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ON THE ANTISYMMETRY OF GALOIS EMBEDDINGS

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PRG-TR-07-00

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Abstract

A Galois connection is a 'natural' way of relating two partially-ordered spaces (for example semantic models of different expressive power). The most common form of Galois connection, which reflects one model being finer than the other, is that of a Galois embedding. This letter shows a published claim, '*two partially-ordered spaces related by a Galois embedding in each direction are isomorphic*', to be false in general but true in the finite case. The proof we give provides an application of a 'disjoint from' calculus.

Key words: specification languages; formal methods; refinement techniques

1 Introduction

From the viewpoint of 'formal methods', system derivation consists of a sequence of refinements. Each design in the sequence has a denotation in a relevant semantic domain and each refinement step is consistent with the pre-order $\leq$ (representing removal of demonic nondeterminism) on that domain. It is the purpose of the various refinement calculi to abstract that enterprise (i.e. system derivation) of explicit semantic reasoning; and that is achieved by ensuring that each law $A \sqsubseteq B$ of the calculus is sound with respect to the underlying domain in the sense just described: $[A] \leq [B]$ where $[A]$ denotes the semantic denotation of design A.

Thus system derivation, viewed either semantically or syntactically, is founded on a pre-ordered space. It is characterised, amongst comparable engineering enterprises, by the vast spectrum of levels of abstraction which it spans from specification to implementation. The key theoretical and calculational technique in relating models at different levels of abstraction is that of a Galois connection: a pre-ordered equivalent of the well-known concept of adjunction from category theory. Its theoretical strength lies in providing a way to translate between different models; it offers calculational facility by subsuming the technique of data representation. The concept is, by its simplicity, widely applicable; but as a consequence its theory is relatively restricted. The purpose of this note is to demonstrate that one less result is true for Galois connections than was previously supposed and to provide an alternative.

The equivalence with respect to the refinement pre-order identifies (observably) equivalent systems. Although equivalence can be taken as primitive for theoretical purposes (e.g. bisimulation in CCS [13]), for the reasons explained above it is the pre-order which is primitive when considering system derivation. Nonetheless it is important to identify the equivalence generated by the pre-order. Thus we are interested in *antisymmetry* properties of Galois connections: if there is a Galois connection from each of two structures to the other, to what extent are the two structures the same?

In view of the goal-directed nature of system derivation the most common appearance of the symmetric notion of a Galois connection is in situations where one semantic domain is embedded in the other. We call such a Galois connection a *Galois embedding* [10]. In this paper we show that there is no antisymmetry property for arbitrary Galois connections. Furthermore we show the natural conjecture for Galois embeddings ([3], section 7) to be false; but we provide a simple replacement and mention a novel alternative kind of result involving extended orders.

The first application of Galois connections in Computer Science is suspected by Backhouse [1] to have been, without explicit reference to the concept, by Hartmanis and Stearns [7]. In the form of Galois embeddings, Galois connections seem first to have been applied explicitly by Cousot and Cousot [3], since when the concept seems to have caught on, particularly in work from Eindhoven and Oxford. Recent theoretical contributions motivated by Computer Science are due to Melton *et al* [11,12]. For a variety of recent applications in Computer Science we refer to [6,4,9,1,8,10], the last of which we find a convenient reference for general theory.

In section 2 we summarise notation and definitions; in section 3 we introduce the property of antisymmetry and show that it fails comprehensively for Galois connections; in section 3.1 we provide a counterexample to antisymmetry of Galois embeddings and show in section 3.2 that the problem is at least as hard as the Cantor-Schröder-Bernstein theorem from set theory; and in section 4 we provide a replacement positive result.

2 Definitions and notation

We adopt the convention (from Z) that a conjunction $X \wedge Y$ is written

$$\begin{pmatrix} X \\ Y \end{pmatrix} .$$

We write the conditional 'p if r else q' as $p \lhd r \rhd q$; and we write $p \Rrightarrow q$ [$p \equiv q$] to mean that the implication $p \Rightarrow q$ [equivalence $p \Leftrightarrow q$] is a theorem.

Application of a function f to an argument x is written $f.x$ and the lifting of f which maps subsets of its domain to subsets of its range is also written f but its application is distinguished by square brackets:

$$f.[E] \;\hat{=}\; \{f.e \mid e \in E\} .$$

The identity function on X is written 1_X .

By a *partially-ordered space* $(A, \leq_A)$ we mean, as usual, that $\leq_A$ is a reflexive, antisymmetric and transitive relation on A .

Suppose that $(A, \leq_A)$ and $(B, \leq_B)$ are partially-ordered spaces. A function $f : A \to B$ is *monotone* [*isotone*] iff

$$x \leq_A y \Rrightarrow f.x \leq_B f.y$$
$$[\, x \leq_A y \;\equiv\; f.x \leq_B f.y \,] .$$

Partially-ordered spaces are isomorphic, written $(A, \leq_A) \;\cong\; (B, \leq_B)$, means that there is an isotone bijection from A to B .

By a *Galois connection* between partially-ordered spaces $(A, \leq_A)$ and $(B, \leq_B)$ we mean a pair of monotone funtions $\varepsilon : A \to B$ and $\pi : B \to A$ for which these (pointwise) inequalities hold:

$$\begin{pmatrix} \pi \circ \varepsilon & \geq_A & 1_A \\ \varepsilon \circ \pi & \leq_B & 1_B \end{pmatrix} .$$

In that case we write

$$gc(\varepsilon, \pi; (A, \leq_A), (B, \leq_B)) .$$

By a *Galois embedding* we mean a Galois connection for which $\pi \circ \varepsilon \;=\; 1_A$, in which case we write

$$ge(\varepsilon, \pi; (A, \leq_A), (B, \leq_B)) .$$

Standard theory shows that a Galois connection is a Galois embedding iff either ε is injective or π is surjective.

3 Antisymmetry

It is easy to see that if a single pair of functions forms a Galois connection in each direction then each function is in fact an isomorphism:

Proposition.

$$\begin{pmatrix} gc(\varepsilon, \pi; (A, \leq_A), (B, \leq_B)) \\ gc(\pi, \varepsilon; (B, \leq_B), (A, \leq_A)) \end{pmatrix}$$
$$\Rrightarrow$$
$$\varepsilon : (A, \leq_A) \cong (B, \leq_B) .$$

Proof. By definition of Galois connection it follows that

$$\begin{pmatrix} \pi \circ \varepsilon = 1_A \\ \varepsilon \circ \pi = 1_B \end{pmatrix} ,$$

from which elementary set theory shows ε to be an isotone bijection from A onto B . □

However much less can be said if the hypothesis is weakened to *two* pairs of connections:

$$\begin{pmatrix} gc(\varepsilon_0, \pi_0; (A, \leq_A), (B, \leq_B)) \\ gc(\varepsilon_1, \pi_1; (B, \leq_B), (A, \leq_A)) \end{pmatrix} .$$

Indeed assuming that the partially-ordered spaces contain a maximum $\top$ and minimum $\perp$ then the weakened hypothesis is satisfied by the trivial connections:

$$\begin{aligned} &\varepsilon_0,\ \pi_1 : A \to B \\ &\pi_0,\ \varepsilon_1 : B \to A \\ &\varepsilon_0.x \;\hat{=}\; \perp_B \\ &\pi_0.y \;\hat{=}\; \top_A \\ &\varepsilon_1.y \;\hat{=}\; \perp_A \\ &\pi_1.x \;\hat{=}\; \top_B . \end{aligned}$$

It is still tempting to conjecture (cf. [3], section 7) that isomorphism follows from the stronger Galois embedding property (which rules out the previous counterexample):

$$\left(\begin{array}{l} ge(\varepsilon_0, \pi_0; (A, \leq_A), (B, \leq_B)) \\ ge(\varepsilon_1, \pi_1; (B, \leq_B), (A, \leq_A)) \end{array}\right) \qquad (1)$$
$$\Rrightarrow$$
$$(A, \leq_A) \cong (B, \leq_B)$$

Unfortunately implication (1) too is false, as we now see.

3.1 Counterexample

A counterexample to implication (1) is provided by the following.

The partially-ordered spaces

We write $(\mathbb{N}, \leq)$ for the partially-ordered space of natural numbers and suppose that the subset $\{0, 1, 2\}$ inherits its order from $\mathbb{N}$. Define a partially-ordered space:

$$X \;\hat{=}\; \{(0,0)\} \cup (\mathbb{N} \times \{1,2\})$$
$$(a,b) \leq_X (c,d) \;\hat{=}$$
$$\left(\begin{array}{l} a = c \\ b \leq d \end{array}\right) \vee \left(\begin{array}{l} a = 0 \\ b = 0 \end{array}\right) .$$

We are interested in two subsets of X with their inherited orders:

$$A \;\hat{=}\; X$$
$$B \;\hat{=}\; X \setminus \{(0,2)\}$$

(see figure 1). Evidently each is a partially-ordered space.

The first Galois embedding

The first Galois embedding is defined:

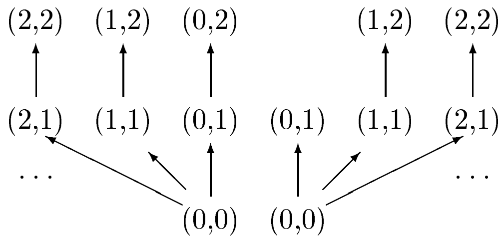

Fig. 1. Partially-ordered spaces $(A, \leq_A)$ on the left and $(B, \leq_B)$ on the right

$$\varepsilon_0 : A \to B$$
$$\pi_0 : B \to A$$
$$\varepsilon_0.(a,b) \;\hat{=}\; (0,0) \lhd\; b = 0 \;\rhd (a{+}1, b)$$
$$\pi_0.(c,d) \;\hat{=}\; (0,0) \lhd\; c = 0 \;\rhd (c{-}1, d) .$$

Claim. $ge(\varepsilon_0, \pi_0; (A, \leq_A), (B, \leq_B))$.
Proof. Firstly $\pi_0 \circ \varepsilon_0 = \imath_A$, for

$$\begin{array}{ll} & \pi_0 \circ \varepsilon_0.(a,b) \\ = & \text{definitions of } \varepsilon_0 \text{ and } \circ \\ & \pi_0.((0,0) \lhd\; b = 0 \;\rhd (a{+}1, b)) \\ = & \text{application distributes } \lhd\ \rhd \\ & \pi_0.(0,0) \lhd\; b = 0 \;\rhd \pi_0.(a{+}1, b) \\ = & \text{definition of } \pi_0 \\ & (0,0) \lhd\; b = 0 \;\rhd (a,b) \\ = & \text{definitions of } \lhd\ \rhd \text{ and } A \\ & (a,b) . \end{array}$$

Secondly $\varepsilon_0 \circ \pi_0 \leq \imath_B$, for

$$\begin{array}{ll} & \varepsilon_0 \circ \pi_0.(c,d) \\ = & \text{definitions of } \pi_0 \text{ and } \circ \\ & \varepsilon_0.((0,0) \lhd\; c = 0 \;\rhd (c{-}1, d)) \\ = & \text{application distributes } \lhd\ \rhd \\ & \varepsilon_0.(0,0) \lhd\; c = 0 \;\rhd \varepsilon_0.(c{-}1, d) \\ = & \text{definition of } \varepsilon_0 \\ & (0,0) \lhd\; c = 0 \;\rhd (c,d) \\ \leq_B & \text{definitions of } \lhd\ \rhd \text{ and } B \\ & (c,d) . \end{array}$$

In particular equality fails at (just) $(0,1)$. $\square$

The second Galois embedding

The second Galois embedding is defined using (vertically-) nested conditionals

$$
\begin{array}{l}
\varepsilon_1 : B \to A \\
\pi_1 : A \to B \\
\varepsilon_1.(c,d) \ \hat{=} \\
\qquad (0,0) \lhd \ d=0 \ \rhd (0,2) \\
\qquad \lhd \ c=0 \ \rhd \\
\qquad (c,d) \\
\pi_1.(a,b) \ \hat{=} \\
\qquad (0,0) \lhd \ b \le 1 \ \rhd (0,b{-}1) \\
\qquad \lhd \ a=0 \ \rhd \\
\qquad (a,b) \ .
\end{array}
$$

(Note: simplification of the definition of ε_1 leads to a more intricate proof.)

Claim. $ge(\varepsilon_1, \pi_1; (B, \le_B), (A, \le_A))$.

Proof. Firstly $\pi_1 \circ \varepsilon_1 = \imath_B$, for

$$
\begin{array}{ll}
& \pi_1 \circ \varepsilon_1.(c,d) \\
= & \text{arguing as above} \\
& \pi_1.(0,0) \lhd \ d=0 \ \rhd \pi_1.(0,2) \\
& \quad \lhd \ c=0 \ \rhd \\
& \quad \pi_1.(c,d) \\
= & \text{definition of } \pi_1 \\
& (0,0) \lhd \ d=0 \ \rhd (0,1) \\
& \lhd \ c=0 \ \rhd \\
& (c,d) \\
= & \text{definitions of } \lhd \ \rhd \text{ and } B \\
& (c,d) \ .
\end{array}
$$

Secondly $\varepsilon_1 \circ \pi_1 \le \imath_A$, for

$$
\begin{array}{ll}
& \varepsilon_1 \circ \pi_1.(a,b) \\
= & \text{application distributes } \lhd \ \rhd \\
& \varepsilon_1.(0,0) \lhd \ b \le 1 \ \rhd \varepsilon_1.(0,b{-}1) \\
& \lhd \ a=0 \ \rhd \\
& \varepsilon_1.(a,b) \\
= & \text{definition of } \varepsilon_1 \\
& (0,0) \lhd \ b \le 1 \ \rhd (0,2) \\
& \lhd \ a=0 \ \rhd \\
& (a,b) \\
\le & \text{definitions of } \lhd \ \rhd \text{ and } A \\
& (a,b)
\end{array}
$$

with equality failing again at just $(0,1)$. $\square$

Claim. $(A, \le_A) \not\cong (B, \le_B))$.

Proof. If $f : A \to B$ is an isomorphism of partially-ordered spaces then $a <_A b \Rrightarrow f.a <_B f.b$. For

$$
\begin{array}{ll}
& a <_A b \\
\equiv & \text{definition of } <_A \\
& \begin{pmatrix} a \le_A b \\ a \ne b \end{pmatrix} \\
\Rrightarrow & f \text{ monotone and injective} \\
& \begin{pmatrix} f.a \le_B f.b \\ f.a \ne f.b \end{pmatrix} \\
\Rrightarrow & \text{definition of } <_B \\
& f.a <_B f.b \ .
\end{array}
$$

For each $n{:}\mathbb{N}$, in space A

$$(0,0) <_A (n,1) <_A (n,2) \ ,$$

hence in space B

$$f.(0,0) <_B f.(n,1) <_B f.(n,2) \ .$$

Since the only possibility is

$$f.(0,0) = (0,0)$$

we infer, for each $n{:}\mathbb{N}$,

$$(0,0) <_B f.(n,1) <_B f.(n,2) \ .$$

But that means the range of f cannot contain the maximal chain

$$(0,0) <_B (0,1)$$

contradicting surjectivity of f . $\square$

That completes the verification of the counterexample to (1), which also shows that implication (1) fails even restricted to A and B satisfying both the ascending chain condition and descending chain condition (i.e. for which every strictly increasing or decreasing chain is finite).

3.2 Cantor-Schröder-Bernstein

In section 3.1 a counterexample has been provided to a published result and in section 4 a restricted case of that result is proved. The present section is included for methodological purposes: to show that even in the restricted case the kind of reasoning originally published is insufficient. Without it there may linger a suspicion that proof of the theorem in section 4 ought to be trivial. We show that is not the case by showing that to establish implication (1) for given spaces A and B is at least as hard as proof, for such spaces, of the Cantor-Schröder-Bernstein theorem from set theory.

Suppose that the hypotheses of the Cantor-Schröder-Bernstein theorem hold: A and B are sets and there is an injection f from A to B and an injection g from B to A. We wish to show that if spaces A and B (with partial orders defined below) satisfy implication (1) for some embeddings ε_i and projections π_i $(i = 0, 1)$ then the conclusion to the Cantor-Schröder-Bernstein theorem holds: there is a bijection from A onto B.

First we make the two sets into partially-ordered spaces in a uniform way: introduce a top, a bottom, and place all the elements of the set between the two so that they are pairwise incomparable. The result is, for sets A and B, complete lattices which we denote $(A_L, \leq_A)$ and $(B_L, \leq_B)$.

Next we extend the two functions in a uniform way: top maps to top, bottom to bottom and otherwise the extension behaves like the original function:

$$f_L : A_L \to B_L$$

$$f_L.a \;\hat{=}\; \begin{cases} \perp_B & \text{if } a = \perp_A \\ \top_B & \text{if } a = \top_A \\ f.a & \text{otherwise.} \end{cases}$$

The resulting function f_L is injective and universally $\vee$-junctive:

$$\forall E \subseteq A \bullet f_L.(\vee E) = \vee f_L.[E] .$$

Thus by the theory of Galois connections (for example [10], section 4.1.3, 'adjunction' theorem) f_L has an adjoint: there is a function ϕ_L completing a Galois connection with f_L. Since f_L is injective the connection is in fact a Galois embedding ([10], section 4.3.2, 'ge equivalence' theorem): $ge(f_L, \phi_L; A_L, B_L)$.

Similarly for $g_L : B_L \to A_L$ and its adjoint γ_L yielding $ge(g_L, \gamma_L; B_L, A_L)$.

So an application of (1) yields an isomorphism of partially-ordered spaces from A_L onto B_L. That restricts to a bijection from A onto B as required.

4 Positive results

In this section we prove that implication (1) holds if A is finite. The proof is an interesting application of the 'disjoint-from' calculus.

Theorem. Assume $(A, \leq_A)$ and $(B, \leq_B)$ are partially-ordered spaces with A finite and with isotone functions $\varepsilon_0 : A \to B$ and $\varepsilon_1 : B \to A$. Then

$$\varepsilon_1 : (B, \leq_B) \cong (A, \leq_A)\,.$$

Proof. Elementary reasoning shows that an isotone function is injective. So it remains to show that the isotone injection ε_1 is surjective. Writing

$$Q \;\hat{=}\; A \setminus \varepsilon_1.[B]$$

we show $Q = \varnothing$. On the assumption that $Q \neq \varnothing$ we shall deduce that A is infinite, a contradiction.

We show A is infinite by showing that it contains an infinite sequence of strictly-increasing subsets. To do so we reason that a certain sequence of sets is pairwise disjoint. Accordingly we write $E \diamond F$ to mean that sets E and F are disjoint:

$$E \diamond F \;\hat{=}\; (E \cap F = \varnothing)\,.$$

We use the following simple laws of the 'disjoint-from' calculus (overline denoting complement).

$$\overline{E} \diamond E \quad (2)$$

$$E' \diamond F \;\Leftarrow\; \begin{pmatrix} E \diamond F \\ E' \subseteq E \end{pmatrix} \quad (3)$$

$$F \subset (E \cup F) \;\Leftarrow\; \begin{pmatrix} E \diamond F \\ E \neq \varnothing \end{pmatrix} \quad (4)$$

$$E \diamond (F \cup G) \;\equiv\; \begin{pmatrix} E \diamond F \\ E \diamond G \end{pmatrix} \quad (5)$$

$$f.[E] \diamond f.[F] \;\Leftarrow\; \begin{pmatrix} E \diamond F \\ f \text{ injective} \end{pmatrix} \quad (6)$$

Finally we write ε for the composition $\varepsilon_1 \circ \varepsilon_0$. We note that ε is injective on A and satisfies, for $n\!:\!\mathbb{N}$,

$$\varepsilon^{n+1}.[Q] \subseteq \varepsilon_1.[B] = \overline{Q}. \qquad (7)$$

With those preliminaries we now define, for $n\!:\!\mathbb{N}$,

$$Q_n \;\hat{=}\; \cup\{\varepsilon^i.[Q] \mid 0 \leq i \leq n\}$$

and prove by induction that

$$\forall n\!:\!\mathbb{N} \;\bullet\; \varepsilon^{n+1}.[Q] \diamond Q_n\,.$$

For the base case we have

$$\begin{array}{ll}
\varepsilon^1.[Q] \diamond Q_0 & \\
\equiv & \text{definitions} \\
\varepsilon.[Q] \diamond Q & \\
\Leftarrow & \text{laws (3) and (7)} \\
\overline{Q} \diamond Q & \\
\Leftarrow & \text{law (2)} \\
\textit{true}\,. &
\end{array}$$

For the step case, writing $\forall_{a \leq i < b} \bullet P$ to mean

$$\forall i \bullet (a \leq i < b) \Rightarrow P,$$

we have

$$\begin{array}{ll}
\varepsilon^{n+1}.[Q] \diamond Q_n & \\
\equiv & \text{definition and law (5)} \\
\begin{pmatrix} \varepsilon^{n+1}.[Q] \diamond Q \\ \forall_{0 \leq i \leq n-1} \bullet \varepsilon^{n+1}.[Q] \diamond \varepsilon^{i+1}.[Q] \end{pmatrix} & \\
\Leftarrow & \varepsilon \text{ injective and law (6)} \\
\begin{pmatrix} \varepsilon^{n+1}.[Q] \diamond Q \\ \forall_{0 \leq i \leq n-1} \bullet \varepsilon^{n}.[Q] \diamond \varepsilon^{i}.[Q] \end{pmatrix} & \\
\Leftarrow & \text{law (5) and definition} \\
\begin{pmatrix} \varepsilon^{n+1}.[Q] \diamond Q \\ \varepsilon^{n}.[Q] \diamond Q_{n-1} \end{pmatrix} & \\
\Leftarrow & \text{induction hypothesis} \\
\varepsilon^{n+1}.[Q] \diamond Q & \\
\Leftarrow & \text{laws (3) and (7)} \\
\overline{Q} \diamond Q & \\
\Leftarrow & \text{law (2)} \\
\textit{true}\,. &
\end{array}$$

That completes the induction. Since Q is nonempty, the sets $\varepsilon^{n+1}.[Q]$ are nonempty. Hence, we can apply law (4) and deduce from the definition of Q_n that the sequence $(Q_n)_{n:\mathbb{N}}$ is an infinite strictly-increasing sequence of subsets of A as required to complete the proof. □

The result we seek is an immediate consequence of that theorem:

Corollary.

$$\left(\begin{array}{l} ge(\varepsilon_0, \pi_0; (A, \leq_A), (B, \leq_B)) \\ ge(\varepsilon_1, \pi_1; (B, \leq_B), (A, \leq_A)) \\ A \text{ finite} \end{array}\right)$$
$$\Rrightarrow$$
$$(A, \leq_A) \cong (B, \leq_B)\,.$$

We have shown in section 3.1 that the ascending chain condition (ACC) and descending chain condition (DCC) together still do not suffice to ensure (1). However a more intricate argument [2] shows that in that case the orders $\leq_A$ and $\leq_B$ are at least consistent with each other in the sense that they may be extended to make A and B isomorphic.

5 Conclusions

Implication (1) says that the relationship, between partially-ordered spaces, of being related by a Galois embedding is antisymmetric on isomorphism classes. Although we have shown that to be false in general, we have shown it to hold in the finite case. We have also shown that result to be best possible in the sense that the hypothesis of finiteness cannot be weakened to 'satisfies the ascending or descending chain condition'. The results are of interest since partially-ordered spaces underpin any program-refinement calculus, and antisymmetry captures models having equivalent semantic content.

The proof we have given uses the 'disjoint-from' calculus. Although simple, that calculus was found to be extremely helpful in formulating the proof. Rather than suppress it as an extraneous ingredient we have retained it in the hope that others may find it useful in similar situations (including arguments like the proof of Lagrange's theorem, in logic where $p \diamond q$ is defined to be $\neg\exists x \bullet p(x) \wedge q(x)$, and in relational algebra).

6 Acknowledgements

The letter has been improved as a result of comments and corrections by the editor (Roland Backhouse) and referees, to whom the authors are most grateful.

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