Programming Research Group

B-SIDES

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Abstract

a. not concrete; theoretical. n. **summary**.

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This is the way, step inside.
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– Joy Division, from "Atrocity Exhibition."

$\rightarrow$

Technical mastery is a virtuoso on piano who can play any piece of music set before them note perfect; Beauty is Chopin. Technical mastery is definition, theorem, proof, solution; Beauty is mathematics. Technical mastery is learned by repetition; Beauty is imagination. The planet is too happy about technical mastery and too pissed off about beauty.

One reason is that nearly all of the planet is ruled by inhibition: Either people are afraid to express the truest that they know and feel, or they are afraid to act according to it. Too often this fear corrupts the soul, leaving in its wake the horror that is the reasonable, sensible and consistent "human." And therein lies the problem: The planet is too quick to turn its back on *irrationality,* one of the primary causes of beautiful things.

It seems natural that a scientist, perhaps more than most, would be well-versed in these simple matters. For the record, those who earn their living conducting the business of science are fortunate indeed; let us hope that they always remember what that business is: The pursuit, exploration and proliferation of beauty.

This is a collection of ideas from October 2000 to January 2001. I do not claim that they are beautiful, interesting, or for that matter, even worth reading. They are simply things I wanted to write, and questions I wanted to ask.

1 The continuity of the partial recursive functions

"Computability implies continuity." There are definitely settings in which this is true; there are other settings in which it is not true. But for a small number of extremists, it is not enough that they are able to work in a setting where the beautiful idea embodied by this phrase holds: They also *insist* that we *follow* them there. That to me is as good a definition as any of 'fanatic.' (Or is that just "annoying?")

The renee equation $\varphi = \delta + \varphi \circ r$ is a model of iteration [8]. Here are some basic things we understand about it so far:

(i) The iteration $\bigsqcup f^n(\perp)$ of a Scott continuous map f on a domain implicitly gives rise to a renee equation.

(ii) By using the renee equation as a model of iteration, various partial orders on the set $\mathbb{N} \cup \{\perp\}$ give rise to different classes of computable functions, the partial recursive functions among them.

One of the curious things about (ii) is that while the theory behind $\varphi = \delta + \varphi \circ r$ crucially relies on topological aspects of domains, computable functions are usually not "continuous."

The question we consider here is: Is there a partial order $\leq$ on $\mathbb{N} \cup \{\perp\}$ such that (a) The computable functions can be captured using the renee equation to model iteration, and (b) That the computable functions are continuous with respect to $\leq$? We prove that the answer to this question is no. Thus, computability, even at the most basic level, *can* be understood order theoretically, despite the *impossibility* of "Computability implies continuity."

To be entirely thorough, we first answer the following question: Are there any *non-trivial* topologies at all on the *set* $\mathbb{N} \cup \{\perp\}$ with respect to which the partial recursive functions are continuous? The answer is yes, there are exactly two of them, and lucky for us, each arises as Scott continuity with respect to a certain partial order on $\mathbb{N} \cup \{\perp\}$. In the "first order view" of recursion supported by the renee equation, each of these domains yield an amount of computational power equivalent to a prescribed set of initial functions closed under composition with *no iteration.*

This proves that in the setting where the partial recursive functions naturally reside, the set of natural numbers, that it is not possible to give an order theoretic formulation of iteration based on the renee equation which makes all computable functions continuous. Nevertheless, by (ii), *it is possible* to give an order theoretic formulation of computability. We just can't have continuity. Notice also that, by (i), the standard view in semantics is *not excluded* by the renee equation; it is subsumed by it.

The result is not surprising, and to be honest, it feels silly to be saying all this, but: "Computability implies continuity" is a symptom of a particular mathematical formulation of computability (i.e., the "second order" view of recursion.) It is not, mathematically speaking, a universal truth applicable in all settings. Nevertheless, the characterization of the two topologies mentioned above is definitely a lot of fun. Plus, a new issue emerges, directly relevant to the standard view of recursion in semantics: The potential to *prove* that the phrase "Computability implies continuity" *necessitates* the use of domains in their usual sense, i.e., the extension order on partial functions. We comment on this at the end.

The partial recursive functions

Definition 1.1 Let $\mathbb{N}_\perp$ denote the set $\mathbb{N} \cup \{\perp\}$, where $\perp$ is an element that does not belong to $\mathbb{N}$.

For instance, $\perp = \{\mathbb{N}\}$ is such an element.

Definition 1.2 A *partial function* on the naturals is a function

$$f : \mathbb{N}^n \to \mathbb{N}_\perp,$$

where $n \geq 1$. We say that f is *undefined* at x exactly when $f(x) = \perp$.

Thinking of f as an algorithm, $f(x) = \perp$ intuitively means that the program f crashed when we sent it input x.

Definition 1.3 The *composition* of a partial map $f : \mathbb{N}^n \to \mathbb{N}_\perp$ with partial mappings $g_i : \mathbb{N}^k \to \mathbb{N}_\perp$, $1 \leq i \leq n$, is the partial map

$$f(g_1, \cdots, g_n) : \mathbb{N}^k \to \mathbb{N}_\perp$$

$$f(g_1, \cdots, g_n)(x) = \begin{cases} f(g_1(x), \cdots, g_n(x)) & \text{if } (\forall i)\ g_i(x) \neq \perp; \\ \perp & \text{otherwise.} \end{cases}$$

That is, if in the process of trying to run the program f, the computation of one of its inputs fails, then the entire computation fails.

Definition 1.4 A partial map $f : \mathbb{N}^{n+1} \to \mathbb{N}_\perp$ is defined by *primitive recursion* from $g : \mathbb{N}^n \to \mathbb{N}_\perp$ and $h : \mathbb{N}^{n+2} \to \mathbb{N}_\perp$ if

$$f(\bar{x}, y) = \begin{cases} g(\bar{x}) & \text{if } y = 0; \\ h(\bar{x}, y-1, f(\bar{x}, y-1)) & \text{otherwise.} \end{cases}$$

where we have written $\bar{x} \in \mathbb{N}^n$.

This computationally is a counting loop. Notice that the rule of composition given above insists that in order for $f(\bar{x}, y)$ to be defined, f must be defined at $(\bar{x}, z)$, for all $z < y$.

Definition 1.5 The *minimization* of a partial function $f : \mathbb{N}^{n+1} \to \mathbb{N}_\perp$ is the partial function $\mu f : \mathbb{N}^n \to \mathbb{N}_\perp$ given by

$$\mu f(x) = \min\{y \in \mathbb{N} : (\forall z < y)\ f(x, z) \neq \perp \ \&\ f(x, y) = 0\}$$

with the convention that $\mu f(x) = \perp$ if no such y exists.

The intent of minimization is that the least y such that $f(x,y) = 0$ is calculable according to the natural algorithm: We begin iterating at 0, if $f(x,0) = 0$, this is the answer. If $f(x,0) = \perp$, the program has crashed, so the output is $\perp$. Otherwise, we proceed to $f(x,1)$, and so on, in the same fashion. Hence the requirement that $f(x,z) \neq \perp$, for all $z < y$: Without it, we have no assurance that y was found using the algorithm above.

Definition 1.6 The class of *partial recursive functions* on the naturals is the smallest collection of partial maps $f : \mathbb{N}^n \to \mathbb{N}_\perp$ which contains the zero function, the successor, the projections, and is closed under composition, primitive recursion, and minimization.

Topology

A topology τ on a set X is a subset $\tau \subseteq \mathcal{P}(X)$ closed under finite intersections and arbitrary unions. Every topology τ on a set X lies between a *smallest* topology $\{\emptyset, X\}$ and a largest topology $\mathcal{P}(X) = \{A : A \subseteq X\}$. A topology τ is *nontrivial* when it is neither of these two.

Let D denote the set $\mathbb{N}_\perp$. A partial recursive function $f : \mathbb{N}^n \to \mathbb{N}_\perp$ is regarded a map of type $D^n \to D$ by sending the *undefined* elements $D^n \setminus \mathbb{N}^n$ of D^n to $\perp$.

Definition 1.7 A function $f : D^n \to D$ is *partial recursive* if $f|_{\mathbb{N}^n} : \mathbb{N}^n \to \mathbb{N}_\perp$ is partial recursive and $f(D^n \setminus \mathbb{N}^n) = \{\perp\}$.

There is a natural one-to-one correspondence between partial recursive maps of type $\mathbb{N}^n \to \mathbb{N}_\perp$ and those of type $D^n \to D$. The question is then: Are there any nontrivial topologies on D such that all partial recursive functions $f : D^n \to D$ are continuous?

The inapplicability of classical topology

Let σ be a *nontrivial* topology on $\mathbb{N}_\perp$ that makes all the partial recursive functions continuous. A *nontrivial* open set $X \in \sigma$ is one for which we have $X \neq \emptyset$ and $X \neq \mathbb{N}_\perp$.

Lemma 1.8 *The following are equivalent:*

(i) *There is a nontrivial open set $X \in \sigma$ with $\perp \in X$.*

(ii) *The set $\{\perp\}$ is open in σ.*

(iii) *For every $X \subseteq \mathbb{N}$, the set $X \cup \{\perp\}$ is open in σ.*

Proof. (i) $\Rightarrow$ (ii) Let $X \in \sigma$ with $\perp \in X$. Then because X is not all of $\mathbb{N}_\perp$, there is some $n \in \mathbb{N}$ with $n \notin X$. The constant map $c : \mathbb{N}_\perp \to \mathbb{N}_\perp$, $c(x) = n$ if $x \in \mathbb{N}$, is continuous with respect to σ. Thus, $c^{-1}(X) = \{\perp\}$ is open in σ.

(ii) $\Rightarrow$ (iii) Let $n \in \mathbb{N}$. Then the function $f : \mathbb{N}_\perp \to \mathbb{N}_\perp$ given by

$$f(x) = \begin{cases} \perp & \text{if } x \in \{n, \perp\}; \\ x & \text{otherwise}; \end{cases}$$

is partial recursive and thus continuous with respect to σ. Since $\{\perp\}$ is open, $f^{-1}(\{\perp\}) = \{n, \perp\}$ is open. By closure under arbitrary unions, $X \cup \{\perp\}$ is open in σ for any $X \subseteq \mathbb{N}$.

(iii) $\Rightarrow$ (i) Let $X = \emptyset$. $\square$

Lemma 1.9 *The following are equivalent:*

(i) *There is a nontrivial open set $X \in \sigma$ with $\perp \notin X$.*

(ii) *The set $\{\perp\}$ is closed in σ.*

(iii) *For every $X \subseteq \mathbb{N}$, the set X is open in σ.*

Proof. (i) $\Rightarrow$ (ii) Let $X \in \sigma$ with $\perp \notin X$. Then because $X \neq \emptyset$, it must contain some $n \in \mathbb{N}$. The constant map $c : \mathbb{N}_\perp \to \mathbb{N}_\perp$, $c(x) = n$ if $x \in \mathbb{N}$, is continuous with respect to σ. Thus, $c^{-1}(X) = \mathbb{N}$ is open in σ, which means that $\{\perp\}$ is closed.

(ii) $\Rightarrow$ (iii) Let $n \in \mathbb{N}$. Then the function $f : \mathbb{N}_\perp \to \mathbb{N}_\perp$ given by

$$f(x) = \begin{cases} n & \text{if } x = n; \\ \perp & \text{otherwise}; \end{cases}$$

is partial recursive and hence continuous with respect to σ. Since $\{\perp\}$ is closed, $\mathbb{N}$ is open, so $f^{-1}(\mathbb{N}) = \{n\}$ is open. By closure under arbitrary unions, X is open in σ for every $X \subseteq \mathbb{N}$.

(iii) $\Rightarrow$ (i) Let $X = \mathbb{N}$. Then X is a nontrivial open set with $\perp \notin \mathbb{N}$. $\square$

Theorem 1.10 *If every partial recursive is continuous with respect to a nontrivial topology on $\mathbb{N}_\perp$, then either $\{\perp\}$ is open or $\{\perp\}$ is closed, but not both.*

Proof. Let σ be a nontrivial topology on $\mathbb{N}_\perp$ that makes all partial recursive functions continuous. Let $X \in \sigma$ be a nontrivial open set. If $\perp \in X$, then $\{\perp\}$ is open, by Lemma 1.8(i). If $\perp \notin X$, then $\{\perp\}$ is closed, by Lemma 1.9(i). But more is true: Not only must $\{\perp\}$ be open or closed, it can never be both.

Indeed, suppose that $\{\perp\}$ is open and closed in σ. Let $X \subseteq \mathbb{N}_\perp$. If $\perp \in X$, then it is open by Lemma 1.8(iii), while if $\perp \notin X$, it must be open by Lemma 1.9(iii). Then $\sigma = \mathcal{P}(\mathbb{N}_\perp)$, standing in contrast to the nontriviality of σ. $\square$

Corollary 1.11 *The only T_1 topology on $\mathbb{N}_\perp$ that makes every partial recursive function continuous is the discrete topology $\mathcal{P}(\mathbb{N}_\perp)$.*

Proof. Let σ be a nontrivial T_1 topology on $\mathbb{N}^{\perp}$ that makes all partial recursive functions continuous. Since σ is T_1, every point $\{x\}$ of $\mathbb{N}_{\perp}$ is closed in σ.

In particular, the set $\{0\}$ is closed in σ. But the constant map zero : $\mathbb{N}_{\perp} \to \mathbb{N}_{\perp}$ is partial recursive, so by continuity, the set

$$\text{zero}^{-1}(\{0\}) = \mathbb{N}$$

is closed, forcing its complement $\{\perp\}$ open. Thus, the set $\{\perp\}$ is both open and closed. By Theorem 1.10, σ must be trivial. But of the two trivial topologies on $\mathbb{N}_{\perp}$, only one of these is T_1. Naturally, this is the discrete topology $\mathcal{P}(\mathbb{N}_{\perp})$. □

Then no classical topology will make the partial recursive functions continuous in a nontrivial way.

Topologies for partial recursive functions

We now prove that there are two candidates for a nontrivial topology on $\mathbb{N}_{\perp}$ that will make the partial recursives continuous.

Proposition 1.12 *Let σ be a nontrivial topology on $\mathbb{N}_{\perp}$ that makes all partial recursive functions continuous. If $\{\perp\}$ is open in σ, then*

$$\sigma = \{\emptyset\} \cup \{X \cup \{\perp\} : X \subseteq \mathbb{N}\}.$$

If $\{\perp\}$ is closed in σ, then

$$\sigma = \{\mathbb{N}_{\perp}\} \cup \{X : X \subseteq \mathbb{N}\}.$$

Proof. If $\{\perp\}$ is open in σ, then the inclusion $\{\emptyset\} \cup \{X \cup \{\perp\} : X \subseteq \mathbb{N}\} \subseteq \sigma$ is immediate by Lemma 1.8(iii). For the other, let $X \in \sigma$ be a nontrivial open set. If $\perp \notin X$, then by Lemma 1.9(ii), the set $\{\perp\}$ is closed. But this contradicts Theorem 1.10. Then $\perp \in X$.

If $\{\perp\}$ is closed in σ, then the inclusion $\{\mathbb{N}_{\perp}\} \cup \{X : X \subseteq \mathbb{N}\} \subseteq \sigma$ is immediate by Lemma 1.9(iii). For the other, let $X \in \sigma$ be a nontrivial open set. If $\perp \in X$, then by Lemma 1.8(ii), the set $\{\perp\}$ is open. But this contradicts Theorem 1.10. Then $\perp \notin X$. □

We name these two as follows.

Definition 1.13 The topology σ_+ is

$$\sigma_+ := \{\emptyset\} \cup \{X \cup \{\perp\} : X \subseteq \mathbb{N}\}$$

and the topology σ_- is

$$\sigma_- := \{\mathbb{N}_{\perp}\} \cup \{X : X \subseteq \mathbb{N}\}.$$

Corollary 1.14 *If the partial recursives are continuous with respect to a nontrivial topology σ on $\mathbb{N}_\perp$, then either $\sigma = \sigma_+$ or $\sigma = \sigma_-$.*

Proof. By Theorem 1.10, either $\{\perp\}$ is open or $\{\perp\}$ is closed. By Prop. 1.12, either $\sigma = \sigma_+$ or $\sigma = \sigma_-$. $\square$

But what are these topologies? And do they make the partial recursives continuous?

The necessity of domains

Example 1.15 The ω-algebraic domain $\mathbb{N}_\perp$. The elements of the set $\mathbb{N} \cup \{\perp\}$ are ordered according to

$$x \sqsubseteq y \Leftrightarrow x = y \text{ or } x = \perp.$$

We call $\mathbb{N}_\perp$ the set of flat naturals with least element $\perp$.

Example 1.16 The ω-algebraic domain $\mathbb{N}^\perp$. The elements of the set $\mathbb{N} \cup \{\perp\}$ are ordered according to

$$x \sqsubseteq y \Leftrightarrow x = y \text{ or } y = \perp.$$

Thus, $\mathbb{N}^\perp$ is the set of flat naturals with $\perp$ as a greatest element.

Theorem 1.17 *There are exactly two nontrivial topologies with respect to which all partial recursive functions are continuous. The first is the Scott topology σ_- on the domain $\mathbb{N}_\perp$; the second is the Scott topology σ_+ on the domain $\mathbb{N}^\perp$.*

Proof. Let D be the domain $\mathbb{N}_\perp$. Any $f : D^n \to D$ with $f(D^n \setminus \mathbb{N}^n) = \{\perp\}$ is monotone. Because all elements of D^n are compact, for all $n \geq 1$, such an f is Scott continuous. In particular, each partial recursive $f : D^n \to D$ is Scott continuous. Since $\{\perp\} = \downarrow\{\perp\}$ is Scott closed in D, Prop. 1.12 implies that the Scott topology on $\mathbb{N}_\perp$ is σ_-.

A similar argument applied to the domain $D = \mathbb{N}^\perp$ shows that each partial recursive $f : D^n \to D$ is Scott continuous, and since $\uparrow\{\perp\} = \{\perp\}$ is Scott open in D, Prop. 1.12 implies that the Scott topology on $\mathbb{N}^\perp$ is σ_+. $\square$

Corollary 1.18 *There are four topologies on the set $\mathbb{N}_\perp$ which make the partial recursive functions continuous: $\{\emptyset, \mathbb{N}_\perp\}$, σ_-, σ_+ and $\mathcal{P}(\mathbb{N}_\perp)$.*

In addition, the two topologies σ_- and σ_+ can be characterized as follows.

Corollary 1.19 *The unique nontrivial topology on $\mathbb{N}_\perp$ which makes all partial recursive functions continuous and open is σ_+.*

Corollary 1.20 *The unique nontrivial topology on $\mathbb{N}_\perp$ which makes all partial recursive functions continuous and closed is σ_-.*

Finally, a partial recursive map is *total* iff it is a closed mapping with respect to σ_+ iff it is an open mapping with respect to σ_-.

Problem 1.21 Extend the results about $\mathbb{N}_\perp$ to function spaces to *prove* that the extension order on $[\mathbb{N} \to \mathbb{N}_\perp]$ is inevitable in recursion theory if one wants the phrase "Computability implies continuity." Perhaps it can be argued that any topology on $[\mathbb{N} \to \mathbb{N}_\perp] = (\mathbb{N}_\perp)^{\mathbb{N}}$ which makes the recursive functionals continuous must arise as the product topology on $(\mathbb{N}_\perp)^{\mathbb{N}}$, where the topology on the set $\mathbb{N}_\perp$ is either σ^+ or σ_-. Perhaps a more subtle path is needed.

2 A method for zero finding faster than bisection without differentiability

One of the great problems in the history of mathematics is the zero finding problem: Given a real valued function f on an interval $[a, b]$, find a *zero* of f, that is, a number x such that $f(x) = 0$. Evariste Galois proved that one must resort to algorithms in solving this problem by showing that polynomials of degree five and higher have no solution by radicals, i.e., their zeroes are not in general expressible by a formula.

If one assumes nothing about f except continuity, then there are many senses in which *the bisection method* is the optimal algorithm for zero finding ([2][7]). However, for a class of Lipschitz mappings [3], the bisection method is no longer optimal. But Lipschitz mappings have derivatives almost everywhere [13]. In addition, the optimal algorithm makes use of the Lipschitz constant [3], which is a bound on its derivative. Another case in which bisection is not optimal is the class of convex mappings [6]. But there again, one finds that convex mappings are differentiable everywhere except on a countable set [13]. These two examples raise the following question: If we have a nontrivial class $\mathcal{C}$ of functions and a zero finding algorithm which is better than the bisection for the members of $\mathcal{C}$, must the functions in $\mathcal{C}$ possess some amount of differentiability? In short, is differentiability in some form necessary in order to beat the bisection method?

We prove that the answer to this question is no. For the class of *Hölder continuous* mappings, which contains all of the well-known examples of nowhere differentiable functions, including those arising in the analysis of Brownian motion and fractals [5], we use domain theory and measurement to design and analyze a new algorithm for zero finding which is better than the bisection method at every iteration.

Example 2.1 The interval domain

$$\mathbf{I}\mathbb{R} := \{[a, b] : a, b \in \mathbb{R} \ \& \ a \leq b\}$$

ordered under reverse inclusion

$$[a, b] \sqsubseteq [c, d] \Leftrightarrow [c, d] \subseteq [a, b]$$

is a continuous dcpo with measurement $\mu : \mathbf{I}\mathbb{R} \to [0, \infty)^* :: [a, b] \mapsto b - a$.

Now that we have objects to compute with and a quantitative notion of approximation, what we need is a notion of process.

Definition 2.2 A *splitting* $s : D \to D$ on a poset D is a map such that $x \sqsubseteq s(x)$ for all $x \in D$. Write $\mathrm{fix}(s) = \{x \in D : s(x) = x\}$ for the *set* of fixed points of s.

There are many fixed point theorems about splittings [8]. The following one will be used later on to establish the correctness of the new zero finding algorithm.

Proposition 2.3 *Let D be a domain with a measurement $\mu \to \sigma_D$ and $I \subseteq D$ be a set closed under suprema of increasing sequences. If $s : I \to I$ is a splitting such that $\mu s \leq c \cdot \mu$, for a constant $0 \leq c < 1$, then*

$$\bigsqcup_{n \geq 0} s^n(x) \in \mathrm{fix}(s) \subseteq \max(D),$$

for all $x \in I$.

Proof. Because s is a splitting and maps I to itself, the sequence $(s^n(x))$ is increasing and belongs to I. Then $\bigsqcup s^n(x) \in I$. But

$$\bigsqcup s^n(x) \sqsubseteq s(\bigsqcup s^n(x))$$

and

$$\mu(\bigsqcup s^n(x)) = \lim_{n \to \infty} \mu s^n(x) \leq \lim_{n \geq \infty} c^n \cdot \mu x = 0,$$

which means $\bigsqcup s^n(x) \in I \cap \max(D)$ and thus a fixed point of s. $\square$

Numerical methods as splittings on domains

We now consider how the theory of the last section applies to zero finding.

Definition 2.4 We use the following operators on $\mathbf{I}\mathbb{R}$:

- $l : \mathbf{I}\mathbb{R} \to \mathbb{R} :: [a, b] \mapsto a$
- $m : \mathbf{I}\mathbb{R} \to \mathbb{R} :: [a, b] \mapsto (a + b)/2$
- $r : \mathbf{I}\mathbb{R} \to \mathbb{R} :: [a, b] \mapsto b$

These are abbreviated $l_x := l(x)$, $r_x := r(x)$ and $m_x := m(x)$.

Example 2.5 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map on the real line and

$$C(f) = \{[a,b] : f(a) \cdot f(b) \leq 0\}.$$

The map $\mathrm{split}_f : C(f) \to C(f)$ given by

$$\mathrm{split}_f(x) = \begin{cases} [l_x, m_x] & \text{if } [l_x, m_x] \in C(f); \\ [m_x, r_x] & \text{otherwise;} \end{cases}$$

is a splitting with

$$\mu\, \mathrm{split}_f(x) \leq \frac{1}{2} \cdot \mu x.$$

By Prop. 2.3,

$$\bigsqcup_{n \geq 0} \mathrm{split}_f^n(x) \in \mathrm{fix}(\mathrm{split}_f),$$

for all $x \in C(f)$. But $\mathrm{fix}(\mathrm{split}_f) = \{[r] : f(r) = 0\}$, which means that iterating split_f is a scheme for calculating a zero of f. This numerical technique is called *the bisection method.*

Well, almost. Let's take a closer look at things. In the zero finding problem, the desired result is a *number* that approximates the zero r, not an interval. In practice, we calculate a small enough interval x, and then choose a point like m_x as the approximation to r. The true reason that split_f is an algorithm for computing r is that if we begin with any $x \in C(f)$, and then choose any sequence $x_n \in \mathrm{split}_f^n(x)$, we *always* have

$$|x_n - r| \leq \mu\, \mathrm{split}_f^n(x) \leq \frac{\mu x}{2^n},$$

and hence $x_n \to r$ in the *usual topology* on the real line.

Then what we need to know is that computation on a domain actually corresponds to computation in reality. For the splittings of Prop. 2.3, the following result confirms exactly this.

Proposition 2.6 *Let D be a domain with a measurement $\mu \to \sigma_D$, $I \subseteq D$ a set closed under suprema of increasing sequences and $s : I \to I$ a splitting with $\mu s \leq c \cdot \mu$ for a constant $0 \leq c < 1$. Then for all $x \in I$, if $x_n \in \uparrow s^n(x) \cap \ker \mu$, we have*

$$x_n \to \bigsqcup_{n \geq 0} s^n(x) \in \mathrm{fix}(s) \subseteq \ker \mu,$$

in the relative Scott topology on $\ker \mu$.

Proof. Let $a \ll \bigsqcup s^n(x)$. Then $\Uparrow a$ is Scott open and so there is an integer k such that

$$n \geq k \Rightarrow s^n(x) \in \Uparrow a \Rightarrow x_n \in \Uparrow a \Rightarrow x_n \in \Uparrow a \cap \ker \mu,$$

which finishes the proof. $\square$

For instance, in the case of the interval domain $\mathbf{I}\mathbb{R}$, we have

$$\ker \mu = \max(\mathbf{I}\mathbb{R}) = \{[x] : x \in \mathbb{R}\} \simeq \mathbb{R}$$

where the homemorphism is between the relative Scott topology on $\ker \mu$ and the usual topology on the real line. Thus, to say that $\bigsqcup_{n\geq 0} \mathrm{split}_f^n(x)$ computes a zero of f means exactly the same thing as it does in numerical analysis.

Other examples of numerical methods which can be modeled as a splitting on a domain include the golden section search, the regula-falsi method, the secant method and any contraction on a complete metric space. For more on this we refer the reader to the seventh chapter of [8] and to [11].

A method for zero finding without derivatives

We will design the method for Hölder continuous functions which have a simple zero on a compact interval $[a, b]$.

Definition 2.7 A map $f : [a, b] \to \mathbb{R}$ is *Hölder continuous* if there are positive constants $c > 0$ and $\alpha > 0$ such that

$$|f(x) - f(y)| \leq c \cdot |x - y|^\alpha$$

for all $x, y \in [a, b]$.

Definition 2.8 A function $f : [a, b] \to \mathbb{R}$ has a *simple zero* $r \in [a, b]$ if

$$\operatorname{sgn} f(x) = \operatorname{sgn}(x - r)$$

for $x \in [a, b]$, where $\operatorname{sgn}(x) = x/|x|$ for $x \neq 0$, and $\operatorname{sgn}(0) = 0$. Write

$$\square f := \{x \in \mathbf{I}\mathbb{R} : [a, b] \sqsubseteq x \sqsubseteq [r]\}$$

for the set of intervals where f changes sign.

Then a function has a simple zero r if it is positive to the right of r and negative to the left of r. For instance, if $f : [a, b] \to \mathbb{R}$ has a simple zero r on $[a, b]$, then the bisection $\mathrm{split}_f : \square f \to \square f$ can be written compactly as

$$\mathrm{split}_f(x) = \begin{cases} [l_x, m_x] & \text{if } f(m_x) > 0; \\ [m_x, r_x] & \text{otherwise.} \end{cases}$$

This will help us understand its relation to the following.

Theorem 2.9 *Let $f : [a,b] \to \mathbb{R}$ be a Hölder continuous map with simple zero r. Then iterating the splitting $s_f : \Box f \to \Box f$ given by*

$$s_f(x) = \begin{cases} \left[l_x, m_x - (f(m_x)/c)^{1/\alpha}\right] & \text{if } f(m_x) > 0; \\ \\ \left[m_x + (|f(m_x)|/c)^{1/\alpha}, r_x\right] & \text{otherwise;} \end{cases}$$

is an algorithm for computing r. That is,

$$\bigsqcup_{n \geq 0} s_f^n(x) = [r],$$

for all $x \in \Box f$. Thus, for all $x \in \Box f$, if $x_n \in s_f^n(x)$ for each n, then $x_n \to r$.

Proof. The set $\Box f = \uparrow[a,b] \cap \downarrow[r]$ is closed under suprema of increasing sequences as the intersection of two such sets. We now show that s_f is a splitting on $\Box f$. Let $x \in \Box f$. Then $l_x \leq r \leq r_x$. We consider two cases separately.

If $f(m_x) > 0$, then $r < m_x$, as r is a simple zero of f. By Hölder continuity,

$$f(m_x) = |f(m_x) - f(r)| \leq c \cdot |m_x - r|^\alpha = c \cdot (m_x - r)^\alpha,$$

from which we easily obtain

$$l_x \leq r \leq m_x - (f(m_x)/c)^{1/\alpha},$$

which gives $x \sqsubseteq s_f(x) \in \Box f$.

If $f(m_x) \leq 0$, then $m_x \leq r$, as r is a simple zero of f. By Hölder continuity,

$$|f(m_x)| = |f(m_x) - f(r)| \leq c \cdot |m_x - r|^\alpha = c \cdot (r - m_x)^\alpha,$$

from which we easily obtain

$$m_x + (|f(m_x)|/c)^{1/\alpha} \leq r \leq r_x,$$

which gives $x \sqsubseteq s_f(x) \in \Box f$.

Then s_f is a splitting on $\Box f$ with $\mu s_f \leq \mu/2$. The result now follows from Prop. 2.3 and Prop. 2.6. $\Box$

A comparison with the bisection

If s_1 and s_2 are two algorithms, then a natural intuition stemming from domain theory is to say that s_2 is a better algorithm than s_1 if

$$s_1 \sqsubseteq s_2 \equiv (\forall x)\, s_1(x) \sqsubseteq s_2(x).$$

But this isn't theory – the idea that this should be true for two numerical methods is almost difficult to fathom.

In the analysis of numerical techniques, one is not normally able to make bold statements like "Algorithm 1 is better than Algorithm 2 always and there is nothing more to be said." For instance, sometimes the bisection method is better than Newton's method, if the derivatives of a function are difficult (or impossible) to calculate, while an advantage of Newton's method is its quadratic convergence when close enough to the root.

But if we have two numerical methods s_1 and s_2 and we are able to write $s_1 \sqsubseteq s_2$, then assuming they both do the same amount of work at each iteration (function evaluations), we are in the nice situation of being able to say that algorithm s_2 is *always preferable* to algorithm s_1.

Proposition 2.10 *Let $f : [a, b] \to \mathbb{R}$ be a Hölder continuous map with a simple zero r. Then* $\mathrm{split}_f \sqsubseteq s_f$ *and for any $x \in \Box f$,*

$$\mu s_f(x) = \mu\,\mathrm{split}_f(x) - \left(\frac{|f(m_x)|}{c}\right)^{1/\alpha} \leq \mu\,\mathrm{split}_f(x),$$

with equality only in the unlikely event that $m_x = r$.

Proof. The first claim is obvious but its significance for numerical analysis is worth pointing out. For the second, first notice that

$$\mu s_f(x) = \begin{cases} \frac{\mu x}{2} - \left(\frac{f(m_x)}{c}\right)^{1/\alpha} & \text{if } f(m_x) > 0; \\ \\ \frac{\mu x}{2} - \left(\frac{|f(m_x)|}{c}\right)^{1/\alpha} & \text{otherwise,} \end{cases}$$

which is to say that

$$\mu s_f(x) = \frac{\mu x}{2} - \left(\frac{|f(m_x)|}{c}\right)^{1/\alpha} = \mu\,\mathrm{split}_f(x) - \left(\frac{|f(m_x)|}{c}\right)^{1/\alpha}.$$

Then if $\mu s_f(x) = \mu\,\mathrm{split}_f(x)$, we must have

$$\left(\frac{|f(m_x)|}{c}\right)^{1/\alpha} = 0 \Rightarrow f(m_x) = 0 \Rightarrow m_x = r,$$

as r is the only zero of f on $[a, b]$. $\Box$

For instance, if r is a computable irrational and we begin with an input x having rational endpoints, then s_f is a strict improvement over the bisection.

Corollary 2.11 *Let $f : [a, b] \to \mathbb{R}$ be a Hölder continuous map with a simple irrational zero r. Then for any $x \in \Box f$ with rational endpoints $l_x, r_x \in \mathbb{Q}$,*

$$\mu s_f^n(x) < \mu\,\mathrm{split}_f^n(x),$$

for all iterations $n \geq 1$.

Proof. If we have an input x with rational endpoints $l_x, r_x \in \mathbb{Q}$, then its midpoint m_x is rational and hence $m_x \neq r$. By Prop. 2.10,

$$\mu s_f(x) < \mu\,\mathrm{split}_f(x),$$

and so at iteration $n \geq 1$ we have

$$\mu s_f^n(x) \leq \frac{\mu s_f(x)}{2^{n-1}} < \frac{\mu\,\mathrm{split}_f(x)}{2^{n-1}} = \mu\,\mathrm{split}_f^n(x),$$

proving the claim. $\square$

And in general we can see the same is true anytime the input interval does not contain r as its midpoint: Once s_f gains an advantage over the bisection, it keeps this advantage forever. While the domain theoretic assertion $\mathrm{split}_f \sqsubseteq s_f$ is certainly a strong one for numerical methods, when taken on its own, it leaves something to be desired: How are we to know the inputs where they are equal? Even if we know that $\mathrm{split}_f \sqsubseteq s_f$ and $\mathrm{split}_f \neq s_f$, they may only differ on a single input, which doesn't say very much.

This provides a simple and clear example of the "extra something" that measurement adds to the standard order theoretic setting: split_f and s_f are equal iff their measures are iff one can magically choose an input whose midpoint is the zero (which is tantamount to guessing the answer). In short, we have a precise, quantitative account of their relation to one another.

Weierstrass's function

The function introduced in 1872 by Weierstrass,

$$f(x) = \sum_{n=0}^{\infty} a^n \cos\left(b^n \pi x\right),$$

is nowhere differentiable for $0 < a < 1$ and b an odd integer with $ab > 1 + 3\pi/2$. It is Hölder continuous [14] with $\alpha = \log(1/a)/\log b$.

Closing remarks

We have used domain theory and measurement as a language for expressing zero finding algorithms which renders the verification process systematic and *uniform:* It enables us to turn the question of correctness into one about fixed points for *all* zero finding methods, whereas this is only normally achieved in numerical analysis for one point schemes like Newton's method.

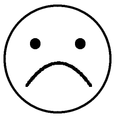
Finally, the method easily extends to the case where $|fx - fy| \leq c \cdot g(|x - y|)$, for a left invertible $g : [0, \infty) \to [0, \infty)$ satisfying $g(0) = 0$.

Problem 2.12 Prove or disprove the following **conjecture**: The method given here is *optimal* in the sense of the numerical accuracy minimax problem for the class of Hölder continuous functions with a simple zero.

Anonymity I: Gmaj7 Gm6 F$^\sharp$7 Bm A

> *Comments for the author: It is not true that all continuous functions between complete metric spaces preserve distance.*
> – from a referee report.

The chords in the title of this section are background music you can play while singing the referee's comment. This song is called "Anonymity," because only after hearing it does one truly develop a complete understanding of what it means to be anonymous. It could very well be the saddest song ever written. Can you hear it?



Damn, that's sad.

Dear Sir, I have a complaint.
– radiohead, from "Stop Whispering."

To elaborate slightly, we are *not* encouraging subversive action against referees. We do however strongly encourage people to respond to insensitivity with kindness. For instance, try giving the gift of music.

3 Techniques for constructing measurements on domains

We consider various techniques for constructing measurements on domains. One of these, in the discussion of recursive domain equations, ends up providing motivation for a domain of measurements. One has to be careful with this idea though, it has various interpretations, each of which make sense. Another, more abstract in spirit, is concerned with the existence of measurements on countably based domains. Here we get to briefly reconsider a problem that I will miss terribly when its gone: The G_δ lemma.

Products

Use max, + or any other expression that mimics a metric yielding the Euclidean topology.

Recursive domain equations

A theory about how to measure the canonical solution of a recursive domain equation just seems like a fun and valuable idea. For example, consider an innocent question like " How do we measure $[D \to E]$?" Suppose we have a countably infinite basis $\{b_n\}$ for D.

Then a good first step is remember that a continuous map is determined by the values it takes on a basis – but there are only countably many of these! We then define

$$\lambda : [D \to E] \to \prod_{i \in \mathbb{N}} E$$

$$\lambda(f) = (f(b_0), f(b_1), f(b_2), \cdots)$$

and – presto! – we have a continuous order reflecting map that must measure $[D \to E]$.

We can go further if we like, and if we do, good things happen: (1) What's $\prod E$, and (2) How do we measure *it*? Here, we intend the infinite product of domains to be modelled by $[\mathbb{N} \to E]$ in the pointwise order on functions, where $\mathbb{N}$ is ordered flatly. It is a domain iff E is an L-domain.

So how do we measure it? For this, we realize that $[\mathbb{N} \to E]$ is a solution of

$$X \simeq E \times X.$$

Suppose λ is a measurement on E. *If* we had a measurement μ on X, then since X is a product of domains, we expect that a measurement for it can be obtained by adding the measurements on its factors, which are E and X, respectively. Then μ should satisfy

$$\mu = \lambda \circ \pi_1 \varphi + \mu \circ \pi_2 \varphi,$$

where φ is an isomorphism. That is, the measurement μ is a fixed point of $f(\mu) = \lambda + \mu$. This implies that μ is an infinite sum of $\lambda's$, and the only way that can make sense is if $\lambda = 0$ which means E is discrete.

Next trick, if μ is a measurement on X, then so is $\mu/2$. Then we expect the canonical fixed point of

$$f(\mu) = \lambda + \frac{\mu}{2}$$

to yield a measurement. But this one can be solved! We get

$$\mu x = \sum_{n \geq 0} \frac{\lambda x}{2^n},$$

which leads to the following conjecture (cleverly disguised as a theorem):

Theorem 3.1 *Let E be an L-domain with $\perp$ and a map λ that measures X. Then $\mu : [\mathbb{N} \to E] \to [0, \infty)^*$ given by*

$$\mu x = \sum_{n \geq 0} \frac{\lambda x(n)}{2^n}$$

measures all f with $\mathrm{Im}(f) \subseteq X$.

Corollary 3.2 *The map* $\bar{\mu} : [D \to E] \to [0, \infty)^*$ *by*

$$\bar{\mu}(f) = \mu(f(b_0), f(b_1), f(b_2), \cdots)$$

measures all f *with* $\mathrm{Im}(f) \subseteq X$ *and* λ *measures* $X \subseteq E$.

Then here is what ought to happen: Each recursive domain equation, represented by functor F, should yield an operator f on the domain of measurements; the canonical fixed point of f is a measurement on the least fixed point of F. What does all this mean? That's the point of the next question.

Problem 3.3 Provide the theory that let's us write "fix(f) is a measurement on fix(F)."

***Note**: Dieter Spreen (Siegen) has shown how to measure limits of ω-chains of domains. In those notes, he also considers measuring the function space. The notes here were written before his, but I am happy crediting the result to him (I will *never* type this proof in; sometimes daydreaming is enough). I suspect also that his work on ω-chains should be useful in the derivation of f from a given recursive domain equation F.

A construction for semantic domains

In this section, let D be an ω-algebraic dcpo with an infinite basis of compact elements $\{b_n\}_{n\geq 0}$ and compact Lawson topology. These domains are often referred to as *coherent* in the literature.

Now what we want to do is construct a measurement on a given semantic domain. When working at the level of theory, a good first step is to aim for constructing a measurement into a domain like $(\mathcal{P}\omega, |\cdot|)$. Once we know the domain that will be used to measure D, the next step should *always* be to think about the maximal elements: They should have measure zero.

But since we are mapping into $\mathcal{P}\omega$, this means we want to think of them as having measure $\mathbb{N}$. Here's a simple idea that does the trick nicely.

Lemma 3.4 *For an element* $x \in D$,

$$x \in \max D \Leftrightarrow (\forall n \in \mathbb{N})\, b_n \sqsubseteq x \text{ or } x \in D \setminus \mathrm{Cl}_\sigma(\uparrow b_n)$$

We now use this characterization of maximal elements to design a measurement.

Lemma 3.5 *The map* $f : D \to \mathcal{P}\omega$ *given by*

$$f(x) = \{n \in \mathbb{N} : b_n \sqsubseteq x\}$$

is Scott continuous.

Lemma 3.6 *The map* $g : D \to \mathcal{P}\omega$ *given by*

$$g(x) = \{n \in \mathbb{N} : x \in D \setminus \mathrm{Cl}_\sigma(\uparrow b_n)\}$$

is Scott continuous.

Theorem 3.7 *The map* $\mu : D \to [0,1]^*$ *given by*

$$\mu(x) = |f(x) \cup g(x)|$$

is a Scott continuous map that measures D *with* $\ker \mu = \max D$.

Proof. Let $\lambda x = f(x) \cup g(x)$. The continuity of f and g implies that of λ. We are going to show that $\lambda \to \sigma_D$. Let U be a Scott open set around $x \in D$. Then there is a compact element $b_k \in U$ with $b_k \sqsubseteq x$. Then set $\varepsilon := \{k\}$. We claim

$$x \in \lambda_\varepsilon(x) \subseteq \uparrow b_k \subseteq U.$$

First, $\{k\} \ll \lambda(x)$ since $b_k \sqsubseteq x$. Thus, $x \in \lambda_\varepsilon(x)$.

Now let $a \in \lambda_\varepsilon(x)$. Then $a \sqsubseteq x$ and $\varepsilon \ll \lambda(a)$. The latter implies that either

$$a \in D \setminus \mathrm{Cl}_\sigma(\uparrow b_k) \text{or } a \in \uparrow b_k.$$

But the first of these is impossible since $b_k, a \sqsubseteq x$. Then $b_k \sqsubseteq a$ which puts $a \in \uparrow b_k \subseteq U$. This proves $\lambda \to \sigma_D$.

Then $\mu = |\lambda|$ is continuous and measures D. Finally,

$$\mu x = 0 \Leftrightarrow f(x) \cup g(x) = \mathbb{N} \Leftrightarrow x \in \max D,$$

which finishes the proof. □

It can then be shown that by applying this technique to the upper powerdomain of a semantic domain, $\mathcal{P}_U(D)$, that one actually obtains a measurement with all the same properties but that also satisifes the triangle inequality.

Lemma 3.8 *Let* D *be an* ω*-algebraic Scott domain whose maximal elements* $\max D$ *are* $\sqcap$*-primes: That is, for all* $x, y \in D$,

$$x \sqcap y \sqsubseteq z \in \max D \Rightarrow x \sqsubseteq z \text{ or } y \sqsubseteq z.$$

Then the measurement $\mu : D \to [0,1]^*$ *satisfies*

$$\mu(x \sqcap y) \leq \mu(x) + \mu(y) - \mu(x \sqcup y)$$

for any pair x, y *of elements bounded from above.*

Proof. The map f always preserves binary infima. In this case, however, g does as well. For the latter, we have $g(x \sqcap y) \subseteq g(x) \cap g(y)$ by monotonicity. For the other inclusion, let $n \in g(x) \cap g(y)$. If $n \notin g(x \sqcap y)$, then $\uparrow b_n \cap \uparrow (x \sqcap y) \neq \emptyset$, and so this intersection contains a maximal element $z \in \max D$, by the directed completeness of D. But then z is prime so either $x \sqsubseteq z$ or $y \sqsubseteq z$. This leads to

$$n \notin g(x) \text{ or } n \notin g(y),$$

as then either $z \in \uparrow b_n \cap \uparrow x$ or $z \in \uparrow b_n \cap \uparrow y$. Thus, g preserves binary infima.

Finally, we can verify the triangle inequality for μ. □

This leads to a nice measurement on the upper powerdomain of a compact domain.

Theorem 3.9 *Let D be an ω-algebraic dcpo which is Lawson compact. Then there is a measurement $\mu : \mathcal{P}_U(D) \to [0,1]^*$ with $\mu \to \sigma_{\mathcal{P}_U(D)}$ and $\ker \mu = \max \mathcal{P}_U(D)$. Furthermore, it also satisfies the triangle inequality*

$$\mu(x \sqcap y) \leq \mu(x) + \mu(y) - \mu(x \sqcup y),$$

for any two elements x, y bounded from above.

Proof. If D is ω-algebraic, then so is $\mathcal{P}_U(D)$. Because D is Lawson compact, $\mathcal{P}_U(D)$ is a Scott domain, and its maximal elements are $\sqcap$-prime ($\sqcap$ is set theoretic union). Now we construct μ as in Lemma 3.8. $\Box$

It is strange to see a measurement with the triangle inequality on the upper powerdomain, and this really just goes to show how silly existence theorems can be.

Example 3.10 Given $\mu s = 1/2^{|s|}$ on the domain of bit streams Σ^∞, use it to explicitly derive $\bar{\mu}$ on $\mathcal{P}_U(\Sigma^\infty)$ which has all the properties given in the last theorem.

Later, the fact that adjunctions (Galois connections) are measurements will be used to demonstrate the difficulty in obtaining such a measurement $\bar{\mu}$ in a hierarchical manner.

A construction for countably based domains

The one interesting thing about the construction given for semantic domains in the last section is that it indirectly teaches us something new about the class of countably based domains in general. For this, we only have to broaden our view of things slightly.

Lemma 3.11 *If U is an open subset of a topological space X, then the set $U \cup (X \setminus \mathrm{Cl}(U))$ is nonempty, open and dense.*

Theorem 3.12 *Let D be an ω-continuous dcpo. Then there is a measurement $\mu : D \to [0,\infty)^*$ with $\mu \to \sigma_D$ such that $\ker \mu$ is dense in D.*

Proof. We're not interested in the case of a finite basis. If D has an infinite basis $\{b_n\}$, the sets

$$U_n := \Uparrow b_n \cup (D \setminus \mathrm{Cl}(\Uparrow b_n))$$

are nonempty, open and dense. The argument from the case of semantic domains still applies, proving that

$$\mu x = \{n : x \in U_n\}$$

defines a measurement $\mu \to \sigma_D$ from D to $\mathcal{P}\omega$. As before, we can assume $\mu : D \to [0,\infty)^*$. The interesting part is that $\ker \mu$ is *dense*. For the proof,

$$\ker \mu = \bigcap_{n \geq 1} U_n,$$

which is a countable intersection of nonempty, open dense sets. As *everyone knows,* the Scott topology on a domain is Baire, so $\ker \mu$ must also be dense. $\square$

The denseness of $\ker \mu$ requires only that D in the Scott topology is a Baire space. If the space D is Choquet complete and second countable, which it is if D is an ω-continuous dcpo, then the space $\ker \mu$ should be Polish and zero-dimensional. The fact that the last result only makes use of abstract topological ideas, and does not seem to require the full structure of domains, may provide some hope in trying to solve the next one.

Problem 3.13 Prove or disprove that we can construct μ so that $\ker \mu = \max(D)$ on any ω-continuous dcpo. Prove or disprove the G_δ lemma too.

Notice that the G_δ lemma means that there is a sequence of dense Scott open sets whose intersection is exactly $\max(D)$. So the last theorem may be a good place to start.

4 Adjunctions, measurements, the upper powerdomain.

The central question here is abstract and almost strange: How can we tell when a construction is likely to yield a measurement? Its subtle. Some constructions irreversibly destroy information; this destructive nature appears to be explainable mathematically. To illustrate this, we show how a seemingly "natural" construction for a measurement on the upper powerdomain is limited in a manner that cannot be avoided. The reason is the existence of an underlying ep-pair, which is a special type of measurement that is only capable of measuring so much. This also calls our attention to the fact that *adjunctions* are measurements.

Adjunctions and embedding-projection pairs

Our notation for adjunctions is slanted toward embedding-projection pairs and measurements (apologies).

Definition 4.1 An *adjunction* between continuous dcpo's D and E is a pair of monotone maps $(e : E \to D, p : D \to E)$ with $1_E \sqsubseteq p \circ e$ and $e \circ p \sqsubseteq 1_D$.

Whenever we discusss adjunctions, we implicitly assume that they are defined between continuous dcpo's. See [1] about this, and for the following.

Lemma 4.2 *If (e, p) is an adjunction, then e is Scott continuous. The map p is Scott continuous iff e preserves the $\ll$ relation.*

Thus, we say that an adjunction is *continuous* if p is Scott continuous.

Definition 4.3 An *embedding-projection pair* is a continuous adjunction $(e : E \to D, p : D \to E)$ such that $p \circ e = 1_E$.

If (e,p) is an embedding-projection pair (*ep-pair*), then the *projection* p is surjective and the *embedding* e is injective. Then e is a continuous, order-reflecting mapping, which explains the following.

Lemma 4.4 *Let* (e,p) *be an ep-pair. Then the image* $\mathrm{Im}(e)$ *of the embedding is closed under directed suprema.*

Theorem 4.5 *Let* (e,p) *be a continuous adjunction. Then* p *measures* $\mathrm{Im}(e)$. *Further, if* (e,p) *is an ep-pair, then* $\mathrm{Im}(e)$ *is the largest set that* p *can measure.*

Proof. First we show that $p \to \sigma_{\mathrm{Im}(e)}$. Let U be a Scott open set containing $y = e(x) \in \mathrm{Im}(e)$. By Lemma 4.2 and the continuity of e, $e(\mathord{\downarrow\!\!\downarrow} x)$ is a directed subset of $\mathord{\downarrow\!\!\downarrow} e(x)$ with supremum $e(x)$. Then since U is Scott open, there is $\varepsilon \ll x$ such that $e(\varepsilon) \ll e(x)$ and $e(\varepsilon) \in U$. We claim that

$$y \in p_\varepsilon(y) \subseteq \mathord{\uparrow\!\!\uparrow} e(\varepsilon) \subseteq U.$$

For $y \in p_\varepsilon(y)$, we simply note that $\varepsilon \ll x \sqsubseteq p(ex) = p(y)$, using $1_E \sqsubseteq p \circ e$.

Now let a be any element of $p_\varepsilon(y)$. Then $a \sqsubseteq y$ and $\varepsilon \ll p(a)$. Since e preserves the $\ll$ relation, $e(\varepsilon) \ll e(pa) \sqsubseteq a$, using $e \circ p \sqsubseteq 1_D$. Thus, $y \in p_\varepsilon(y) \subseteq U$, establishing $p \to \sigma_{\mathrm{Im}(e)}$.

To see that $\mathrm{Im}(e)$ is the *largest* such set for an ep-pair, suppose p measures $X \subseteq D$. Let $x \in X$ and $a \ll x$. Then there is ε in E with $x \in p_\varepsilon(x) \subseteq \mathord{\uparrow\!\!\uparrow} a$. In particular, $\varepsilon \ll p(x)$, so by interpolation, there is δ with $\varepsilon \ll \delta \ll p(x)$. But then we have

$$e(\delta) \ll e(px) \sqsubseteq x,$$

since e preserves $\ll$ and $e \circ p \sqsubseteq 1_D$, and also

$$\varepsilon \ll \delta \sqsubseteq p(e\delta),$$

using $1_E \sqsubseteq p \circ e$. Then $e(\delta) \in p_\varepsilon(x) \subseteq \mathord{\uparrow\!\!\uparrow} a$.

We have then shown that for every $a \ll x$, there is $b \in \mathrm{Im}(e)$ with $a \ll b \ll x$. This implies $\mathord{\downarrow\!\!\downarrow} x \cap \mathrm{Im}(e)$ is directed with supremum x. By Lemma 4.4, for an *ep-pair*, $\mathrm{Im}(e)$ is closed under directed suprema. Hence $x \in \mathrm{Im}(e)$. $\square$

The embedding e is also a measurement. We defer proof of this until the next section where we are able to give a good explanation of *why*. The conclusion that p measures a largest set is particularly strong (and useful), as we will see in the next section.

Corollary 4.6 *Let* (e,p) *be an ep-pair. Then* p *measures its entire domain iff it is an isomorphism with inverse* e.

The upper powerdomain

Definition 4.7 The collection of nonempty Scott compact upper sets of a continuous dcpo D is denoted $\mathcal{P}_U(D)$.

A measurement on a domain leads to what *appears* to be a natural measurement $\bar{\mu}$ on $\mathcal{P}_U(D)$. When proving theorems about $\bar{\mu}$ in this and the next section we always assume that a *domain* satisfies $(\forall x, y \in D)(\exists z \in D)\, z \sqsubseteq x, y$.

Theorem 4.8 *If $\mu : D \to [0,\infty)^*$ measures $X \subseteq D$, then $\bar{\mu} : \mathcal{P}_U(D) \to [0,\infty)^*$ given by*

$$\bar{\mu}(K) = \inf\{\mu x : \uparrow\! x \sqsubseteq K \ \&\ x \in D\}$$

measures $\bar{X} = \{\uparrow\! x : x \in X\}$.

Proof. First, $\bar{\mu}$ is defined as the infimum of a *nonempty* set of nonnegative reals. By compactness of K and continuity of D, there is a nonempty finite set $F \subseteq D$ such that $K \subseteq \uparrow\! F$. By the definition of *domain,* there is $x \in D$ with $x \sqsubseteq y$ for all $y \in F$. Then $\uparrow\! x \sqsubseteq K$ in $\mathcal{P}_U(D)$.

Next, $\bar{\mu}$ is Scott continuous. The monotonicity of μ is clear, so for directed $S \subseteq \mathcal{P}_U(D)$ we have $\bar{\mu}(\bigsqcup S) \leq \inf\{\bar{\mu}(K) : K \in S\}$. For the opposite inequality, let $\uparrow\! x \sqsubseteq \bigsqcup S$. Let $a \ll x$. Then $\uparrow\! a \ll \bigsqcup S$. Thus, there is $K \in S$ with $\uparrow\! a \sqsubseteq K$, and we may write

$$\inf\{\bar{\mu}(K) : K \in S\} \leq \mu(\bar{K}) \leq \mu a.$$

But a was an *arbitrary* member of $\twoheaddownarrow\! x$, so the continuity of μ gives

$$\inf\{\bar{\mu}(K) : K \in S\} \leq \inf\{\mu a : a \ll x\} = \mu x$$

and hence

$$\inf\{\bar{\mu}(K) : K \in S\} \leq \inf\{\mu x : \uparrow\! x \sqsubseteq \bigsqcup S\} = \bar{\mu}(\bigsqcup S),$$

establishing that $\bar{\mu}$ is continuous.

Finally, $\bar{\mu}$ measures $\bar{X}$. Let $\uparrow\! x \in U$ with $U \subseteq \mathcal{P}_U(D)$ Scott open. Then there is $y \ll x$ with $\twoheaduparrow(\uparrow\! y) \subseteq U$. Since $x \in X$, there is $\varepsilon > 0$ such that $x \in \mu_\varepsilon(x) \subseteq \twoheaduparrow y$. We claim that

$$\uparrow\! x \in \bar{\mu}_\varepsilon(\uparrow\! x) \subseteq \twoheaduparrow(\uparrow\! y).$$

First, $\bar{\mu}(\uparrow\! x) = \mu x < \varepsilon$, so $\uparrow\! x \in \bar{\mu}_\varepsilon(\uparrow\! x)$. Now let $K \in \bar{\mu}_\varepsilon(\uparrow\! x)$. Then $K \sqsubseteq \uparrow\! x$ and $\bar{\mu}(K) < \varepsilon$. Thus, there is $z \in D$ with

$$\uparrow\! z \sqsubseteq K \sqsubseteq \uparrow\! x \quad \text{and} \quad \mu z < \varepsilon,$$

which puts $z \in \mu_\varepsilon(x) \subseteq \twoheaduparrow y$ giving $\uparrow\! y \ll \uparrow\! z \sqsubseteq K$ and hence $K \in \twoheaduparrow(\uparrow\! y)$. Then $\uparrow\! x \in \bar{\mu}_\varepsilon(\uparrow\! x) \subseteq \twoheaduparrow(\uparrow\! y) \subseteq U$, which means $\bar{\mu}$ measures $\bar{X}$. $\square$

The last theorem implicitly assumes that any two elements of D are bounded from below by another.

Corollary 4.9 *Let D be a domain with a measurement μ such that* $\ker \mu = \max D$. *Then there is a measurement*

$$\bar{\mu} : \mathcal{P}_U(D) \to [0,\infty)^*$$

with $\ker \bar{\mu} = \max \mathcal{P}_U(D)$.

Proof. Use $\max \mathcal{P}_U(D) = \{\{x\} : x \in \max D\}$ and $\bar{\mu}(\uparrow x) = \mu x$ for all $x \in D$. $\Box$

But now a question arises: Even if $\mu \to \sigma_D$, the last result only guarantees that $\bar{\mu}$ measures a comparatively small subset of $\mathcal{P}_U(D)$. Does $\bar{\mu}$ ever measure the entirety of $\mathcal{P}_U(D)$?

The destructive nature of composition

The first result is important in its own right, it says that composing measures of information content is implicitly a destructive act: $\mu \circ p$ measures only as much as p measures.

Theorem 4.10 *Let the composition $\mu \circ p : D \to E$ be a map which measures $X \subseteq D$. If μ is Scott continuous, then p measures X as well.*

Proof. Let $x \in X$ and $U \subseteq D$ be Scott open with $x \in U$. Since $\mu \circ p$ measures X, there is δ in E with

$$x \in (\mu \circ p)_\delta(x) \subseteq U.$$

Then $\delta \ll \mu(px)$. By the continuity of μ, there is ε with

$$px \in \twoheaduparrow\varepsilon \subseteq \mu^{-1}(\twoheaduparrow\delta).$$

We claim that $x \in p_\varepsilon(x) \subseteq U$. First, $x \in p_\varepsilon(x)$, since $\varepsilon \ll px$. Moreover, if $y \in p_\varepsilon(x)$, then

$$y \sqsubseteq x \;\&\; \varepsilon \ll py \Rightarrow y \sqsubseteq x \;\&\; \delta \ll \mu(py) \Rightarrow y \in (\mu \circ p)_\delta(x) \subseteq U,$$

which finishes the proof. $\Box$

Corollary 4.11 *A continuous order reflecting map $\mu : D \to E$ measures D.*

Corollary 4.12 *The embedding in an ep-pair measures its entire domain.*

Proof. Let (e, p) be an ep-pair. Then $p \circ e = 1_E$. But 1_E measures all of E. By Theorem 4.10, e does too. $\Box$

We are now ready to explore the limitations of $\bar{\mu}$ on $\mathcal{P}_U(D)$.

Definition 4.13 A continuous dcpo is *bounded complete* if each pair of elements with an upper bound has a supremum.

Then in a bounded complete *domain,* each nonempty compact set has an infimum (again, we assume that pairs of elements are bounded from below).

Proposition 4.14 *Let D be a bounded complete domain with a measurement μ. Then the largest subset of $\mathcal{P}_U(D)$ that $\bar{\mu}$ can measure is $\{\uparrow x : x \in D\}$.*

Proof. First, for any $x \in \mathcal{P}_U(D)$, we have

$$\bar{\mu}(x) = \mu(\sqcap x) = \mu \circ p(x),$$

where $p : \mathcal{P}_U(D) \to D :: x \mapsto \sqcap x$ is the map taking a nonempty compact upper set to its infimum. But this map is a projection.

Indeed, $e : D \to \mathcal{P}_U(D) :: x \mapsto \uparrow x$ is a monotone map with $p \circ e = 1_D$ and $e \circ p \sqsubseteq 1_{\mathcal{P}_U(D)}$, which means (e, p) is an adjunction. But e also preserves the $\ll$ relation, so by Lemma 4.2, p is continuous. Thus, (e, p) is an ep-pair.

Now suppose $\bar{\mu}$ measures X. Since $\bar{\mu} = \mu \circ p$ and μ is continuous, p must measure X, by Theorem 4.10. But any set that p can measure is contained in $\mathrm{Im}(e) = \{\uparrow x : x \in D\}$, according to Theorem 4.5. $\square$

Here is what is so fascinating about this result to me: The ability of $\bar{\mu}$ to measure $\mathcal{P}_U(D)$ does not really depend on μ, even though it is clearly defined from μ! No matter how nice of a measurement μ is, the map $\bar{\mu}$ can only do so much. It is the *construction* which is limited; this limitation is independent of μ.

The *reason* for this limitation is that it is based on an adjunction which throws out all information in a compact set: It tries to measure compact sets in D with points. And if you try to measure compact sets with points, you can only measure points. These results point to the fact that something interesting has to be done to measure the entire upper powerdomain.

Example 4.15 Consider $\bar{\mu}$ on $\mathcal{P}_U([S])$ derived from the length function μ on the domain of lists $[S]$. But notice: $[S]$ is *not* bounded complete!

Problem 4.16 Study fractals using the upper powerdomain by realizing them as unique (nonmaximal) fixed points of operators on $\mathcal{P}_U(D)$, where D is a model of a space. The idea is to apply one of the measurement based results that yield nonmaximal unique fixed points.

Question 4.17 Why do some constructions fail, and others not? The existence of underlying ep-pairs is one reason; what other mathematical ideas can we use to explain why certain constructions don't measure domains well?

5 Nonarchimedean measurement

This refers to the case of a measurement μ whose distance function

$$d(x, y) = \inf\{\mu z : z \sqsubseteq x, y\}$$

is an ultrametric on $\ker \mu$. Examples of domains like these include the domain of bit streams, and the domain of quantum two states. The proofs of all results in this section existed at one moment in time, but that moment is now gone.

Ultraposets

Proposition 5.1 *Let D be a domain. Then for a measurement $\mu \to \sigma_D$, the following are equivalent:*

(i) *For all pairs x, y bounded from above, there is $z \sqsubseteq x, y$ with*

$$\mu z \leq \max\{\mu x, \mu y\}$$

(ii) *For all x, y, z, $d(x,y) \leq \max\{d(x,z), d(z,y)\}$*

(iii) *For all pairs x, y bounded from above,*

$$\mu(x \sqcap y) = \max\{\mu x, \mu y\}$$

(iv) *For all pairs x, y bounded from above,*

$$\mu(x \sqcup y) = \min\{\mu x, \mu y\}$$

In either case, for all pairs $x, y \in D$ with an upper bound, we have $x \sqsubseteq y$ or $y \sqsubseteq x$.

This takes us naturally to the following.

Definition 5.2 An *ultraposet* is a poset in which every pair of elements bounded from above is comparable.

Theorem 5.3 *An ultraposet is continuous.*

Definition 5.4 An *ultradomain* is an ultraposet that is also a dcpo.

Because ultraposets need not be algebraic, the next result is worth pointing out.

Proposition 5.5 *The space of maximal elements in an ultraposet is zero-dimensional and Tychonoff.*

Expect that a space is the set of maximal elements of an ultraposet iff it is nonarchimedean. The term 'nonarchimedean' is from the work of a topologist named P.Nyikos.

Measurements on ultraposets

There are many ways to vary the following idea if one is only interested in models of spaces.

Definition 5.6 Let D be a poset. An *ultrameasurement* on D is a measurement $\mu \to \sigma_D$ such that for all pairs x, y bounded from above,

$$\mu(x \sqcap y) = \max\{\mu x, \mu y\}.$$

In particular, for an ultrameasurement μ, the restriction of $d(\mu)$ to $\ker\mu \times \ker\mu$ is an ultrametric yielding the relative Scott topology on $\ker\mu$.

Proposition 5.7 *Let D be an ultraposet. For a Scott continuous mapping $\mu : D \to [0,\infty)^*$, the following are equivalent:*

(i) *The map μ is strictly monotone.*

(ii) *The map μ is an ultrameasurement.*

In fact, we are in the rare situation that the structure of the domain and nature of the measurement determine one another.

Theorem 5.8 *Let D be an ω-continuous poset. Then the following are equivalent:*

(i) *D is an ultraposet.*

(ii) *There is an ultrameasurement $\mu : D \to [0,\infty)^*$ with $\ker\mu = \max(D)$.*

(iii) *Every Scott continuous, strictly monotone map $\mu : D \to [0,\infty)^*$ is an ultrameasurement.*

Models by ultradomains

Notice that the G_δ lemma holds for continuous posets P such that $\twoheaduparrow x \neq \emptyset$ for $x \in P \setminus \max P$, with $\max(P)$ developable. Just kidding – see [9] for a proof of this.

Theorem 5.9 *For an ultraposet P, the following are equivalent:*

(i) *The set $\max(P)$ is a G_δ subset of P.*

(ii) *There is a measurement $\mu : P \to [0,\infty)^*$ with $\ker\mu = \max(P)$.*

(iii) *The space $\max(P)$ is ultrametrizable and every isolated point of $\max(P)$ is the supremum of an increasing sequence of approximations.*

Corollary 5.10 *Let P be a first countable ultraposet. Then $\max(P)$ is a G_δ subset of P iff it is ultrametrizable.*

Theorem 5.11 *A space X is ultrametrizable iff there is a poset P with an ultrameasurement μ such that $X \simeq \ker\mu = \max(P)$.*

Theorem 5.12 *A space X is completely ultrametrizable iff there is a dcpo D with an ultrameasurement μ such that $X \simeq \ker\mu = \max(D)$.*

Hopefully, this has provided a little insight into the nature of zero-dimensionality. Now we turn to the problem we think the previous results may be of use in solving.

Królowa lodu

Topology is useful for doing more than simply proving theorems that good society isn't interested in. For example:

> When the topologist *separated* from his wife,
> He could not *Baire* the thought that it was for life,
> So he spoke *continuously* to his only dear friend,
> Who reassured him "grief is *normal* when important things end."
> They met one day in a fancy restaurant,
> where he pleaded with her on bending knees;
> And she responded, "I need my *space*,"
> then waved her hand, "*Čech*, please."
> – November 2000.

This we could interpret as someone having read a little too much topology. Instead, we take it as evidence that sometimes life can appear strangely mathematical. What we are interested in now is the converse: Settings where mathematics seems to reflect simple truths from life.

Now, we know about models D of spaces $X = \max(D)$ that have measurements $\ker \mu = X$. In topology, there is a class of mappings called *perfect*: Closed mappings defined on Hausdorff spaces whose fibers are compact. Jimmie Lawson gave a talk once where he discussed a way of extending the notion of perfect mappings to the domain theoretic setting. It is then natural to wonder about *models with perfect measurements.* It's almost human. Win a minor victory for humanity and solve the following problem over the class of semantic domains:

Problem 5.13 Prove that models with perfect measurements are in 1-1 correspondence with the Stone spaces (i.e., the *closed, bounded, zero-dimensional* subsets of the real line).

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Appendix: The basic elements

Let $(P, \sqsubseteq)$ be a partially ordered set or *poset* [1]. A nonempty subset $S \subseteq P$ is *directed* if $(\forall x, y \in S)(\exists z \in S)\, x, y \sqsubseteq z$. The *supremum* $\bigsqcup S$ of $S \subseteq P$ is the least of its upper bounds when it exists. A *dcpo* is a poset in which every directed set has a supremum.

For elements x, y of a dcpo D, we write $x \ll y$ iff for every directed subset S with $y \sqsubseteq \bigsqcup S$, we have $x \sqsubseteq s$, for some $s \in S$.

Definition 5.14 Let $(D, \sqsubseteq)$ be a dcpo. We set

- $\twoheaddownarrow x := \{y \in D : y \ll x\}$ and $\twoheaduparrow x := \{y \in D : x \ll y\}$
- $\downarrow x := \{y \in D : y \sqsubseteq x\}$ and $\uparrow x := \{y \in D : x \sqsubseteq y\}$

and say D is *continuous* if $\twoheaddownarrow x$ is directed with supremum x for each $x \in D$.

When it is impossible to tell what we mean by 'domain,' take the following as the default.

Definition 5.15 A *domain* is a continuous dcpo.

The *Scott topology* on a domain D has as a basis all sets of the form $\twoheaduparrow x$ for $x \in D$. A function $f : D \to E$ between domains is *Scott continuous* if it reflects Scott open sets. This is equivalent to saying that f is *monotone,*

$$(\forall x, y \in D)\, x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y),$$

and that it *preserves directed suprema*:

$$f(\bigsqcup S) = \bigsqcup f(S),$$

for all directed $S \subseteq D$.

Definition 5.16 A *basis* for a domain D is a subset $B \subseteq D$ such that $B \cap \twoheaddownarrow x$ is directed with supremum x, for all $x \in D$.

Definition 5.17 A domain is *ω-continuous* if it has a countable basis.

Definition 5.18 An element x in a domain D is *compact* if $x \ll x$. The set of compact elements is denoted by $K(D)$. A domain D is *algebraic* if $K(D)$ forms a basis for D. An algebraic domain is *ω-algebraic* if its set of compact elements is countable.

Definition 5.19 A continuous dcpo is *compact* if it is Scott compact and the intersection of any two Scott compact upper sets is Scott compact.

The terminology "compact" is based on the fact that the above condition is equivalent to compactness in the Lawson topology.

Definition 5.20 A domain is *semantic* if it is ω-algebraic and compact.

By carefully analyzing the behavior of a while loop which generates successively better approximations to some desired ideal object, we become aware of the fact that our ability to distinguish between algorithms which progress and those which do not depends in a subtle way on the relationship between the *ε-approximations* around a point x,

$$\mu_\varepsilon(x) := \{y \in D : y \sqsubseteq x \;\&\; \varepsilon \ll \mu y\},$$

where $\mu : D \to E$ is a continuous map between domains, and our qualitative understanding expressed by $\sqsubseteq$. The study of measurement connects this pragmatic desire with an element of the imagination: "information content." Notice that the former is real, meaning that it is specific and concrete in purpose; the latter is anyone's guess.

Definition 5.21 A continuous map $\mu : D \to E$ between domains is said to *measure* the set $X \subseteq D$ if for all $x \in X$ and all Scott open sets $x \in U \subseteq D$, we have $x \in \mu_\varepsilon(x) \subseteq U$, for some $\varepsilon \ll \mu x$. We sometimes write $\mu \to \sigma_X$.

The terminology "induces the Scott topology near X" is used in [8], but we prefer to reserve it mostly for topological discussions. One of the fundamental properties of measurements is *strict monotonicity.*

Proposition 5.22 *Let $\mu : D \to E$ be a map that measures $X \subseteq D$. Then for all $x \in D$ and $y \in X$, we have $x \sqsubseteq y$ and $\mu x = \mu y \Rightarrow x = y$.*

The case $E = [0, \infty)^*$, the nonnegative reals in their *opposite* order, warrants special attention.

Definition 5.23 A *measurement* is a continuous map $\mu : D \to [0, \infty)^*$ that measures $\ker \mu = \{x \in D : \mu x = 0\}$.

By Prop. 5.22, note that $\ker \mu \subseteq \max D = \{x \in D : \uparrow\!x = \{x\}\}$. That is, an element with *no uncertainty* is maximal in the information order.

Acknowledgements

pray for a man in the middle
one that talks like doolittle
– *The Pixies, from "Mr. Grieves"*

This is the point of the paper where unknown researchers usually mention well-known researchers in an effort to impress people. For instance, say a young researcher has had a five second encounter with some maybe one day historic figure. You can bet that their next paper will have a sentence like "I thank *** for useful discussions." To which discussion are you referring? Do you mean the enlightening exchange that took place when one of them said "Excuse me" while reaching for more wine at the conference?

First, I thank a guy you never heard of, Bob Coecke, whose talent for creating frowns has not gone unnoticed. An example of his artistry appears in the section on anonymity.

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$\leftarrow$

We shall have to go into this question of reality again. Before we do that, I should like to develop the theory further.
– Dirac, from the lectures on field theory.