t \in \uparrow r \cap \max(\Omega^n)$, using Lemma 4.9 and $t_1 \subseteq r^+$. Then by the definition of s, $s \sqsubseteq t$, while Lemma 4.9 gives $t_1 \subseteq s^+$. But then, because s^+ is a subspace, we clearly have

$$r^+ = \mathsf{span}(\{t_1 : t \in S\}) \subseteq s^+,$$

which proves $r = s$. Thus, $\mathsf{spec}(r|e) = \mathsf{spec}(s|e)$ is irreducible in Δ^n.

(iii) $\Rightarrow$ (i) Let $r = \perp$. Then $\uparrow\perp \cap \max(\Omega^n) = \max(\Omega^n)$. If $s \sqsubseteq \max(\Omega^n)$ (pointwisely) then by Lemma 4.9 it follows that

$$\mathcal{H}^n = \{t_1 \mid t \in \max(\Omega^n)\} \subseteq s^+ .$$

Thus $s = \perp$ so $s \sqsubseteq r$ and as such since trivially $\perp \sqsubseteq \max(\Omega^n)$ (pointwisely) we conclude $\perp = \bigwedge\big(\uparrow\perp \cap \max(\Omega^n)\big)$.

Let $\mathsf{spec}(r) = \{0, \lambda\}$. First, $r \sqsubseteq \uparrow r \cap \max(\Omega^n)$ (pointwisely) is again trivial. Second, let $s \sqsubseteq \uparrow r \cap \max(\Omega^n)$ (pointwisely). Then,

$$\uparrow r \cap \max(\Omega^n) \subseteq \uparrow s \cap \max(\Omega^n)$$

so it follows by Lemma 4.9 that

$$\begin{aligned} r^+ &= \bigcup\{t_1 \mid t \in \uparrow r \cap \max(\Omega^n)\} \\ &\subseteq \bigcup\{t_1 \mid t \in \uparrow s \cap \max(\Omega^n)\} \\ &= s^+ . \end{aligned}$$

Now define a labeling e that satisfies

- $e_1, \ldots, e_k \subseteq r^+$ for $k := \mathsf{dim}(r^+)$,
- $e_{k+1}, \ldots, e_{k+l} \subseteq (r^+)^\perp \cap s^+$ for $l := \mathsf{dim}(s^+) - \mathsf{dim}(r^+)$, and,
- $e_{k+l+1}, \ldots, e_n \subseteq (s^+)^\perp$ where $1 - l - k = 1 - \mathsf{dim}(s^+)$.

We have $[r, e] = [s, e] = 0$, while Lemma 4.7 gives $\mathsf{spec}(s|e) \sqsubseteq \mathsf{spec}(r|e)$ since $\mathsf{spec}(r|e)$ is irreducible in Δ^n. Thus, $s \sqsubseteq r$. □

Theorem 4.11 *For any $n \geq 2$,*

$$\mathrm{Ir}(\Omega^n)^*_\perp \simeq \mathbb{L}^n .$$

Proof. An order isomorphism $\varphi : \mathrm{Ir}(\Omega^n)^* \to \mathbb{L}^n \setminus \{0\}$ is given by

$$\varphi(r) = r^+.$$

For its surjectivity, given any $A \in \mathbb{L}^n \setminus \{0\}$, define an irreducible quantum state $r : \{0, \lambda\} \to \mathbb{L}^n$ by $r_\lambda = A$ and $r_0 = A^\perp$, where $\lambda = 1/\mathsf{dim}(A) > 0$. Then $\varphi(r) = A$. The fact that it is an order isomorphism follows straightforwardly from quantum degeneration (Lemma 3.24). □

The particular nature of this proof, which essentially relies on how we recover $\mathcal{P}\{1, \ldots, n\}$ from Δ^n, exhibits how much of the structure of Ω^n is already present in the partial order on Δ^n.

To summarize, we are able to recover $\mathbb{L}^n$, the basic quantum structure from which all other are derivable, from the domain of quantum states in a purely order theoretic manner. Here is an analogy worth remembering: Ω^n is to $\mathbb{L}^n$ as density operators are to pure states. More to the point, in view of the fact that

- The canonical order theoretic structure corresponding to quantum mechanics in terms of only pure states is $\mathbb{L}^n$,

we are tempted to claim that

- The canonical order theoretic structure corresponding to quantum mechanics in terms of density operators is Ω^n.

In short, because the density operator formulation offers a more complete picture than simply working with pure states, the domain Ω^n offers a more complete picture than the lattice $\mathbb{L}^n$.

Finally, let us add one last twist to the story: Not only does this more complete picture emerge as the result of *commutative considerations,* but any natural approach to ordering states which allows non-commutativity seems destined to fail!

Fact 4.12 *If we define $r \sqsubseteq s$ for $r, s \in \Omega^n$ by either*

(i) *"there exists a labeling e such that $\mathsf{spec}(r|e) \sqsubseteq \mathsf{spec}(s|e)$ in Δ^n," or*

(ii) *"for all labelings e we have $\mathsf{spec}(r|e) \sqsubseteq \mathsf{spec}(s|e)$ in Δ^n,"*

where e does not necessarily commute with r and s, then in both cases, the relation $\sqsubseteq$ is not an information order.

Justification. We will only provide explicit proofs of the following partial statements for the case of $n = 2$ (arguments in higher dimensions are essentially of the same nature):

- In case (i), all states (including bottom) are above all pure states.
- In case (ii), no state (including bottom) is strictly below a pure state.

Let r be a pure state with $\psi \in r_1$ and let $\psi^\perp \in r_0$. Then there exists a labeling e such that $\psi + \psi^\perp \in e_1$ and $\psi - \psi^\perp \in e_2$. For this labeling e we have $\langle r|e_1\rangle = \langle r|e_2\rangle = 1/2$, that is, $\mathsf{spec}(r|e) = \perp$ in Δ^2. Thus, for all $s \in \Omega^2$ we have $\mathsf{spec}(r|e) \sqsubseteq \mathsf{spec}(s|e)$ in Δ^2. In case (i) this implies $r \sqsubseteq s$ in Ω^2. In case (ii) this implies that we cannot have $s \sqsubset r$ in Ω^2. □

Thus, the commutativity implied by the existence of a joint labeling in the spectral order seems unavoidable if one wants to obtain a non-trivial partial order. This can be physically explained as follows: Quantum mechanics bears as one of its most fundamental principles that the maximal knowledge an observer can have about a system at a single point in time amounts to knowing the values of a class of observables that constitute a maximally commuting family; any knowledge beyond this is forbidden. Thus, on the assumption that a partial order on quantum states should make statements about knowledge we possess about a system, commutativity at some level is probably unavoidable.

5 Applications

We consider some basic applications of classical and quantum states.

5.1 A calculus for noise

One of the basic ideas in the measurement formalism [8] is that one can differentiate functions $f : D \to E$ between collections of informative objects with respect to underlying notions of content. Speaking abstractly, it offers a definition of "informatic rate of change," i.e., the *rate* at which (the content of) the output of a process changes with respect to (the content of) its input.

As we have seen, the domains of classical and quantum states have many natural notions of content, so in principle we ought to be able to study informatic rates of change in these settings as a means of improving our understanding about the behavior of various phenomena.

One such example arises easily in the study of noise: By modelling the effect of noise as a selfmap on classical or quantum states, we can apply the informatic derivative with respect to a preferred notion of content μ to gain a precise measure of the effect a given form of noise f has on a given state σ. For ease of exposition, we illustrate the idea on Δ^2. Here are some natural candidates for μ:

- $\mu x = 1 - x^+$
- $\mu x = 2x^+x^-$
- $\mu x = -x^+ \log x^+ - x^- \log x^-$ (Shannon entropy)

We'll use the first since it is the simplest.

Definition 5.1 A *noise operator* is a function $f : \Delta^2 \to \Delta^2$ such that $f\sigma \sqsubseteq \sigma$.

The intuition in this definition is that noise *qualitatively* increases uncertainty. Now, suppose a system is in state σ when it suffers an unwanted interaction with its environment, which changes its state to $f\sigma$. How can we measure the effect of the noise on the state of the system?

First, we write down a 'grammar' which allows for the description of noise: A simple class of noise operators $\mathbb{N}$ is

- $\perp, 1 \in \mathbb{N}$
- $f, g \in \mathbb{N} \Rightarrow f \circ g \in \mathbb{N}$
- $f, g \in \mathbb{N} \Rightarrow pf + (1-p)g \in \mathbb{N}$ for $p \in [0, 1]$,
- $f, g \in \mathbb{N}$ & $f \sqsubseteq g \Rightarrow pf^* + (1-p)g \in \mathbb{N}$ for $p \in [0, 1/2]$,

where $*$ is the involution $(x, y)^* = (y, x)$. It is straightforward to check that the class of noise operators on Δ^2 are closed under the operations mentioned above.

Now the effect that channel $f \in \mathbb{N}$ has on state σ can be systematically calculated as follows:

Theorem 5.2 *If $f, g \in \mathbb{N}$, then*

- $d(\perp)_\mu(\sigma) = 0,$
- $d(1)_\mu(\sigma) = 1,$
- $d(f \circ g)_\mu(\sigma) = df_\mu(g\sigma) \cdot dg_\mu(\sigma),$
- $d(pf + (1-p)g)_\mu(\sigma) = pdf_\mu(\sigma) + (1-p)dg_\mu(\sigma),$
- $d(pf^* + (1-p)g)_\mu(\sigma) = (1-p)dg_\mu(\sigma) - pdf_\mu(\sigma),$

for any $\sigma \neq \perp$.

This theorem allows us to verify *inductively* that $df_\mu(\sigma)$ is a measure of *reliability.* For instance, if $df_\mu(\sigma) = 0$, then the noise f has had a very strong effect on σ (as a channel, f is *unreliable* for the transmission of σ), while if $df_\mu(\sigma) = 1$, we intuitively expect $f(\sigma) = \sigma$, i.e., f is completely reliable.

Lemma 5.3 If f is a noise operator and $f\sigma = \sigma$, then either $df_\mu(\sigma) \geq 1$ or it does not exist.

We now have a fun and *systematic* approach to an interesting problem: Determining the states that a particular type of noise does not affect.

Example 5.4 Consider the depolarization of a classical state,

$$f\sigma = p\perp + (1-p)\sigma.$$

For $\sigma \neq \perp$, we have

$$df_\mu = pd(\perp)_\mu + (1-p)d(1)_\mu = 1 - p,$$

so the only unaffected state is $\perp$ for $p > 0$.

Example 5.5 The effect of a magnetic field on data stored on a disk is

$$f\sigma = p\sigma^* + (1-p)\sigma.$$

For $\sigma \neq \perp$, we have

$$df_\mu = -pd(1)_\mu + (1-p)d(1)_\mu = 1 - 2p.$$

Thus, if you are a state, it is better to be depolarized than flipped.

In quantum mechanics, the study of noise and how to beat it is called *decoherence.* In the quantum case, some neat measures of content arise, corresponding to the classical ones:

- $\mu x = 1 - x^+ \quad \Longrightarrow \quad \mu\rho = 1 - \mathrm{spec}(\rho)^+$,
- $\mu x = 2x^+x^- \quad \Longrightarrow \quad \mu\rho = 1 - \mathrm{tr}(\rho^2)$,
- $\mu x = -x^+ \log x^+ - x^- \log x^- \quad \Longrightarrow \quad \mu\rho = -\mathrm{tr}(\rho \log \rho)$.

For consistency, we use the first one here as well.

Example 5.6 Depolarization of quantum states is

$$f(r) = t \cdot \frac{I}{n} + (1-t)r = t\bot + (1-t)r.$$

Once again, $df_\mu(r) = 1-t$. But the reason is physical. For instance, in the two dimensional case we have

$$f(r) = (1-t)r + \frac{t}{3}(\sigma_x r \sigma_x + \sigma_y r \sigma_y + \sigma_z r \sigma_z)$$

It affects the entire state in a uniform way. Quantum bit/phase flipping, by contrast, only affects 'part' of r. Things get more interesting then.

5.2 The axioms of domain theory

This work led to the introduction of a new class of domains, the *exact domains.* We will show in this section that exact domains offer a new perspective on the more traditional, *continuous* domains [1]. With the benefit of this new point of view, it then becomes possible to ask certain foundational questions that some domain theorists may find intriguing.

Recall that in the study of approximation on classical states, we learned that $x \ll y$ is a statement which implicitly carries a specific *context.* In order to conclude $x \ll z$ when $y \sqsubseteq z$, we need to know that the statement $y \sqsubseteq z$ is being made in the same context as $x \ll y$. Aside from the case when x approximates a pure state (Prop. 2.39), there is another way of ensuring this: If *all* entities involved (x, y, z) can be regarded as necessary for a *single* state ($\Uparrow z \neq \emptyset$).

Proposition 5.7 (Context) *For all $x, y, z \in \Delta^n$, if $x \ll y \sqsubseteq z$ and $\Uparrow z \neq \emptyset$, then $x \ll z$.*

Proof. First we prove that if $x, y, z \in \Lambda^n$ with $x \ll y \sqsubseteq z$ in Λ^n and $z_i > 0$ for all i, then $x \ll z$ in Λ^n. Let $z = \bigsqcup w_k$ where $(w_k)_{k \geq 1}$ is increasing in Λ^n. Now we proceed just as in the proof of Theorem 2.36. For $x_i = x_{i+1} > 0$, we can take $k_i = 1$, since the monotonicity of w_k implies

$$\frac{x_i}{x_{i+1}} = 1 \leq \frac{\pi_i(w_k)}{\pi_{i+1}(w_k)}$$

for all k, while in the case of $x_i > x_{i+1} > 0$, degeneration (Lemma 2.13) gives $y_i > y_{i+1} > 0$, which accounts for the strict inequality in

$$\frac{x_i}{x_{i+1}} < \frac{y_i}{y_{i+1}} \leq \frac{z_i}{z_{i+1}} = \lim_{k\to\infty} \frac{\pi_i(w_k)}{\pi_{i+1}(w_k)}.$$

The definition of limit again makes it clear that the required k_i exists.

More generally, if $x \ll y \sqsubseteq z \ll w$ in Δ^n, we use Prop. 2.35(ii) to prove $x \ll z$. First, $z \in \Delta^n_\sigma \Rightarrow x \in \Delta^n_\sigma$ follows from Lemma 2.34(i) since $z_i > 0$ for all i using Lemma 2.37 and $z \ll w$. And second, since Prop. 2.35(ii) gives $r(x) \ll r(y) \sqsubseteq r(z)$ in Λ^n, our opening argument now applies leaving $r(x) \ll r(z)$ in Λ^n. $\square$

The value of this observation is that it provides a theoretical explanation for why the approximation relation on Δ^n is interpolative:

Lemma 5.8 *If D is an exact dcpo such that for all $x, y, z \in D$,*

$$x \ll y \sqsubseteq z \Rightarrow x \ll z,$$

whenever $\Uparrow\!z \neq \emptyset$, then $\ll$ is interpolative. Moreover, a dcpo is continuous iff it is exact and $x \ll y \sqsubseteq z \Rightarrow x \ll z$ for all $x, y, z \in D$.

Proof. The proof given in [1] applies unchanged. $\square$

From this we can see that exact domains require *precision* when reasoning about approximation. By contrast, the single most important aspect of approximation on a continuous domain is *not* that it is interpolative [1], but rather that it is *context independent.* The present work seems to provide sufficient impetus for investigating domains beyond the continuous variety.

5.3 Qualitative measures of entanglement

Quantum entanglement is the essential feature in quantum communication schemes and quantum cryptographic protocols that distinguishes them from their classical counterparts. For the particular dialectics used here we refer to the standard literature on the matter.

We illustrate by means of a series of examples how the results of this paper can be applied to the study of entanglement. A full development on the matter is in preparation.

Example 5.9 *Measures of entanglement of bipartite quantum systems.* Let $\mathcal{H}^n$ be a n-dimensional complex Hilbert space. According to Schmidt's biorthogonal decomposition theorem [15], any bipartite state $\Psi \in \mathcal{H}^n \otimes \mathcal{H}^n$ can be rewritten as

$$\Psi = \sum_i c_i \psi_i \otimes \phi_i$$

with (ψ_i) and (ϕ_i) orthonormal bases of $\mathcal{H}^n$ and (c_i) positive real coifficients which are as a set uniquely defined. In particular we have $\sum_i c_i^2 = 1$ due to normalization of Ψ, so every $\Psi \in \mathcal{H}^n \otimes \mathcal{H}^n$ defines a unique classical state $c := (c_i^2)$. We can then qualitatively measure entanglement using the dcpo Λ^n as

$$\mathsf{Ent} : \mathcal{H}^n \otimes \mathcal{H}^n \to \Lambda^n : \Psi \mapsto r(c)$$

where r is the usual retraction on classical states.

Moreover, every measure of content

$$\mu : \Lambda^n \to [0,1]^*$$

gives rise to a quantitative measure of entanglement

$$\mu \cdot \mathsf{Ent} : \mathcal{H}^n \otimes \mathcal{H}^n \to [0,1]^* .$$

When taking as μ Shannon entropy we find the usual quantitative measure of entanglement for bipartite quantum systems.

The maximal element of Λ^n then encodes the non-entangled states, that is, the pure tensors $\psi \otimes \phi$. The minimal element of Λ^n then encodes the maximally entangled state, that is

$$\sum_i \frac{1}{\sqrt{n}} \psi_i \otimes \phi_i \in \mathcal{H}^n \otimes \mathcal{H}^n$$

which does not depend on the choice of bases.

Since $\mu : \Lambda^2 \to [0,1]^*$ is a duality, using Λ^2 rather than in $[0,1]^*$ doesn't teach us much for the case $n = 2$, that is, for a pair of qubits. For qutrits however, $n = 3$, we capture essential qualitative differences by valuating in Λ^3.

Consider for example the state

$$\mathsf{S} := \frac{1}{\sqrt{2}}(\psi_1 \otimes \phi_1 + \psi_2 \otimes \phi_2) \in \mathcal{H}^3 \otimes \mathcal{H}^3,$$

that is,

$$\mathsf{S} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

The state S is entangled but this entanglement has essentially a qubit nature, that is, we can express the state by only using a subbase of $\mathcal{H}^3$ that contains two vectors. In particular, the entanglement coincides with that of the EPR or singlet state so it is maximal as qubit entanglement.

On the other hand, the states

$$\mathsf{T}_q := q(\psi_1 \otimes \phi_1) + \frac{1-q}{2}(\psi_2 \otimes \phi_2 + \psi_3 \otimes \phi_3) \in \mathcal{H}^3 \otimes \mathcal{H}^3$$

for $1/3 < q < 1$, that is,

$$\mathsf{T}_q := q(|00\rangle) + \frac{1-q}{2}(|11\rangle + |22\rangle)\,,$$

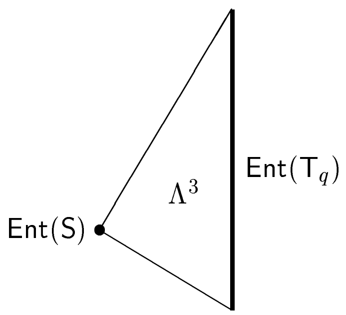
exhibits genuine qutrit entanglement.

Unfortunately, for q ranging in $(\frac{1}{3}, 1)$ Shannon entropy ranges in $(0,1)$ so some T_q have entropy higher than S and some have entropy less than S. The valuation $\mu \cdot \mathsf{Ent}$ as such doesn't capture the qualitative feature that distinquishes between maximal qubit-type entanglement and essentially qutrit type entanglement.

However, Λ^3 does. Indeed, consider

$$\mathsf{Ent}(\mathsf{S}) = r\left(\frac{1}{2}, \frac{1}{2}, 0\right) \quad \text{and} \quad \mathsf{Ent}(\mathsf{T}_q) = r\left(q, \frac{1-q}{2}, \frac{1-q}{2}\right) ,$$

that is,



in graphical terms. Since there is no value for $q \in (\frac{1}{3}, 1)$ for which we have that $(1/2, 1/2, 0)$ and $(q, (1-q)/2, (1-q)/2)$ compare in Λ^3, it follows for all T_q with $q \in (\frac{1}{3}, 1)$ that

$$\mathsf{Ent}(\mathsf{S}) \not\sqsubseteq \mathsf{Ent}(\mathsf{T}_q) \quad \text{and} \quad \mathsf{Ent}(\mathsf{T}_q) \not\sqsubseteq \mathsf{Ent}(\mathsf{S}) .$$

The states $\Psi \in \mathcal{H}^3 \otimes \mathcal{H}^3$ for which we have $\mathsf{Ent}(\Psi) \sqsubseteq \mathsf{Ent}(\mathsf{S})$ are those which are such that

$$\mathsf{Ent}(\Psi) = r\left(q, \frac{1-q}{2}, \frac{1-q}{2}\right)$$

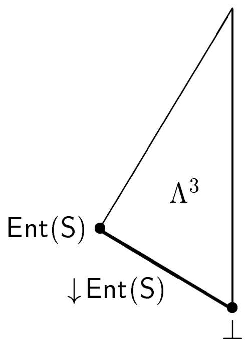
for $0 \leq q \leq 1/3$, that is, that are convex combinations of S and the maximally entangled state in $\mathcal{H}^3 \otimes \mathcal{H}^3$,

$$\frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle) ,$$

for which we set

$$\top := \mathsf{Ent}\left(\frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)\right) .$$

Graphically,



where $\downarrow\mathsf{Ent}(\mathsf{S})$ is the lower set of $\mathsf{Ent}(\mathsf{S})$ in Λ^3.

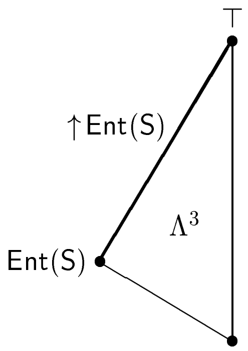
The states $\Psi \in \mathcal{H}^3 \otimes \mathcal{H}^3$ for which we have $\mathsf{Ent}(\mathsf{S}) \sqsubseteq \mathsf{Ent}(\Psi)$ are those which are such that

$$\mathsf{Ent}(\Psi) = r\,(q, 1-q, 0)$$

for $0 \leq q \leq 1/2$, that is, convex combinations of S and the minimally entangled state in $\mathcal{H}^3 \otimes \mathcal{H}^3$ (the pure tensor $|00\rangle$), for which we set

$$\top := \mathsf{Ent}(|00\rangle)\,.$$

Graphically,



where $\uparrow\mathsf{Ent}(\mathsf{S})$ is the upper set of $\mathsf{Ent}(\mathsf{S})$ in Λ^3.

We can now refine our qualitative representation of entanglement for bipartite states using the order on quantum states.

Example 5.10 *Qualitative entanglement of bipartite quantum systems.* In Example 5.9, the quantitative valuation $\mu \cdot \mathsf{Ent}$ with μ Shannon entropy, that is, the usual valuation attributed to a bipartite quantum system in order to measure entanglement, can equivalently be defined as the von Neumann entropy of one of the quantum states $\rho_1(\Psi)$ or $\rho_2(\Psi)$ for $\Psi \in \mathcal{H}^n \otimes \mathcal{H}^n$ that arise by tracing over the other system.

Explicitly, for $\Psi = \sum_i c_i \psi_i \otimes \phi_i$ we obtain

$$\rho_1(\Psi) := \mathsf{tr}_2(\Psi) = \begin{pmatrix} c_1^2 & & 0 \\ & \ddots & \\ 0 & & c_n^2 \end{pmatrix} \text{ in } (\psi_i)$$

$$\rho_2(\Psi) := \mathsf{tr}_1(\Psi) = \begin{pmatrix} c_1^2 & & 0 \\ & \ddots & \\ 0 & & c_n^2 \end{pmatrix} \text{ in } (\phi_i)\,.$$

Since the diagonals coincide, von Neumann entropy coincides and in either case gives the same value.

This implies that we can refine the valuation of entanglement Ent in Example 5.9 as

$$\mathsf{Ent}^\Omega : \mathcal{H}^n \otimes \mathcal{H}^n \to \Omega^n \times \Omega^n : \Psi \mapsto (\rho_1(\Psi), \rho_2(\Psi))$$

where $\Omega^n \times \Omega^n$ is ordered pointwisely, that is,

$$(r_1, r_2) \sqsubseteq (s_1, s_2) \ \Leftrightarrow\ r_1 \sqsubseteq r_2 \text{ and } s_1 \sqsubseteq s_2\,.$$

On $\Omega^n \times \Omega^n$ we can then define as a measure of content

$$\mu_{1,2} : \Omega^n \times \Omega^n \to [0,1]^* : (r_1, r_2) \mapsto \frac{\mu(r_1) + \mu(r_2)}{2}$$

where μ is von Neumann entropy. This results in a quantitative measure of entanglement on $\mathcal{H}^n \otimes \mathcal{H}^n$ that exactly coincides with the usual one. Indeed,

$$\mu_{1,2}\left(\mathsf{Ent}^\Omega(\Psi)\right) = \frac{\mu_1(\rho_1(\Psi)) + \mu_2(\rho_2(\Psi))}{2} = \mu(\rho_1(\Psi)) = \mu(\rho_2(\Psi))\,.$$

Note here in particular that Ent^Ω "almost" turns the states in $\mathcal{H}^n \otimes \mathcal{H}^n$ into a domain by setting

$$\Psi \sqsubseteq \Phi \ \Leftrightarrow\ \mathsf{Ent}^\Omega(\Psi) \sqsubseteq \mathsf{Ent}^\Omega(\Phi)\,.$$

We obtain a preorder that has pure tensors as maximal elements and that has $\perp$ as a minimum.

We however lose some anti-symmetry in this passage. In particular, when considering the Schmidt base, the order loses track of relative phases between base vectors. Indeed,

$$\mathsf{Ent}^\Omega\left(\psi_1 \otimes \phi_1 + \psi_2 \otimes \phi_2\right) = \mathsf{Ent}^\Omega\left(\psi_1 \otimes \phi_1 + i\psi_2 \otimes \phi_2\right)$$

although

$$\mathsf{ray}\left(\psi_1 \otimes \phi_1 + \psi_2 \otimes \phi_2\right) \neq \mathsf{ray}\left(\psi_1 \otimes \phi_1 + i\psi_2 \otimes \phi_2\right)$$

so these vectors do not encode the same state.

However, this can be fixed by taking into account their phases in defining the order. We will provide the details in a future paper.

Pure tensors avoid this since

$$\psi \otimes (i\phi) = (i\psi) \otimes \phi = i(\psi \otimes \phi)$$

for which we have

$$\mathsf{ray}\,(i\psi \otimes \phi) = \mathsf{ray}\,(\psi \otimes \phi)\ .$$

The maximally entangled states do not depend on the bases at all.

The essential difference between the qualitative valuations Ent^Ω and Ent is the fact that Ent^Ω takes into account the identity of pure tensors above.

Example 5.11 *Qualitative entanglement of multipartite quantum systems.* In Example 5.9 we measured entanglement of bipartite quantum systems using unicity of the coefficients in the Schmidt biorthogonal decomposition. There however does not exist a similar construction for arbitrary multipartite sytems, that is, there is no Schmidt-type decomposition theorem for arbitrary $\mathcal{H}^n \otimes \ldots \otimes \mathcal{H}^n$.

In particular, up to now there was not even a satisfactory notion of maximal entanglement e.g. see [10]. Indeed, when considering three partite qubit states, for the Greenberger-Horn-Zeilinger state [6]

$$\mathsf{GHZ} := \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

and the W-state

$$\mathsf{W} := \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$$

there are conflicting arguments about which one is maximally entangled. The general favourite is however GHZ in particular in view of its maximal violation of certain type of inequalities (e.g. Bell's) that are characteristic for entanglement.

The solution of this conflict lies in specification of a context with respect to which one measures entanglement, in the sense of Example 5.10.

We propose here a qualitative measure for multipartite entanglement that favours GHZ as the maximally entangled state, allong the lines of the valuation in Example 5.10 for bipartite entanglement.

Define

$$\mathsf{Ent}^\Omega : \mathcal{H}^n \otimes \ldots \otimes \mathcal{H}^n \to \Omega^n \times \ldots \times \Omega^n : \Psi \mapsto \Big(\rho_1(\Psi), \ldots, \rho_m(\Psi)\Big)$$

where $\rho_i(\Psi)$ arises by tracing over all systems except the ith. We can do this for example by considering the Schmidt decomposition for $\mathcal{H}^n \otimes (\mathcal{H}^n \otimes \ldots \otimes \mathcal{H}^n)$ where the single Hilbert space encodes the ith system.

We then obtain for the above examples that

$$\mathsf{Ent}^\Omega(\mathsf{GHZ}) = \left(\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}\right)$$

since we have

$$\mathsf{GHZ} = \frac{1}{\sqrt{2}}(|0\rangle|00\rangle + |1\rangle|11\rangle)$$

with respect to the 1st component and

$$\mathsf{Ent}^{\Omega}(\mathsf{W}) = \left(\begin{pmatrix} 2/3 & 0 \\ 0 & 1/3 \end{pmatrix}, \begin{pmatrix} 2/3 & 0 \\ 0 & 1/3 \end{pmatrix}, \begin{pmatrix} 2/3 & 0 \\ 0 & 1/3 \end{pmatrix}\right)$$

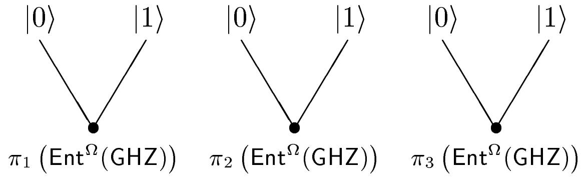
since for example

$$\mathsf{W} := \frac{\sqrt{2}}{\sqrt{3}}|0\rangle \left(\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)\right) + \frac{1}{\sqrt{3}}|1\rangle|00\rangle$$

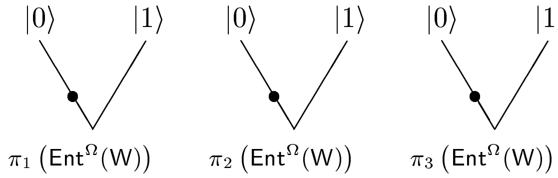
and as such it follows that

$$\mathsf{Ent}^{\Omega}(\mathsf{GHZ}) \sqsubset \mathsf{Ent}^{\Omega}(\mathsf{W})\,.$$

Depicting only the part of Ω^2 containing the relevant pure states $|0\rangle$ and $|1\rangle$ here, that is, a copy of Δ^2, this represents graphically as $\mathsf{Ent}^{\Omega}(\mathsf{GHZ})$



versus $\mathsf{Ent}^{\Omega}(\mathsf{W})$



where the maps π_1, π_2 and π_3 represent the components of Ent^{Ω}.

We can define a quantitative measure of entanglement on $\mathcal{H}^n \otimes \ldots \otimes \mathcal{H}^n$ via composition of Ent^{Ω} and

$$\mu_{1,\ldots,m} : \Omega^n \times \ldots \times \Omega^n \to [0,1]^* : (r_1, \ldots, r_n) \mapsto \frac{1}{m}\sum_i \mu(r_i)$$

where μ is again von Neumann entropy. We obtain as such the desired values on pure tensors and the maximally entangled state. In particular do we obtain

$$\mu_{1,2,3}\left(\mathsf{Ent}^{\Omega}(\mathsf{GHZ})\right) = 1\,.$$

When one prefers to abstract over the identity of the pure tensors above, it is clear that all the above still holds by substituting Λ^n for Ω^n, that is, Ent for Ent^{Ω}.

6 Acknowledgement

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