Programming Research Group

TRACTABLE CONSTRAINTS CLOSED UNDER A BINARY OPERATION

Andrei A. Bulatov
Department of Algebra and Discrete Mathematics
Ural State University, Ekaterinburg, Russia
e-mail: Andrei.Bulatov@usu.ru

Peter Jeavons
Computing Laboratory
University of Oxford, UK
e-mail: Peter.Jeavons@comlab.ox.ac.uk

PRG-TR-12-00

Oxford University Computing Laboratory
Wolfson Building, Parks Road, Oxford OX1 3QD

Abstract

Many combinatorial search problems can be expressed as instances of the "constraint satisfaction problem" (CSP). This class of problems is known to be NP-complete in general, so to ensure tractability it is natural to consider restricted subproblems in which the constraints have certain specified forms. The algebraic approach to the CSP maintains that certain algebraic invariance properties of constraints can be used to determine the complexity of these restricted problems.

In 1978 Schaefer classified the complexity of all subproblems of the CSP which arise from restricting the possible constraint types on a two-element domain. This description can be derived by considering the algebraic structures known as minimal clones on a two-element set, hence, from the point of view of determining complexity, minimal clones are of particular importance.

In the present paper we prove that restricted versions of the CSP in which the constraints are invariant under the term operations of a conservative commutative groupoid are tractable. Using this result, we classify the complexity of all subproblems of the CSP associated with minimal clones on a four-element set.

1 Introduction

The constraint satisfaction problem provides a general framework in which a wide variety of combinatorial problems can be expressed in a natural way. The aim in a constraint satisfaction problem is to assign values to a given set of variables such that certain constraints are satisfied on the values that can be assigned simultaneously to some subsets of variables [16, 14]. A standard satisfiability problem [6], where the variables must be assigned Boolean values, gives a typical example of such a problem.

As is well known, the constraint satisfaction problem is NP-hard in general [16, 14]. However, certain subproblems defined by restrictions on the possible forms of constraints have been shown to be tractable [3, 4, 10, 11, 12, 16, 25, 26].

One fundamental open problem in this area is to characterise those restrictions on the form of constraints which give rise to tractable problem classes. This problem was completely solved for the important special case of Boolean constraints by T.J.Schaefer in 1978 [22]. He proved that there are six tractable classes of Boolean constraints and any problem involving constraints not contained in one of these six classes is NP-complete. This result is known as Schaefer's Dichotomy Theorem for Boolean constraints.

In the same paper [22] Schaefer raised the question of how this result can be generalised to the case of a larger domain. Some progress has been made since 1978. A number of tractable families of constraints have been found for both finite and infinite domains. Many of them have been characterised in terms of algebraic invariance properties by Jeavons and co-authors [11, 9, 10, 8]. Using techniques from logic programming and group theory, Feder and Vardi [5] have identified three large families of tractable constraints which contain all of Schaefer's six classes.

In [7] and then in [2, 1], a strong connection was discovered between the complexity of these restricted subproblems of the constraint satisfaction problem and properties of objects arising in general algebra such as clones and universal algebras.

This paper is a part of a long-term project that aims to use algebraic methods to obtain a complete classification of constraints over a finite domain with respect to tractability. Here we focus on the *weakest possible* restrictions on the form of the constraints, and seek to classify the complexity of the corresponding problem classes. From the algebraic point of view such weakest possible restrictions correspond to *minimal clones* which have been intensively studied since 1983 [21, 20, 23].

Schaefer's Dichotomy Theorem can easily be derived from the well known description of minimal clones on a two-element set [19] and known tractable families of constraints. Moreover, one can obtain a very elegant restatement of this Theorem in algebraic terms which suggests a likely conjecture for a more general theorem [2, 1] (see Theorem 14 below). In view of this, one may expect that the investigation of the complexity of constraints corresponding to minimal clones in the general case will be rather useful.

Minimal clones generated by a binary operation provide the most difficult type of minimal clones: they are the only type whose complexity is still unknown. Inside this class the clones generated by a conservative binary operation play one of the central roles. A binary operation $*$ is said to be *conservative* if $a * b \in \{a, b\}$, for any a, b. A set A endowed with a binary conservative operation is called a *conservative groupoid*.

We characterise in this paper for the first time which finite conservative groupoids give rise to tractable constraint classes:

Theorem 1 *A finite conservative groupoid gives rise to a tractable class of constraints if it is commutative. Otherwise, the corresponding class is NP-complete.*

Furthermore, by using this new dichotomy result we completely characterise all tractable minimal clones on a four-element set.

The paper is organised as follows. In Section 2 we state basic definitions and results about the constraint satisfaction problem, introduce clones and universal algebras, state theorems which provide the connection between constraint satisfaction problems, clones and finite algebras, and introduce all the required algebraic machinery. Then, in Section 3 we prove the main result about conservative groupoids. Finally, in Section 4 we apply the known and the new results to the investigation of minimal clones on a four-element set.

2 Definitions

We recall some notions and notation which have been used in [2, 1] and will be used constantly throughout this paper.

2.1 The Constraint Satisfaction Problem

The set of all n-tuples of elements of D, for any set D and any natural number n, is denoted by D^n. We will usually denote tuples in boldface. The value of the ith coordinate position of a tuple $\mathbf{t} \in D^n$ is denoted by t_i. The tuple $\mathbf{t}$ may then be written in the form $(t_1, \ldots, t_n)$. A tuple whose components are indexed by elements of a set I will also be written as $(a_i)_{i \in I}$. An n-ary *relation* is a subset of D^n. The set of all finitary relations over D is denoted by $\mathbf{R}_D$.

Definition 2 *An instance of a* constraint satisfaction problem *consists of*

- *a finite set of variables, V;*
- *a finite domain of values, D;*
- *a set of constraints $\{C_1, \ldots, C_q\}$.*
- *Each constraint C_i is a pair (S_i, R_i), where S_i is a list of variables of length m_i, called* the constraint scope*, and R_i is an m_i-ary relation over the domain D, called* the constraint relation*.*

The length of the tuples in a constraint relation of a given constraint will be called the *arity* of that constraint. A *solution* of a constraint satisfaction problem is a function from the set of variables to the domain such that the image of each constraint scope is an element of the corresponding constraint relation.

Definition 3 *For a given set of relations, Γ,* CSP(Γ) *is defined to be the decision problem with*

INSTANCE: *An instance, P, of a constraint satisfaction problem, in which all constraint relations are elements of Γ.*
QUESTION: *Does P have a solution?*

Suppose first, that Γ is a finite set of relations. Then, if there exists an algorithm which solves every problem instance of CSP(Γ) in polynomial time, then we say that CSP(Γ) is a *tractable* problem, and Γ is a *tractable set* of relations. In the case when Γ is infinite we must distinguish two kinds of tractability.

Definition 4 *A set of relations, Γ, is said to be* globally tractable *if there exists an algorithm which solves every problem instance from* CSP(Γ) *in polynomial time. The set Γ is called* (locally) tractable *if every finite subset of Γ is tractable.*

A set of relations Γ is said to be NP-complete *if there exists a finite set $\Gamma' \subseteq \Gamma$ such that* CSP(Γ') *is NP-complete.*

For the case of a 2-element domain, T.J.Schaefer [22] has given a complete classification of sets of relations with respect to tractability.

2.2 Operations and clones of operations

By an n-ary operation f we mean a mapping $f\colon D^n \to D$. The set of all operations over D will be denoted by $\mathbf{O}_D$. An operation $f(x_1, \ldots, x_n)$ is called a *projection* if there is an i in the range 1 to n such that

$$f(x_1, \ldots, x_n) = x_i.$$

The *composition* of an n-ary operation f and m-ary operations $g_1, \ldots, g_n$ is defined to be the m-ary operation

$$f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m)).$$

A set of operations is said to be a *clone of operations* (or, simply, a *clone*) if it contains all of the projections and is closed under composition.

A clone $\mathbf{C}$ is said to be *generated* by a set of operations $F \subseteq \mathbf{O}_D$ (or, F *generates* $\mathbf{C}$) if $\mathbf{C}$ is the least clone containing F. In other words, $\mathbf{C}$ consists of all operations that can be derived from operations of F and projections by composition.

We say that an operation $f(x_1, \ldots, x_n)$ *preserves* an m-ary relation ϱ if, for every $(a_{11}, \ldots, a_{m1}), \ldots, (a_{1n}, \ldots, a_{mn}) \in \varrho$ we have

$$f\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} f(a_{11}, a_{12}, \ldots, a_{1n}) \\ f(a_{21}, a_{22}, \ldots, a_{2n}) \\ \vdots \\ f(a_{m1}, a_{m2}, \ldots, a_{mn}) \end{pmatrix} \in \varrho.$$

The set of all relations which are preserved by each operation from a set $C \subseteq \mathbf{O}_D$ is denoted by $\operatorname{Inv} C$ and the set of all operations preserving each relation from a set of relations F is denoted by $\operatorname{Pol} F$. It is well known and easily checked that sets of the form $\operatorname{Pol} F$ are clones of operations. Moreover, every clone can be represented in the form $\operatorname{Pol} F$ for a certain set of relations, F (see, e.g. [18]).

As the following result indicates, sets of the form $\operatorname{Inv} C$ are the only sets of relations which need to be studied.

Theorem 5 ([2, 7]) *Let Γ_0 and Γ be sets of relations over a finite set D, such that Γ is finite. If $\Gamma \subseteq \operatorname{Inv}\operatorname{Pol}\Gamma_0$, then* $\mathrm{CSP}(\Gamma)$ *is reducible to* $\mathrm{CSP}(\Gamma_0)$ *in polynomial time.*

Moreover, it follows from this result that for each set of relations Γ, the complexity of $\mathrm{CSP}(\Gamma)$ is determined (up to polynomial-time reduction) by the clone $\operatorname{Pol}\Gamma$.

Definition 6 *A clone $\mathbf{C}$ is said to be* [globally] tractable *if the set of relations* $\operatorname{Inv}\mathbf{C}$ *is [globally] tractable.*

A clone $\mathbf{C}$ is said to be NP-complete *if* $\operatorname{Inv}\mathbf{C}$ *is* NP*-complete.*

For example, for the smallest clone $\mathbf{C}$ on a finite set D (which consists of all the projections) $\operatorname{Inv}\mathbf{C}$ is the set of all relations on D and therefore is NP-complete.

To obtain small NP-complete classes of CSPs one should look for NP-complete clones which are as large as possible. Analogously, to obtain large tractable classes one should search for small tractable clones. In Section 4 we will see that even the smallest non-trivial clones (which we will later refer to as minimal clones) can be tractable.

2.3 Tractable and NP-complete clones

A subproblem of CSP with restricted constraint relations can often be specified more easily or naturally by describing the operations which preserve the constraints rather than by describing the set of relations themselves. In particular, all classes of constraints which are known to be tractable or NP-complete can be described as sets of relations preserved by some small set of operations. To give examples we first need more definitions.

Definition 7 *Let f be an n-ary operation over D.*

If $f(a, \dots, a) = a$, for all $a \in D$, then f is said to be idempotent.

If $f(a_1, \dots, a_n) \in \{a_1, \dots, a_n\}$, for all $a_1, \dots, a_n \in D$, then f is said to be conservative.

If $n = 1$ and f is injective, then f is said to be a permutation.

If there exists an index $i \in \{1, \dots, n\}$ such that $f(a_1, \dots, a_n) = g(a_i)$, for all $a_1, \dots, a_n \in D$, where g is a non-constant unary operation on D, then f is said to be essentially unary.

If $n \geq 3$ and there exists $i \in \{1, \dots, n\}$ such that for all $a_1, \dots, a_n \in D$ we have $f(a_1, \dots, a_n) = a_i$ whenever $|\{a_1, \dots, a_n\}| < n$, but f is not a projection, then f is said to be a semiprojection.

If $n = 3$ and for all $a, a' \in D$ we have $f(a, a, a') = f(a, a', a) = f(a', a, a) = a$, then f is said to be a majority operation.

If $n = 3$ and for all $a_1, a_2, a_3 \in D$ we have $f(a_1, a_2, a_3) = a_1 - a_2 + a_3$, where $+$ and $-$ are the operations of an Abelian group, then f is said to be affine.

If $n = 2$ and for all $a_1, a_2, a_3 \in D$ we have

- $f(f(a_1, a_2), a_3) = f(a_1, f(a_2, a_3))$ *(*associativity*);*
- $f(a_1, a_2) = f(a_2, a_1)$ *(*commutativity*);*
- $f(a_1, a_1) = a_1$ *(*idempotency*),*

then f is said to be a semilattice *operation*[1]*.*

We can often represent tractable or NP-complete sets of relations in the form $\mathrm{Inv}\, C$. A representation in this form is especially nice if C can be chosen to be a one-element set of operations.

Theorem 8 ([10]) *If a clone* **C** *contains any of the following operations*

– a constant operation; or

– a majority operation; or

– a semilattice operation; or

– an affine operation,

then **C** *is tractable.*

[1] In some earlier papers a semilattice operation is called an *ACI* operation. The term semilattice operation arises from an algebraic tradition to call a set endowed with such an operation a *semilattice*.

In all these cases we may indeed suppose that the clone $\mathbf{C}$ is generated by a single operation.

The following theorem is very close to the one proved in [10]. For an operation $f(x_1, \ldots, x_n)$ and $B \subseteq D$ such that $f(a_1, \ldots, a_n) \in B$ whenever $a_1, \ldots, a_n \in B$ (that is f preserves the unary relation B), we denote by $f|_B$ the restriction of f to B.

Theorem 9 *Let a clone $\mathbf{C}$ and a set $B \subseteq D$, $|B| > 1$, be such that $B \in \mathrm{Inv}\,\mathbf{C}$. If for every operation f from $\mathbf{C}$ the restriction $f|_B$ is essentially unary, then $\mathbf{C}$ is* NP*-complete.*

It follows from the results of [19] that every clone $\mathbf{C}$ on a 2-element set can either be generated by unary operations, or contains one of the following operations: disjunction, conjunction, an affine operation or the majority operation. In the latter case $\mathbf{C}$ is tractable by Theorem 8. The former case splits into two subcases. In the first, $\mathbf{C}$ contains a constant operation and is therefore tractable. The second case includes two clones only: the trivial clone containing all projections and the clone generated by the negation operation. Theorem 9 implies that they are both NP-complete. Thus we see that Schaefer's classification can be easily obtained by combining of Theorems 8,9 and the description of clones on a 2-element set.

2.4 Algebras

In this subsection we give basic definitions of standard algebraic constructions. In our definitions we mainly follow [15] and [24].

Definition 10 *An* algebra *is an ordered pair $\mathbb{A} = (A, F)$ such that A is a non-empty set and F is a family of finitary operations on A. The set A is called the* universe *of $\mathbb{A}$, the operations from F are called* basic. *An algebra with a finite universe is referred to as a* finite algebra.

The universe of an algebra $\mathbb{A}$ is denoted by A.

Definition 11 *Let $\mathbb{A} = (A, F)$ be an algebra and B a subset of A such that, for any $f_i \in F$, and for any $b_1, \ldots, b_{n_i} \in B$, we have $f_i(b_1, \ldots, b_{n_i}) \in B$. Then the algebra $\mathbb{B} = (B, F|_B)$, where $F|_B$ consists of restrictions of operations $f_i|_B$ to B, is called a* subalgebra *of $\mathbb{A}$.*

Definition 12 *Let $\mathbb{A} = (A, F)$ be an algebra. The clone generated by F is called the* clone of term operations *of $\mathbb{A}$ and is denoted by* $\mathrm{Clo}(\mathbb{A})$. *Two algebras with the same universe are called* term equivalent *if they have the same clone of term operations.*

In other words, the clone $\mathrm{Clo}(\mathbb{A})$ consists of all operations that can be obtained from F and the projections by means of composition.

Definition 13 *An algebra $\mathbb{A} = (A; F)$ is said to be* [globally] tractable *if* $\mathrm{Clo}(\mathbb{A})$ *is [globally] tractable.*

An algebra $\mathbb{A}$ is said to be NP-complete *if* $\mathrm{Clo}(\mathbb{A})$ *is* NP*-complete.*

Schaefer's result [22] provides a first step towards a classification of finite algebras with respect to complexity, as it yields a complete classification of two-element algebras in this sense.

Theorem 14 (Schaefer) *A two-element algebra $\mathbb{A} = (\{0,1\}, F)$ is tractable if it has a term operation which is not essentially unary. Otherwise $\mathbb{A}$ is NP-complete.*

The next theorem connects the tractability of a finite algebra with the tractability of its subalgebras and *homomorphic images.*

Definition 15 *Algebras $\mathbb{A}_1 = (A; F_1)$ and $\mathbb{A}_2 = (A; F_2)$ are said to be* similar *if there exists a set I such that $F_1 = (f_i^1 \mid i \in I)$, $F_2 = (f_i^2 \mid i \in I)$ and f_i^1, f_i^2 are of the same arity, for each $i \in I$.*

Usually we will treat sets of basic operations of similar algebras as different interpretations of a common set of operation symbols.

Definition 16 *Let $\mathbb{A}_1 = (A_1, F_1)$ and $\mathbb{A}_2 = (A_2, F_2)$ be similar algebras. A map $\varphi : A_1 \to A_2$ is called a* homomorphism *from $\mathbb{A}_1$ to $\mathbb{A}_2$ if $\varphi f_i^1(a_1, \dots, a_{n_i}) = f_i^2(\varphi(a_1), \dots, \varphi(a_{n_i}))$ holds for all $i \in I$ and all $a_1, \dots, a_{n_i} \in A_1$. If the map φ is onto then $\mathbb{A}_2$ is said to be a* homomorphic image *of $\mathbb{A}_1$. A bijective homomorphism is called an* isomorphism, *and if there exists an isomorphism from $\mathbb{A}_1$ to $\mathbb{A}_2$, the algebras $\mathbb{A}_1, \mathbb{A}_2$ are said tobe* isomorphic. *A homomorphic image of a subalgebra of an algebra $\mathbb{A}$ is called a* factor *of $\mathbb{A}$.*

Theorem 17 ([2, 1]) *Let $\mathbb{A}$ be a finite algebra. Then*
(i) if $\mathbb{A}$ is (globally) tractable then so is every subalgebra of $\mathbb{A}$.
(ii) if $\mathbb{A}$ has an NP*-complete subalgebra then $\mathbb{A}$ is* NP*-complete itself.*
(iii) if $\mathbb{A}$ is tractable then so is every homomorphic image of $\mathbb{A}$.

Corollary 18 *If $\mathbb{A}$ is a tractable finite algebra then so is every factor of $\mathbb{A}$.*

The following corollary from Theorem 17 seems to be quite significant, since all known forms of NP-complete CSPs (see [17, 22]) can be obtained using it.

Corollary 19 ([2, 1]) *A finite algebra $\mathbb{A}$ is* NP*-complete if it has a factor $\mathbb{B}$ which contains at least two elements, and whose basic operations are all essentially unary.*

2.5 The multi-sorted CSP

In this subsection we introduce an auxiliary concept, that will be very intensively used in the next section.

By the *Cartesian product* $A_1 \times \ldots \times A_n$ of sets $A_1, \dots, A_n$ we mean the set of all tuples $(a_1, \dots, a_n)$ where $a_1 \in A_1, \dots, a_n \in A_n$. The Cartesian product of sets $A_1, \dots, A_n$ will also be written as $\prod_{i \in \{1,\dots,n\}} A_i$.

Definition 20 *An instance of a* multi-sorted constraint satisfaction problem *consists of*

- *a finite set of variables, V;*
- *a family of sets of values, $\{D_v \mid v \in V\}$;*
- *a set of constraints $\{C_1, \ldots, C_q\}$.*
- *Each constraint C_i is a pair (S_i, R_i), where S_i is a list of variables, called* the constraint scope*, and R_i is a subset of the Cartesian product $\prod_{v \in S_i} D_v$, called* the constraint relation.

Note that, an instance $(V; D; C_1, \ldots, C_q)$ of the standard constraint satisfaction problem (Definition 2) can be treated as a particular case of a multi-sorted constraint satisfaction problem $(V; \{D_v \mid v \in V\}; C_1, \ldots, C_q)$ where $D_v = D$ for all $v \in V$. Conversely, every instance of a multi-sorted constraint satisfaction problem can be reduced to the standard form by putting $D = \bigcup_{v \in V} D_v$ and adding unary constraints of the form $(\{v\}, D_v)$. However, the multi-sorted framework is more flexible and we will use it as the main framework in our proofs.

Throughout this paper we will treat the domains of a multi-sorted constraint satisfaction problem as the universes of similar algebras. This puts some restrictions on the possible forms of the constraints.

Let $D_1, \ldots, D_n$ be sets, $R \subseteq \prod_{\{1,\ldots,n\}} D_i$, and $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$. The *projection* of R onto the set I of coordinates is the set of k-tuples

$$\begin{aligned}\mathrm{pr}_I R \;=\; & \{(a_{i_1}, \ldots, a_{i_k}) \mid \text{there are } a_i \in D_i,\, i \in \{1, \ldots, n\} - I, \\ & \text{such that } (a_1, \ldots, a_n) \in R\}.\end{aligned}$$

If I is a singleton, say $I = \{i\}$, then we will write $\mathrm{pr}_i R$ instead of $\mathrm{pr}_{\{i\}} R$.

Definition 21 *Let $\mathbb{A}_1 = (A_1, F^1), \ldots, \mathbb{A}_n = (A_n, F^n)$ be similar algebras with the index-set F of basic operations. The* direct product *of $\mathbb{A}_1, \ldots, \mathbb{A}_n$ is the algebra $\mathbb{B} = (B; F^B)$ where $B = A_1 \times \ldots \times A_n$ and a (k-ary) basic operation $f^B \in F^B$ is defined as follows*

$$f^B\left(\begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}, \ldots, \begin{pmatrix} a_{1k} \\ \vdots \\ a_{nk} \end{pmatrix}\right) = \begin{pmatrix} f^1(a_{11}, \ldots, a_{1k}) \\ \vdots \\ f^n(a_{n1}, \ldots, a_{nk}) \end{pmatrix}$$

where $a_{11}, \ldots, a_{1k} \in A_1, \ldots, a_{n1}, \ldots, a_{nk} \in A_n$. The direct product of $\mathbb{A}_1, \ldots, \mathbb{A}_n$ is written as $\mathbb{A}_1 \times \ldots \times \mathbb{A}_n$ or $\prod_{i \in \{1,\ldots,n\}} \mathbb{A}_i$.

A subalgebra $\mathbb{C}$ of a direct product $\mathbb{A}_1 \times \ldots \times \mathbb{A}_n$ is said to be a subdirect product *if, for any $i \in \{1, \ldots, n\}$ and any $a \in A_i$ there exist $a_1 \in A_1, \ldots, a_{i-1} \in A_{i-1}, a_{i+1} \in A_{i+1}, \ldots, a_n \in A_n$ such that $(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n) \in C$. In other words, $\mathbb{C}$ is a subdirect product if and only if, for every $i \in \{1, \ldots, n\}$, $\mathrm{pr}_i C = A_i$.*

We will write an instance of a multi-sorted constraint satisfaction problem in the form $(V; \{\mathbb{A}_v \mid v \in V\}; C_1, \ldots, C_q)$ and mean that every constraint relation R_i is a subuniverse of the direct product $\prod_{v \in S_i} \mathbb{A}_v$. Moreover, since every problem instance can be reduced in polynomial time to one that satisfies the property that for any $v \in V$, $\mathrm{pr}_v R_i = \mathrm{pr}_v R_j$ whenever $v \in S_i, S_j$, we may assume that every constraint relation R_i is a subdirect product of $\{\mathbb{A}_v \mid v \in S_i\}$.

As above, a problem instance $(V; \mathbb{A}; C_1, \dots, C_q)$ from the class CSP(Inv Clo($\mathbb{A}$)) can be treated as a particular case of a multi-sorted constraint satisfaction problem. Unfortunately, the converse reduction is not so easy, because there is no general natural way to extend basic operations of algebras $\mathbb{A}_1, \dots, \mathbb{A}_n$ to the union of their universes.

To use multi-sorted problems effectively we need more definitions.

Definition 22 *Let $\mathbb{A} = (A; F)$ be an algebra.*

- *An equivalence relation θ on A is called a* congruence *of $\mathbb{A}$ if each basic operation of $\mathbb{A}$ preserves θ.*
- *The set $A/_\theta$ of all congruence classes of θ is called the* factor set *modulo θ. The equivalence class containing an element $a \in A$ is denoted by $a/_\theta$.*
- *The algebra $\mathbb{A}/_\theta = (A/_\theta; F)$ where each (k-ary) $f \in F$ is interpreted on $A/_\theta$ through the formula*

$$f(a_1/_\theta, \dots, a_k/_\theta) = f(a_1, \dots, a_k)/_\theta, \quad a_1/_\theta, \dots, a_k/_\theta \in A/_\theta,$$

  is called the factor algebra *of $\mathbb{A}$ modulo θ.*

We remark that the factor algebra is a homomorphic image of the original algebra where the homomorphism φ is defined as $\varphi: a \mapsto a/_\theta$.

A congruence θ is said to be *proper* if θ equals neither the total nor the identity relation. An algebra is said to be *simple* if it has no proper congruences.

Definition 23 *Let $P = (V; \{\mathbb{A}_v \mid v \in V\}; C_1, \dots, C_q)$ be an instance of a multi-sorted constraint satisfaction problem and let θ_v be a congruence of $\mathbb{A}_v$, for each $v \in V$. By the* factor problem *of P modulo $\overline{\theta} = \{\theta_v \mid v \in V\}$ we mean the instance $(V; \{\mathbb{A}/_{\theta_v} \mid v \in V\}; C_1/_{\overline{\theta}}, \dots, C_q/_{\overline{\theta}})$ where, for each $i \in \{1, \dots, q\}$, $C_i/_{\overline{\theta}} = (S_i, R_i/_{\overline{\theta}})$ and a tuple $(a_v/_{\theta_v})_{v \in S_i} \in R_i/_{\overline{\theta}}$ if and only if $(a_v)_{v \in S_i} \in R_i$. If only one congruence θ_w of $\{\theta_v \mid v \in V\}$ is distinct from the identity relation, then we will also write $C_i/_{\theta_w}$.*

A factor problem modulo $\overline{\theta}$ is said to be proper *if at least one of $\{\theta_v \mid v \in V\}$ is proper.*

The general scheme for proving our main result will be the following. Suppose we are going to prove tractability of an algebra $\mathbb{A}$. Then we prove first that any multi-sorted problem instance $(V; \{\mathbb{A}_v \mid v \in V\}; C_1, \dots, C_q)$ where all $\mathbb{A}_v$ are simple factors of $\mathbb{A}$ is tractable. This forms the induction base. Then we prove that if all proper factor problems of a multi-sorted instance $P = (V; \{\mathbb{B}_v \mid v \in V\}; C_1, \dots, C_q)$ are tractable and P satisfies a certain additional condition (usually some kind of consistency), then P is tractable itself. Using this fact, we deduce, by induction, that any instance of the form $(V; \mathbb{A}, \dots, \mathbb{A}; C_1, \dots, C_q)$ is tractable.

3 Conservative groupoids

3.1 Preliminary remarks

In this section we shall prove Theorem 1.

First we show that a non-commutative conservative groupoid is NP-complete. Let $\mathbb{A} = (A, *)$ be a conservative groupoid, that is $a * b \in \{a, b\}$ for all $a, b \in A$. We may treat $\mathbb{A}$ as a digraph where the set of nodes is A and a pair (a, b) is an edge if and only if $a * b = b * a = b$. The set of all edges is denoted by E.

Since $\mathbb{A}$ is conservative, every two-element subset of A is a two-element subalgebra whose clone of term operations is generated by an idempotent binary operation. Therefore, by [19], it is term equivalent either to a two-element semilattice or to an algebra all of whose term operations are essentially unary. If $\mathbb{A}$ is not commutative, then for some $a, b \in A$ neither $(a, b) \in E$, nor $(b, a) \in E$, so $(\{a, b\}, *|_{\{a,b\}})$ is of the latter form, and by Corollary 19, $\mathbb{A}$ is NP-complete.

The rest of this section is devoted to the proof that every commutative conservative groupoid is tractable. We will refer to such groupoids as cc-groupoids.

So, in the following we will assume that $\mathbb{A}$ is commutative, and hence for any $a, b \in A$ either (a, b), or (b, a) belongs to E. This implies that the digraph (A, E) is a *tournament* and so consists of a linear ordered family of strongly connected components. We denote the greatest strongly-connected component in this ordering by $\overline{\mathbb{A}}$. We will write $a \leq b$ whenever $a * b = b * a = b$. The relation $\leq$ is a quasi-order on $\mathbb{A}$.

Let $P = (V; \{D_v \mid v \in V\}; C_1, \ldots, C_q)$ be a multi-sorted constraint satisfaction problem, and let $V' \subseteq V$. A *partial solution* of P on V' is a function $\phi: V' \to \bigcup_{v \in V'} D_v$ such that, for any $i \leq q$, $\phi(S_i \cap V') \in \mathrm{pr}_{S_i \cap V'} R_i$. Let us denote the set of all partial solutions on V' (considered as tuples) by $P_{V'}$. In other words,

$$P_{V'} = \bigcap_{l \in \{1,\ldots,q\}} \left(\mathrm{pr}_{V' \cap S_l} R'_l \times \prod_{v \in V' - S_l} \mathbb{A}_v \right).$$

We will call a problem instance P *k-minimal* if, for any k-element subset V' of V, and any $l \in \{1, \ldots, q\}$, we have $\mathrm{pr}_{V' \cap S_l} R_l = \mathrm{pr}_{V' \cap S_l} P_{V'}$.

Any problem instance P can be transformed to an equivalent k-minimal instance P'. To do this we just repeat the following procedure until the obtained problem instance coincides with the previous one: for each three-element set $V' \subseteq V$, calculate the relation $P_{V'}$ and, then, throw out "superfluous" tuples from all the constraint relations. It is easily seen that P' is k-minimal, and the time complexity of this procedure is $O(m^2 n^k)$, where n is the number of variables and m is the total number of tuples in the constraint relations.

Theorem 1 follows from the next theorem which is proved in the next two subsections.

Theorem 24 *Let $P = (V; \{\mathbb{A}_v \mid v \in V\}; C_1, \ldots, C_q)$ be a multi-sorted constraint satisfaction problem where every $\mathbb{A}_v$ is a (non-empty) commutative conservative groupoid. If P is 3-minimal and for any $u, v, w \in V$ there is $j \in \{1, \ldots, q\}$ with $u, v, w \in S_j$, then P has a solution.*

3.2 Simple conservative groupoids

First, we prove Theorem 24 assuming that all the groupoids occurring in the instance are simple.

Lemma 25 *A conservative groupoid $\mathbb{A}$ is simple if and only if, for any proper subset $B \subseteq A$ containing at least two elements, there exist $a, b \in B$ and $c \notin B$ such that $a * c = c * a = c$ and $b * c = c * b = b$.*

Proof. First assume that $\mathbb{A}$ is not simple, and that B is a nontrivial equivalence class of a proper congruence θ. Then we have $a * c \equiv b * c \pmod{\theta}$ for any $a, b \in B$, $c \in A \backslash B$. This means that either $a * c = b * c = c$ or $a * c = a, b * c = b$, which gives the result.

Conversely, let $B \subseteq A$ be such that $|B| \geq 2$ and, for any $c \in A \backslash B$, either $a * c = a$ for all $a \in B$, or $a * c = c$ for all $a \in B$ holds. Then define an equivalence relation θ through the following rule: $(a, b) \in \theta$ if and only if $a = b$ or $a, b \in B$. It is easy to check that θ is a proper congruence of $\mathbb{A}$, and hence $\mathbb{A}$ is not simple. □

Corollary 26 *If $\mathbb{A}$ is simple and $|A| > 2$ then it is strongly connected.*

Lemma 27 *Let $\mathbb{D}$ be a strongly connected subgroupoid of $\mathbb{A}$. Then for every $a, b \in \mathbb{D}$ there exists $n \in \mathbb{N}$ such that the equation*

$$(\dots(a * x_1) * x_2 * \dots * x_{n-1}) * x_n = b$$

is solvable.

Proof. The lemma follows straightforwardly from the definition of strongly connectedness. □

Lemma 28 *Let $\mathbb{D}$ be a subdirect product of cc-groupoids $\mathbb{A}_1, \dots, \mathbb{A}_n$. Then for any $a_1 \in \overline{\mathbb{A}}_1$ there exist $a_2 \in \overline{\mathbb{A}}_2, \dots, a_n \in \overline{\mathbb{A}}_n$ such that $(a_1, a_2, \dots, a_n)^T \in \mathbb{D}$.*

Proof. Since $\mathbb{D}$ is a subdirect product there exist $(a_{11}, \dots, a_{n1})^T, \dots, (a_{1n}, \dots, a_{nn})^T \in \mathbb{D}$ such that $a_{ii} \in \overline{\mathbb{A}}_i$ for all $i \in \{1, \dots, n\}$. Then the element

$$\left(\cdots \left(\left(\begin{array}{c} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{array} \right) * \left(\begin{array}{c} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{array} \right) \right) * \cdots \right) * \left(\begin{array}{c} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{array} \right) = \left(\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array} \right)$$

belongs to $\mathbb{D} \cap (\overline{\mathbb{A}}_1 \times \dots \times \overline{\mathbb{A}}_n)$.

Further, as $\overline{\mathbb{A}}_1$ is strongly connected, there exist $c_1, \dots, c_l \in A_1$ such that $(\dots(b_1 * c_1) * \dots) * c_l = a_1$. Let $c_{12}, \dots, c_{1n}, \dots, c_{l2}, \dots, c_{ln}$ be such that $(c_1, c_{21}, \dots, c_{n1})^T, \dots, (c_l, c_{2l}, \dots, c_{nl})^T \in \mathbb{D}$. Setting $a_i = (\dots(b_i * c_{i1}) * \dots) * c_{il}$, $i \in \{2, \dots, n\}$ we obtain

$$\left(\dots \left(\left(\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array} \right) * \left(\begin{array}{c} c_1 \\ c_{21} \\ \vdots \\ a_{n1} \end{array} \right) \right) * \dots \right) * \left(\begin{array}{c} c_l \\ c_{2l} \\ \vdots \\ c_{nl} \end{array} \right) = \left(\begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_n \end{array} \right)$$

where $a_1 \in \overline{\mathbb{A}}_1, \dots, a_n \in \overline{\mathbb{A}}_n$ as required. $\Box$

Lemma 29 *Let $\mathbb{D}$ be a subdirect product of strongly connected cc-groupoids $\mathbb{A}_1, \mathbb{A}_2$ and assume there exists an element $a \in A_1$ such that $\{a\} \times A_2 \subseteq \mathbb{D}$. Then $\mathbb{D} = A_1 \times A_2$.*

Proof. We prove by induction that $\{c\} \times A_2 \subseteq \mathbb{D}$ for every $c \in A_1$. The inclusion $\{a\} \times A_2 \subseteq \mathbb{D}$ forms the induction base. Further, suppose that there is $d \in A_1$ such that $d \leq c$ and $\{d\} \times A_2 \subseteq \mathbb{D}$. Take $b \in A_2$. For a certain $b_1 \in A_2$, $\left(\begin{array}{c} c \\ b_1 \end{array} \right) \in \mathbb{D}$ and by Lemma 27 there exist $b_2, \dots, b_l \in A_2$ such that $(\dots(b_1 * b_2) * \dots) * b_l = b$. It is easy to see that

$$\left(\dots \left(\left(\begin{array}{c} c \\ b_1 \end{array} \right) * \left(\begin{array}{c} d \\ b_2 \end{array} \right) \right) * \dots \right) * \left(\begin{array}{c} d \\ b_l \end{array} \right) = \left(\begin{array}{c} c \\ b \end{array} \right) \in \mathbb{D}.$$

Since A_1 is strongly connected, this proves the lemma. $\Box$

Our first purpose is to show that a subdirect product of simple cc-groupoids has a very restricted form.

Proposition 30 *Let $\mathbb{D}$ be a subdirect product of simple strongly connected cc-groupoids $\mathbb{A}_1, \mathbb{A}_2$. Then $\mathbb{D}$ is either the graph of a bijective mapping from A_1 to A_2, or $A_1 \times A_2$.*

Proof. Notice first, that if $\mathbb{D}$ is the graph of a mapping $\pi : A_1 \to A_2$, then the relation θ: $(a, b) \in \theta \iff \pi(a) = \pi(b)$ is a congruence of A_1 and, since $\mathbb{A}_1$ is simple, π is a bijection. The same holds if $\mathbb{D}$ is the graph of a mapping from A_2 onto A_1.

For $a \in A_1$ by B_a we denote the set $\{b \mid (a, b)^T \in \mathbb{D}\}$. Suppose $\mathbb{D}$ is not the graph of a bijective mapping. This means that for some $a_1, a_2 \in A_1$ we have $B_{a_1} \cap B_{a_2} \neq \varnothing$. We prove that there exists $c \in A_1$ such that

$$B_c \subseteq \bigcup_{a \in A_1 \setminus \{c\}} B_a.$$

Let $B \subseteq A_1$ be a maximal set such that

$$C = \bigcap_{a \in B} B_a \neq \varnothing$$

and $b \in C$. Since A_1 is simple, there exist $c \in A_1 \backslash B$ and $a_1, a_2 \in B$ such that $a_1 \leq c$, $c \leq a_2$, by Lemma 25. Suppose $d \in B_c \backslash (B_{a_1} \cup B_{a_2})$. We have two cases.

Case 1. $b \leq d$.

In this case $\begin{pmatrix} a_2 \\ b \end{pmatrix} * \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a_2 \\ d \end{pmatrix}$ holds, which contradicts the assumption $d \in B_c \backslash (B_{a_1} \cup B_{a_2})$.

Case 2. $d \leq b$.

Analogously, we have $\begin{pmatrix} a_1 \\ b \end{pmatrix} * \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} c \\ b \end{pmatrix}$, so $b \in B_c$, which contradicts the maximality of B.

Thus, $B_c \subseteq B_{a_1} \cup B_{a_2}$.

In the remaining part we prove by induction on the cardinality of A_1, A_2, that $\mathbb{D} = \mathbb{A}_1 \times \mathbb{A}_2$. The induction base, that is the case $|A_1| = |A_2| = 1$, is obvious. Suppose that the assertion of the proposition holds for any subdirect product $\mathbb{D}'$ of simple commutative conservative groupoids $\mathbb{A}'_1, \mathbb{A}'_2$ such that either $|A'_1| < |A_1|$, or $|A'_2| < |A_2|$.

Define a sequence $\mathbb{B}_0, \mathbb{B}_1, \ldots, \mathbb{B}_k$ of groupoids and a sequence of congruences $\theta_0, \theta_1, \ldots, \theta_k$ where θ_i is a congruence of $\mathbb{B}_i$ through the following rules.

1) $\mathbb{B}_0$ is the greatest strongly connected component of $\mathbb{A}_1 \backslash \{c\}$.

2) Suppose that $\mathbb{B}_i$ is already defined. Let θ_i be its maximal congruence or the identity relation if $\mathbb{B}_i$ is simple.

3) If $\mathbb{B}_i$ is a singleton, then $k = i$ and the process stops. Otherwise set $\mathbb{B}_{i+1}$ to be the greatest strictly connected component of an equivalence class of θ_i.

Further, set $\mathbb{B}'_i = \mathbb{B}_i/_{\theta_i}$ and

$$\mathbb{D}_i = \{(a, b) \mid a = a'/_{\theta_i} \text{ where } (a', b) \in \mathbb{D}\} \subseteq \mathbb{B}'_i \times \mathbb{A}_2.$$

As was proved, for every $b \in \mathbb{A}_2$, there exists $a \in \mathbb{B}_0$ such that $(a, b) \in \mathbb{D}$. Further, since $\mathbb{B}'_0$ is a simple algebra, we obtain that $\mathbb{D}_0$ is either the graph of a bijective mapping, or $\mathbb{D}_0 = \mathbb{B}'_0 \times \mathbb{A}_2$.

Suppose that $i-1$ is the greatest number such that (i) for every $b \in \mathbb{A}_2$ there exists $a \in \mathbb{B}_{i-1}$ where $(a, b) \in \mathbb{D}$ and (ii) $\mathbb{D}_{i-1} = \mathbb{B}'_{i-1} \times \mathbb{A}_2$. Then, for any $a' \in \mathbb{B}'_{i-1}$ we have $\{a'\} \times \mathbb{A}_2 \subseteq \mathbb{D}_{i-1}$, that is for every $b \in \mathbb{A}_2$ there exists $a \in \mathbb{B}_i$ such that $(a, b) \in \mathbb{D}$. Note that if conditions (i),(ii) fail already for $\mathbb{B}_0$, then we may put $i = 0$ in the following. As $\mathbb{B}'_i$ is simple and $i-1$ is the greatest number satisfying the properties (i) and (ii), $\mathbb{D}_i$ must be the graph of a bijective mapping $\pi' \colon B'_i \to A_2$. Define a mapping $\pi \colon B_i \to A_2$ through the rule: $\pi(a) = \pi'(a/_{\theta_i})$. Then we have

$$\mathbb{D} \cap (\mathbb{B}_i \times \mathbb{A}_2) = \left\{ \begin{pmatrix} a \\ \pi(a) \end{pmatrix} \mid a \in \mathbb{B}_i \right\}.$$

Denote by C_b the set $\{a \in A_1 \mid (a, b)^T \in \mathbb{D}\}$. Since $\mathbb{D}$ is not the graph of a mapping from A_2 to A_1, there are $b_1, b_2 \in A_2$ such that $C_{b_1} \cap C_{b_2} \neq \varnothing$. Let $C \subseteq A_2$ be a maximal set such that $\bigcap_{b \in C} C_b \neq \varnothing$ and, say, a belongs to this intersection. If $C = A_2$ then $\{a\} \times \mathbb{A}_2 \subseteq \mathbb{D}$ and by Lemma 29 we are done. Otherwise, simplicity of $\mathbb{A}_2$ implies that there are $b \in A_2 \backslash C$ and $b_1, b_2 \in C$ such that $b_1 \leq b$, $b \leq b_2$. Arguing as above we obtain that $C_b \subseteq C_{b_1} \cup C_{b_2}$. Take an element $d \in C_b \cap B_i$. Without loss of generality we may assume $d \in C_{b_1}$. Therefore $\begin{pmatrix} d \\ b \end{pmatrix}, \begin{pmatrix} d \\ b_1 \end{pmatrix} \in \mathbb{D}$, which contradicts the assumption.

We have proved that $\mathbb{D}_k = \mathbb{B}'_k \times \mathbb{A}_2$. Since $\mathbb{B}_k$ is a singleton, say, $\mathbb{B}_k = \{a\}$ this implies $\mathbb{B}_k = \mathbb{B}'_k$, that is $\{a\} \times \mathbb{A}_2 \subseteq \mathbb{D}$.

To complete the proof we just have to apply Lemma 29. □

Note that the second part of the proof is valid for a subdirect product of arbitrary (not just simple) cc-groupoids.

Corollary 31 *Let $\mathbb{D}$ be a subdirect product of cc-groupoids $\mathbb{A}_1, \mathbb{A}_2$ where $\mathbb{A}_2$ is simple strongly connected and $\mathbb{D}$ is not the graph of any mapping $\pi: A_1 \to A_2$. Then there exists $a \in \mathbb{A}_1$ such that $\{a\} \times \mathbb{A}_2 \subseteq \mathbb{D}$.*

To prove this we just put $\mathbb{B}_0$ equal to $\overline{\mathbb{A}}_1$.

Lemma 32 *Let $\mathbb{D}$ be a subdirect product of simple strongly connected cc-groupoids $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$. If $A_i \times A_j \subseteq \mathrm{pr}_{i,j} D$ for every $i, j \in \{1, 2, 3\}$, then $\mathbb{D} = \mathbb{A}_1 \times \mathbb{A}_2 \times \mathbb{A}_3$.*

Proof. Suppose, without loss of generality, that $|A_1| \leq |A_2| \leq |A_3|$. For $a \in A_1$, $b \in A_2$ set

$$D_a = \{(b_2, b_3) \mid (a, b_2, b_3) \in \mathbb{D}\},$$
$$D'_b = \{(b_1, b_3) \mid (b_1, b, b_3) \in \mathbb{D}\}.$$

Let us assume $D_a = A_2 \times A_3$ for some $a \in A_1$, $a' \geq a$ and $\begin{pmatrix} a' \\ b_2 \\ b_3 \end{pmatrix} \in \mathbb{D}$. Since $\mathbb{A}_2, \mathbb{A}_3$ are strongly connected, for any $\begin{pmatrix} a_2 \\ a_3 \end{pmatrix} \in \mathbb{A}_2 \times \mathbb{A}_3$, there are $\begin{pmatrix} a_{21} \\ a_{31} \end{pmatrix}, \ldots, \begin{pmatrix} a_{2k} \\ a_{3k} \end{pmatrix} \in \mathbb{A}_2 \times \mathbb{A}_3$ such that

$$\left(\cdots \left(\left(\begin{pmatrix} a' \\ b_2 \\ b_3 \end{pmatrix} * \begin{pmatrix} a \\ a_{21} \\ a_{31} \end{pmatrix} \right) \right) * \begin{pmatrix} a \\ a_{22} \\ a_{32} \end{pmatrix} \cdots \right) * \begin{pmatrix} a \\ a_{2k} \\ a_{3k} \end{pmatrix} = \begin{pmatrix} a' \\ a_2 \\ a_3 \end{pmatrix}.$$

Furthermore, strongly connectedness of $\mathbb{A}_1$ implies $\mathbb{A}_1 \times \mathbb{A}_2 \times \mathbb{A}_3 \subseteq \mathbb{D}$.

Now, for every $a \in A_1$, let the set D_a be the graph of a bijective mapping $\pi_a: A_3 \to A_2$ and, analogously, let D'_b be the graph of $\pi'_b: A_3 \to A_1$. This immediately implies $|A_1| = |A_2| = |A_3| = k$ and $|D_a| = |D'_b| = k$ for all $a \in A_1, b \in A_2$. Moreover, by the assumption we have $|\mathrm{pr}_{2,3}\mathbb{D}| = |\mathrm{pr}_{1,3}\mathbb{D}| = k^2$, which means $D_a \cap D_{a'} = \varnothing$, $D'_b \cap D'_{b'} = \varnothing$ whenever $a \neq a'$, $b \neq b'$, $a, a' \in A_1$, $b, b' \in A_2$.

To finish the proof take $b_1, c_1 \in A_1$, $b_2, c_2 \in A_2$ such that $b_1 * c_1 = c_1$, $c_2 * b_2 = b_2$ and $b_1 \neq c_1$, $b_2 \neq c_2$. There exist $b_3, c_3 \in A_3$ such that $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \in \mathbb{D}$. Consequently,

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} * \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} c_1 \\ b_2 \\ b_3 * c_3 \end{pmatrix} \in \mathbb{D}.$$

We have $\pi_{c_1}(c_3) = c_2 \neq b_2 = \pi_{c_1}(b_3 * c_3)$ and $\pi_{b_2}(b_3) = b_1 \neq c_1 = \pi_{b_2}(b_3 * c_3)$. These two inequalities mean that $b_3 * c_3 \neq c_3$ and $b_3 * c_3 \neq b_3$, a contradiction. $\square$

Lemma 33 *Let D be a subdirect product of conservative commutative groupoids $\mathbb{A}_1, \ldots, \mathbb{A}_n$. If $A_i \times A_j \subseteq \mathrm{pr}_{i,j}\mathbb{D}$ for every $i, j \in \{1, 2, \ldots, n\}$, then $\mathbb{D} = \mathbb{A}_1 \times \ldots \times \mathbb{A}_n$.*

Proof. We prove the lemma by induction. The induction base $n = 2, 3$ was proved in Proposition 30 and Lemma 32. Suppose that the lemma holds for each number less than n. Take $a \in A_1$ and denote by D_a the set $\{(b_2, \ldots, b_n) \mid (a, b_2, \ldots, b_n)^T \in \mathbb{D}\}$. By Lemma 32 $A_1 \times A_i \times A_j \subseteq \mathrm{pr}_{1,i,j}\mathbb{D}$ for every $2 \leq i, j \leq n$. Then $A_i \times A_j \subseteq \mathrm{pr}_{i,j} D_a$ and by the induction hypothesis $D_a = A_2 \times \ldots \times A_n$ so the lemma is proved. $\square$

Definition 34 *We say that a relation $R \subseteq A_1 \times \ldots \times A_n$ is* almost trivial *if there exists a partition $\{I_1, \ldots, I_k\}$ of the set $\{1, 2, \ldots, n\}$, such that*

$$R = \mathrm{pr}_{I_1} R \times \ldots \times \mathrm{pr}_{I_k} R$$

where $\mathrm{pr}_{I_j} R = \{(a_{i_1}, \pi_{i_2}(a_{i_1}), \ldots, \pi_{i_l}(a_{i_1})) \mid a_{i_1} \in A_i\}$, $I_j = \{i_1, \ldots, i_l\}$, for certain bijective mappings $\pi_{i_2}: A_{i_1} \to A_{i_2}, \ldots, \pi_{i_l}: A_{i_1} \to A_{i_l}$.

The last lemma gives quite a strong restriction on subdirect products of simple cc-groupoids.

Corollary 35 *A subdirect product of simple strongly connected cc-groupoids is an almost trivial relation.*

Now, suppose that $P = (V; \{\mathbb{A}_v \mid v \in V\}; C_1, \ldots, C_q)$, where each $\mathbb{A}_v$ is a simple cc-groupoid, is 3-minimal. Since each constraint relation is almost trivial, there is a partition θ_i of each constraint scope S_i, $i \leq q$, as stated in Definition 34. Variables v, w are said to be *linked* if they belong to the same block of θ_i for some i. Denote by θ the least equivalence relation on V that contains all pairs of linked variables.

Since P is assumed to be 3-minimal, for any $v \in V$, every $a \in A_v$ is a partial solution for $\{v\}$. It follows immediately from the definition of almost trivial relations that, for any pair $v, w \in V$ from the same block of θ, there exists $\pi_{v,w}: A_v \to A_w$ such that the only extension of a partial solution $a \in A_v$ on v to a partial solution on v, w is $(a, \pi_{v,w}(a))$. Since P is 3-minimal, for any variables u, v, w from the same block of θ, we have $\pi_{u,v} \circ \pi_{v,w} = \pi_{u,w}$.

Therefore a solution to P can be constructed as follows. Let $V_1, \ldots, V_k$ be the blocks of θ.

1) Choose a representative v_i from each block V_i of θ.
2) Assign v_i an arbitrary value $a_i \in A_{v_i}$.
3) For each variable from V_i assign the value $\pi_{v_i,v}(a_i)$.

3.3 General conservative commutative groupoids

In this section we prove Theorem 24 in the case of multi-sorted problem instances over arbitrary cc-groupoids. Take a 3-minimal problem instance $P = (V; \{\mathbb{A}_v \mid v \in V\}$ where each $\mathbb{A}_v$ is a conservative groupoid. First of all we show that P can be transformed to another 3-minimal problem instance which satisfies some additional conditions. The result of this transformation is given in Proposition 41.

Let $\mathbb{D}$ be a subdirect product of cc-groupoids $\mathbb{A}_1, \ldots, \mathbb{A}_n$. In general, the projections $\mathrm{pr}_{\{i,j\}}\mathbb{D}$, $\mathrm{pr}_{\{i,j,k\}}\mathbb{D}$, $i, j, k \in \{1, \ldots, n\}$ are not conservative groupoids. However, one may still define a quasi-order on $\mathrm{pr}_{\{i,j\}}\mathbb{D}$, $\mathrm{pr}_{\{i,j,k\}}\mathbb{D}$ by the rule $a \le b \iff a*b = b$. Moreover, $a * (a * b) = b * (a * b) = a * b$, for any a, b, which implies that $a, b \le a * b$. Therefore there exists a greatest strongly connected component. Let $\mathbb{D}_{i,j}$, $\mathbb{D}_{i,j,k}$ denote the greatest strongly connected components of $\mathrm{pr}_{\{i,j\}}\mathbb{D}$, $\mathrm{pr}_{\{i,j,k\}}\mathbb{D}$ respectively.

As a result of the transformation we get a problem instance P' such that, for any three-element set $V' \subseteq V$, $P'_{V'}$ is strongly connected.

As is easily seen the analogue of Lemma 27 holds for products of cc-groupoids.

Lemma 36 *Let $\mathbb{D}$ be a subdirect product of cc-groupoids. If $\mathbb{D}$ is strongly connected, then for any $a, b \in \mathbb{D}$, there exists n such that the equation*

$$(\ldots(a * x_1) * x_2 * \ldots) * x_n = b$$

has a solution.

Lemma 37 *(1) Let $(a_1, a_2) \in \mathbb{D}_{1,2}$. Then there are $a_3, \ldots, a_n$ such that $(a_1, \ldots, a_n) \in \mathbb{D}$ and $(a_i, a_j) \in \mathbb{D}_{i,j}$ for all $i, j \in \{1, \ldots, n\}$.*

(2) Let $(a_1, a_2, a_3) \in \mathbb{D}_{1,2,3}$. Then there are $a_4, \ldots, a_n$ such that $(a_1, \ldots, a_n) \in \mathbb{D}$ and $(a_i, a_j, a_k) \in \mathbb{D}_{i,j,k}$ for all $i, j, k \in \{1, \ldots, n\}$.

Proof. (1) Choose $\mathbf{a}_{ij}$ such that $\mathrm{pr}_{i,j}\mathbf{a}_{i,j} \in \mathbb{D}_{i,j}$ for every pair $i, j \in \{1, \ldots, n\}$. Then

$$\mathbf{b} = \prod_{\substack{i,j \in \{1,\ldots,n\} \\ i \neq j}} \mathbf{a}_{i,j} = (b_1, \ldots, b_n)$$

satisfies the condition $(b_i, b_j) \in \mathbb{D}_{i,j}$ for any i, j. Since $\mathbb{D}_{1,2}$ is strongly connected, by Lemma 36 there are $a_1^0, \ldots, a_1^k \in \mathbb{A}_1$, $a_2^0, \ldots, a_2^k \in \mathbb{A}_2$ such that $\begin{pmatrix} a_1^i \\ a_2^i \end{pmatrix} \in \mathbb{D}_{i,j}$ and

$(\dots(b_1 * a_1^0) * a_1^1 * \dots) * a_1^k = a_1$, $(\dots(b_2 * a_2^0) * a_2^1 * \dots) * a_2^k = a_2$. There are such $\mathbf{a}^0, \dots, \mathbf{a}^k \in \mathbb{D}$ that $\mathrm{pr}_{1,2}\mathbf{a}^i = \begin{pmatrix} a_1^i \\ a_2^i \end{pmatrix}$. The tuple $\mathbf{a} = (\dots(\mathbf{b} * \mathbf{a}^0) * \mathbf{a}^1 * \dots) * \mathbf{a}^k$ is as required.

(2) The proof of (2) is quite similar. □

Lemma 38 *Let $\mathbb{D}$ be subdirect product of cc-groupoids $\mathbb{A}_1, \mathbb{A}_2$ and let $\overline{\mathbb{D}}$ be the greatest strongly connected component of $\mathbb{D}$. Then the greatest strongly connected component of $\mathbb{A}_1$ equals $\mathrm{pr}_1\overline{\mathbb{D}}$.*

Proof. Obviously, $\overline{\mathbb{D}} \subseteq \overline{\mathbb{A}}_1 \times \mathbb{A}_2$. Take $a \in \overline{\mathbb{A}}_1$ and $\begin{pmatrix} b \\ c \end{pmatrix} \in \overline{\mathbb{D}}$. Since $\overline{\mathbb{A}}_1$ is strongly connected, there are $\begin{pmatrix} b_1 \\ c_1 \end{pmatrix}, \dots, \begin{pmatrix} b_s \\ c_s \end{pmatrix} \in \mathbb{D}$ such that $(\dots(b * b_1) * b_2 \dots) * b_s = a$. Therefore

$$\begin{pmatrix} b' \\ c' \end{pmatrix} = \left(\dots\left(\begin{pmatrix} b \\ c \end{pmatrix} * \begin{pmatrix} b_1 \\ c_1 \end{pmatrix}\right) * \begin{pmatrix} b_2 \\ c_2 \end{pmatrix} \dots\right) * \begin{pmatrix} b_s \\ c_s \end{pmatrix}$$

belongs to $\overline{\mathbb{D}}$ and $b' = a$. The lemma is proved. □

Making use of Lemma 36 the following generalisation of Lemma 38 can be proved in the same way.

Lemma 39 *Let $\mathbb{D}$ be subdirect product of cc-groupoids $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$ and let $\overline{\mathbb{D}}$ be the greatest strongly connected component of $\mathbb{D}$. Then the greatest strongly connected component of $\mathrm{pr}_{\{1,2\}}\mathbb{D}$ equals $\mathrm{pr}_{\{1,2\}}\overline{\mathbb{D}}$.*

Corollary 40 *If $P = (V; \{\mathbb{A}_v \mid v \in V\}; C_1, \dots, C_q)$ is 3-minimal, $D_{u,v}, D_{u,v,w}$ denote the greatest strongly connected components of $P_{u,v}, P_{u,v,w}$, $u, v, w \in V$, then $P' = (V; \{\overline{\mathbb{A}}_v \mid v \in V\}; C'_1, \dots, C'_q)$ where $S'_l = S_l$ and*

$$\begin{aligned} R'_l &= R_l \cap \prod_{u \in S_l} \overline{\mathbb{A}}_u \cap \bigcap_{u,v \in S_l} \left(D_{u,v} \times \prod_{s \in S_l \setminus \{u,v\}} A_s\right) \\ &\cap \bigcap_{u,v,w \in S_l} \left(D_{u,v,w} \times \prod_{s \in S_l \setminus \{u,v,w\}} A_s\right) \end{aligned}$$

is also 3-minimal.

Proof. By Lemmas 37, 38, and 39, $P'_u = \overline{\mathbb{A}}_u = \mathrm{pr}_u P'_{u,v}$, $P'_{u,v} = D_{u,v} = \mathrm{pr}_{\{u,v\}} P'_{u,v,w} = \mathrm{pr}_{\{u,v\}} D_{u,v,w}$, $P'_{u,v,w} = D_{u,v,w}$, for any $u, v, w \in V$. Since P'_u, $P'_{u,v}$, $P'_{u,v,w}$ are exactly the sets of partial solutions on $\{u\}$, $\{u,v\}$, $\{u,v,w\}$ respectively, 3-minimality of P' follows from 3-minimality of P. □

The required assertion now easily follows from what was proved above.

Proposition 41 *Any 3-minimal problem instance* $P = (V; \{\mathbb{A}_v \mid v \in V\}; C_1, \dots, C_q)$ *where each* A_v *is a cc-groupoid can be reduced in polynomial time to a problem instance* $P' = (V; \{\overline{\mathbb{A}}_v \mid v \in V\}; C'_1, \dots, C'_q)$ *such that:*

- $\overline{\mathbb{A}}_v$ *is the greatest strongly connected component of* $\mathbb{A}_v$*, for each* $v \in V$*;*
- P' *is 3-minimal;*
- *for any* $u, v, w \in V$*,* $P'_u, P'_{u,v}, P'_{u,v,w}$ *are strongly connected;*
- *if* P' *has a solution, then* P *does.*

We need two more auxiliary lemmas.

Lemma 42 *Let* $\mathbb{D}$ *be a subdirect product of strongly connected cc-groupoids* $\mathbb{A}_1, \mathbb{A}_2$ *where* $\mathbb{A}_2$ *is simple. Either* $\mathbb{D} = \mathbb{A}_1 \times \mathbb{A}_2$*, or there is a surjective mapping* $\pi\colon A_1 \to A_2$ *such that* $\mathbb{D} = \{(a, \pi(a)) \mid a \in A_1\}$*.*

Proof. If $\mathbb{D}$ is not the graph of a mapping then we are in the conditions of Corollary 31. Therefore there exists $a \in A_1$ such that $\{a\} \times A_2 \subseteq \mathbb{D}$. Since $\mathbb{A}_1$ is strongly connected, by Lemma 29, we get $\mathbb{D} = \mathbb{A}_1 \times \mathbb{A}_2$. □

Lemma 43 *(1) Let* $\mathbb{D}$ *be a subdirect product of strongly connected cc-groupoids* $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$ *where* $\mathbb{A}_1$ *is simple,* $\mathrm{pr}_{2,3}\mathbb{D}$ *is strongly connected, and* $\mathrm{pr}_{1,2}\mathbb{D} = \mathbb{A}_1 \times \mathbb{A}_2$*,* $\mathrm{pr}_{1,3}\mathbb{D} = \mathbb{A}_1 \times \mathbb{A}_3$*. Then* $\mathbb{D} = \mathbb{A}_1 \times \mathrm{pr}_{2,3}\mathbb{D}$*.*

(2) Let $\mathbb{D}$ *be a subdirect product of strongly connected cc-groupoids* $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \mathbb{A}_4$ *where* $\mathbb{A}_1$ *is simple,* $\mathrm{pr}_{2,3,4}\mathbb{D}$ *is strongly connected, and* $\mathrm{pr}_{1,2}\mathbb{D} = \mathbb{A}_1 \times \mathbb{A}_2$*,* $\mathrm{pr}_{1,3}\mathbb{D} = \mathbb{A}_1 \times \mathbb{A}_3$*,* $\mathrm{pr}_{1,4}\mathbb{D} = \mathbb{A}_1 \times \mathbb{A}_4$*. Then* $\mathbb{D} = \mathbb{A}_1 \times \mathrm{pr}_{2,3,4}\mathbb{D}$*.*

Proof. (1) First, we find $b \in A_2$ and $c \in A_3$ such that $\mathbb{A}_1 \times (b, c)^T \subseteq \mathbb{D}$. For an element $c \in A_3$ set $\mathbb{B}_c = \{(a, b) \mid (a, b, c) \in \mathbb{D}\}$. Take $c \in A_3$ such that $\mathrm{pr}_2\mathbb{B}_c$ is minimal with respect to inclusion. Denote $\mathrm{pr}_2\mathbb{B}_c$ by $\mathbb{C}$ and its greatest strongly connected component $\overline{\mathbb{C}}$. Let us denote $\mathbb{B}_c \cap (\mathbb{A}_1 \times \overline{\mathbb{C}})$ by $\mathbb{B}'_c$. It can be easily shown making use of Lemma 38, that since $\mathbb{A}_1 \times \{c\} \subseteq \mathrm{pr}_{1,3}\mathbb{D}$, $\mathrm{pr}_1\mathbb{B}'_c = \mathbb{A}_1$. By Lemma 42, $\mathbb{B}'_c$ is either $\mathbb{A}_1 \times \overline{\mathbb{C}}$, or the graph of a mapping. If the former possibility holds, then we are done. So, suppose $\mathbb{B}'_c = \{(\pi(b), b) \mid b \in \overline{\mathbb{C}}\}$ for a certain surjective mapping $\pi\colon \overline{C} \to A_1$.

Let us set $\mathbb{A}$ to be the set of all $c' \in \mathbb{A}_3$ such that $\mathbb{B}_{c'} \subseteq \mathbb{B}_c$. Notice that for each $c' \in A$, $\mathbb{B}'_c \subseteq \mathbb{B}_{c'}$, because of minimality of $\mathrm{pr}_2\mathbb{B}_c$ and the assumption that $\mathbb{B}'_c$ is the graph of a mapping. Since $\mathrm{pr}_{1,2}\mathbb{D} = \mathbb{A}_1 \times \mathbb{A}_2$, we have $\mathbb{A} \neq \mathbb{A}_3$. Therefore there is $c'' \in \mathbb{A}_3$ such that $c'' \le c'$ for certain $c' \in \mathbb{A}$ and $\begin{pmatrix} a \\ b \\ c'' \end{pmatrix} \in \mathbb{D}$ and either $b \notin \overline{\mathbb{C}}$, or $\pi(b) \neq a$. Notice that $\mathbb{B}_c = \mathbb{B}_{c'}$ Take also $a' \in \mathbb{A}_1$, $a' < a$ and $\begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} \in \mathbb{D}$ with $b' \in \overline{C}$.

CASE 1. $b \notin \mathbb{C}$.

In this case we have $\begin{pmatrix} a \\ b \\ c'' \end{pmatrix} * \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} a \\ b * b' \\ c' \end{pmatrix} \in \mathbb{D}$. Therefore either $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{B}_c$ or $\begin{pmatrix} a \\ b' \end{pmatrix} \in \mathbb{B}'_c$. In both cases we get a contradiction with assumptions.

CASE 2. $b \in \mathbb{C} - \overline{\mathbb{C}}$.

Then $b * b' = b'$, and we have $\begin{pmatrix} a \\ b' \end{pmatrix} \in \mathbb{B}'_c$, a contradiction.

CASE 3. $\pi(b) \neq a$.

Two subcases are possible.

SUBCASE 3A. $b \leq b'$. This subcase is dealt with as above.

SUBCASE 3B. $b' \leq b$.

We have $\begin{pmatrix} a \\ b \\ c'' \end{pmatrix} * \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} a \\ b \\ c' \end{pmatrix}$, that is $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{B}'_c$. Since $a \neq \pi(b)$ this contradicts the assumption.

To complete the proof, one has to make use of Lemma 36.

(2) By (1), $\mathbb{A}_1 \times \mathrm{pr}_{3,4}\mathbb{D} \subseteq \mathrm{pr}_{1,3,4}\mathbb{D}$. Therefore, a proof of the part (2) can be obtained from the proof of (1) by replacing elements c, c', c'' from $\mathbb{A}_3$ with certain elements $(c,d)^T, (c',d')^T, (c'',d'')^T$ from $\mathrm{pr}_{3,4}\mathbb{D}$. □

Now we are in a position to prove Theorem 24.

Proof (of Theorem 24). Take a 3-minimal problem instance $P = (V; \mathbb{A}_1, \ldots, \mathbb{A}_n; C_1, \ldots, C_k)$ where $n = |V|$ and $\mathbb{A}_1, \ldots, \mathbb{A}_n$ are cc-groupoids. We prove by induction on the number of elements in $\mathbb{A}_1, \ldots, \mathbb{A}_n$ that P has a solution.

INDUCTION BASE. All $\mathbb{A}_1, \ldots, \mathbb{A}_n$ are simple or 1-element. The required result follows from the result of the previous subsection.

INDUCTION STEP. Suppose that statement of the theorem holds for all problem instances $P' = (V; \mathbb{B}_1, \ldots, \mathbb{B}_n; C'_1, \ldots, C'_k)$ where $|B_1| \leq |A_1|, \ldots, |B_n| \leq |A_n|$ and at least one inequality is strict. By Proposition 41, we may assume that P satisfies the first three of the properties listed there.

Let us assume that $\mathbb{A}_1$ is not simple and θ is a maximal congruence of $\mathbb{A}_1$. By Lemma 42 for any $v \in V - \{1\}$, $P_{1,v}/_\theta$ is either the direct product of $\mathbb{A}_1/_\theta \times \mathbb{A}_v$, or the graph of a surjective mapping $\pi_v \colon \mathbb{A}_v \to \mathbb{A}_1$. Let W be the set of all $v \in V$ such that $P_{1,v}/_\theta$ is the graph of π_v. Consider the factor problem $(V; \{\mathbb{A}_v/_{\theta_v} \mid v \in V\}; C_1/_\theta, \ldots, C_n/_\theta)$ where

$$\theta_v = \begin{cases} \overline{\theta}, & \text{if } v = 1, \\ \ker \pi_v & \text{if } v \in W, \\ \underline{0} & \text{otherwise,} \end{cases}$$

and $\underline{0}$ denotes the equality relation. The factor problem is 3-minimal, therefore by the induction hypothesis it has a solution. Let $(\mathfrak{a}_1, \dots, \mathfrak{a}_n)$ be a solution. Note that if $v \in W$, then $\mathfrak{a}_v$ is treated as a congruence block of $\mathbb{A}_v$, that is, $\mathfrak{a}_v$ is a subset of A_v in this case. Set $P' = (V; \{\mathbb{A}'_v \mid v \in V\}; C'_1, \dots, C'_q)$ where

$$\begin{aligned} \mathbb{A}'_v &= \begin{cases} \mathfrak{a}_v & \text{if } v \in W, \\ \mathbb{A}_v & \text{otherwise;} \end{cases} \\ R'_i &= R_i \cap \left(\prod_{v \in W \cap S_i} \mathfrak{a}_v \times \prod_{v \in S_i - W} \mathbb{A}_v \right). \end{aligned}$$

Finally, we denote by $R_i/\overline{\theta}$ the factor relation of R_i, that is the set $\{(a_v/\theta_v)_{v \in S_i} \mid (a_v)_{v \in S_i} \in R_i\}$.

To complete the proof it suffices to show that P' has a solution. Since $|\mathfrak{a}_1| < |A_1|$, we may apply the induction hypothesis, and so we must only prove that P' is 3-minimal. Taking $u, v, w \in V$, $i \in \{1, \dots, q\}$, we have to show that

$$\mathrm{pr}_{\{u,v,w\} \cap S_i} R'_i = \left(\mathrm{pr}_{\{u,v,w\} \cap S_i \cap W} R_i \cap \prod_{x \in \{u,v,w\} \cap S_i \cap W} \mathfrak{a}_x \right) \times \mathrm{pr}_{(\{u,v,w\} \cap S_i) - W} R_i.$$

Actually by the definition of P', the left side does not exceed the right one. Therefore, if the equality holds, then

$$P'_{u,v,w} = \left(\mathrm{pr}_{\{u,v,w\} \cap W} P_{u,v,w} \cap \prod_{x \in \{u,v,w\} \cap W} \mathfrak{a}_x \right) \times \mathrm{pr}_{\{u,v,w\} - W} P_{u,v,w}$$

and 3-minimality of P' follows from 3-minimality of P.

Let us denote $V_i = \{u, v, w\} \cap S_i$, $W_i = S_i \cap W$, $W'_i = \{u, v, w\} \cap S_i \cap W$, and $W''_i = (\{u, v, w\} \cap S_i) - W$. Since $\mathrm{pr}_{\{x,y\}} R_i/\overline{\theta}$ is the graph of a bijective mapping, for any $x, y \in W_i$, the groupoids $\mathbb{A}_x/\theta_x$, $\mathbb{A}_y/\theta_y$ are isomorphic, and moreover, $\mathrm{pr}_{W_i} R_i/\overline{\theta}$ is isomorphic to $\mathbb{A}_1/\theta$. Therefore we may treat $R_i/\overline{\theta}$ as a subgroupoid of $\mathbb{A}_1/\theta \times \prod_{x \in S_i - W} \mathbb{A}_x$. By Lemma 43,

$$\mathrm{pr}_{V_i \cup \{1\}} R_i/\overline{\theta} = \mathbb{A}_1/\theta \times \mathrm{pr}_{V_i} R_i$$

if $W'_i = \varnothing$, and

$$\mathrm{pr}_{V_i} R_i/\overline{\theta} = \mathbb{A}_1/\theta \times \mathrm{pr}_{W''_i} R_i$$

otherwise. The required equalities follow immediately. $\square$

4 Minimal clones

4.1 Basic results

As was noted in Section 2.2, small tractable clones are amongst the most desirable objects to find. The smallest non-trivial clones, that is those whose only proper subclone is the clone of projections, are called *minimal.* There exist only finitely many minimal clones on a finite set [21] and every clone contains a minimal one. So, a classification of the minimal clones with respect to tractability allows us to strongly restrict the possible forms of the NP-complete subclasses of the CSP, because a minimal clone contained in an NP-complete clone must be NP-complete itself.

There are 6 minimal clones on a 2-element set. They are the clones generated by each of the 2 constant operations, the clone generated by the disjunction operation, the clone generated by the conjunction operation, the clone generated by the affine operation $x - y + z$, and the one generated by the negation operation. So, we see that in the case of a two-element domain a complete description of tractable clones can be given by a description of the tractable minimal clones.

Up to now there is no complete description of minimal clones over a domain having more than three elements. The main result in the general case is the classification of minimal clones into five types given by I.G.Rosenberg [21].

Theorem 44 (Rosenberg's Five Types Theorem) *If $\mathbf{C}$ is a minimal clone, then it is generated by a single operation f which is one of the following types:*

1) a unary operation;

2) a binary idempotent operation;

3) a majority operation;

4) a semiprojection;

5) $x + y + z$ where $+$ is an operation of an elementary 2-group.

For clones of some of these types, their tractability or NP-completeness follows immediately from Theorems 8 and Corollary 19.

Corollary 45 *A minimal clone generated by a unary constant operation or by a majority operation or by an operation $x + y + z$ of an elementary 2-group is tractable.*

Corollary 46 *A minimal clone generated by a semiprojection or by a unary non-constant operation is* NP*-complete.*

Proof. Let $\mathbf{C}$ be a minimal clone generated by an operation f. If f is unary then we may apply Corollary 19, with $B = D$. Suppose f is a semiprojection and let B be an arbitrary 2-element subset of D. Then $f|_B$ is a projection and for every $\varrho \subseteq B^n$ where $n \in \mathbb{N}$, we have $\varrho \in \operatorname{Inv} f = \operatorname{Inv} \mathbf{C}$. This means that for every $g \in \mathbf{C}$, the restriction $g|_B$ is a projection, and Corollary 19 again gives the result. □

Thus the only case left to study is the case of minimal clones generated by a binary operation (for short, binary minimal clones). This binary operation will be denoted by $*$ and we will write $x * y$ instead of $*(x, y)$. Thus the base set endowed with $*$ is a groupoid. A subset T of a groupoid $\mathbb{S}$ is called a *subgroupoid* if T is closed under the operation. A subgroupoid T such that $T \neq S$ is called *proper.* Recall that groupoids $\mathbb{T} = (T; *_T)$ and $\mathbb{S} = (S; *_S)$ are said to be isomorphic if there exists a bijective mapping $\phi: S \to T$ such that $\phi(a *_S b) = \phi(a) *_T \phi(b)$, for any $a, b \in S$.

4.2 Minimal clones on a four-element set

We heavily use the description of binary minimal clones on a four-element set given in [23]. It is shown there, in particular, that there are 2182 binary minimal clones on a four-element set, but some of the corresponding groupoids are isomorphic. So, all those groupoids are divided into 120 classes such that any two groupoids from the same class are isomorphic.

Almost all of those groupoids have a two-element subgroupoid such that $*$ acts like a projection. By Corollary 19 this means that every such groupoid is NP-complete. Only 12 isomorphic classes have no such subgroupoid. Below we give Cayley tables of their representatives.

(a)

$*$	1	2	3	4
1	1	1	1	1
2	1	2	1	2
3	1	1	3	4
4	1	2	4	4

(b)

$*$	1	2	3	4
1	1	1	1	1
2	1	2	1	4
3	1	1	3	4
4	1	4	4	4

(c)

$*$	1	2	3	4
1	1	1	1	1
2	1	2	2	2
3	1	2	3	2
4	1	2	2	4

(d)

$*$	1	2	3	4
1	1	1	1	1
2	1	2	1	1
3	1	1	3	1
4	1	1	1	4

(e)

$*$	1	2	3	4
1	1	1	1	1
2	1	2	1	1
3	1	1	3	3
4	1	1	3	4

(f)

$*$	1	2	3	4
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
4	1	2	3	4

(g)

$*$	1	2	3	4
1	1	1	1	1
2	1	2	2	2
3	1	2	3	3
4	1	2	3	4

(h)

$*$	1	2	3	4
1	1	1	1	1
2	1	2	2	4
3	1	2	3	3
4	1	4	3	4

(i)

$*$	1	2	3	4
1	1	1	1	4
2	1	2	1	2
3	1	1	3	3
4	4	2	3	4

(j)

$*$	1	2	3	4
1	1	1	1	4
2	1	2	1	2
3	1	1	3	4
4	4	2	4	4

(k)

$*$	1	2	3	4
1	1	1	1	4
2	1	2	2	2
3	1	2	3	3
4	4	2	3	4

(l)

$*$	1	2	3	4
1	1	1	1	4
2	1	2	2	2
3	1	2	3	4
4	4	2	4	4

We are going to prove that all of these 12 groupoids are tractable.

Groupoids (c),(d),(e),(f),(g) are semilattices and therefore tractable by Theorem 8. Groupoids (h),(k) and (l) are conservative and commutative and therefore tractable by Theorem 1. Thus we have to study four remaining groupoids: (a),(b),(i) and (j).

For the remainder of this section we shall assume that every instance of a multi-sorted CSP is 1-minimal, and hence that constraint relations are subdirect products of cc-groupoids.

Proposition 47 *Groupoids (a),(b) are tractable.*

Proof. In both cases, every subgroupoid S has an element a_S such that $a_S * b = b * a_S = a_S$, for all $b \in S$, (such an element is called a *zero-element*). Hence, for a given problem instance $P = (V; \{S_v \mid v \in V\}; C_1, \dots, C_k)$, where each S_v is a certain subgroupoid, the function $\psi: V \to \{1,2,3,4\}$, $\psi(v) = a_{S_v}$ is a solution. □

Proposition 48 *Groupoid (j) is tractable.*

Proof. Denote the groupoid (j) by $\mathbb{A}$. All subgroupoids of $\mathbb{A}$, except the subgroupoid $\mathbb{B}$ with universe $\{1,2,4\}$ and $\mathbb{A}$ itself, have a zero-element. This means that those two subgroupoids give the only trouble for our proof.

CLAIM 1. If D is a subdirect power of $\mathbb{B}$, then D is an almost trivial relation.

This follows immediately from Corollary 35, because $\mathbb{B}$ is a simple strongly connected conservative commutative groupoid.

CLAIM 2. If D is an nth subdirect power of $\mathbb{A}$, then for any $a \in B$ there are $a_2, \dots, a_n \in B$ such that $(a, a_2, \dots, a_n) \in D$.

Take $\mathbf{b}_1 = (a, b_{12}, \dots, b_{1n}), \mathbf{b}_2 = (b_{21}, \dots, b_{2n}), \dots, \mathbf{b}_n = (b_{n1}, \dots, b_{nn}) \in D$ such that $b_{22}, \dots, b_{nn} \in B$. Then the tuple

$$\mathbf{b}' = (\dots(\mathbf{b}_1 * \mathbf{b}_2) * \mathbf{b}_3 \dots) * \mathbf{b}_n = \begin{pmatrix} b'_1 \\ \vdots \\ b'_n \end{pmatrix}$$

belongs to B^n. Finally, there are $c, d \in B$ and $\mathbf{c} = (c, c_2, \dots, c_n), \mathbf{d} = (d, d_2, \dots, d_n) \in D$ such that $(b_1 * c) * d = a$. The tuple

$$\mathbf{a} = (\mathbf{b}' * \mathbf{c}) * \mathbf{d} = \begin{pmatrix} a \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

is as required.

It immediately follows from Claim 2 and Corollary 35 that if D is a subdirect product of $\mathbb{A}_1, \dots, \mathbb{A}_n \in \{\mathbb{A}, \mathbb{B}\}$ then $D \cap B^n$ is an almost trivial relation.

CLAIM 3. Let D be a subdirect product of $\mathbb{A}_1, \dots, \mathbb{A}_k, \mathbb{A}_{k+1}, \dots, \mathbb{A}_n$ where $\mathbb{A}_1, \dots, \mathbb{A}_k \in \{\mathbb{A}, \mathbb{B}\}$ while $\mathbb{A}_{k+1}, \dots, \mathbb{A}_n$ are neither $\mathbb{A}$ nor $\mathbb{B}$. Then for any $(a_1, \dots, a_k) \in \mathrm{pr}_{\{1,\dots,k\}} D \cap B^k$, the tuple $(a_1, \dots, a_k, a_{\mathbb{A}_{k+1}}, \dots, a_{\mathbb{A}_n})$ where $a_{\mathbb{A}_i}$ denotes the zero-element of $\mathbb{A}_i$, belongs to D.

Let $\mathbf{b} = (\dots(\mathbf{c}_1 * \mathbf{c}_2) * \mathbf{c}_3 \dots) * \mathbf{c}_q$ where $\mathbf{c}_1, \dots, \mathbf{c}_q$ are all the elements of D (in arbitrary order). Obviously, $b_j = a_{\mathbb{A}_j}$ for any $j \in \{k+1, \dots, n\}$ and $(b_1, \dots, b_k) \in \mathrm{pr}_{\{1,\dots,k\}} D \cap B^k$. Then, there are $\mathbf{b}_1, \dots, \mathbf{b}_s \in D$ such that for $\mathbf{a} = (\dots(\mathbf{b} * \mathbf{b}_1) * \mathbf{b}_2 \dots) * \mathbf{b}_s$ we have $\mathrm{pr}_{\{1,\dots,k\}} \mathbf{a} = (a_1, \dots, a_k)$ and $a_j = a_{\mathbb{A}_j}$, for any $j \in \{k+1, \dots, n\}$.

Claim 3 implies that 3-minimality of a problem instance from CSP($\mathbb{A}$) ensures global consistency. Actually, one may first tighten the problem instance to a 3-minimal one. Then Claim 3 allows one to assign values to those variables v, whose domain is neither

$\mathbb{A}$ nor $\mathbb{B}$. Finally, one may apply the algorithm from page 16. Therefore CSP($\mathbb{A}$) is tractable. $\square$

Proposition 49 *Groupoid (i) is tractable.*

Proof. Denote the groupoid (j) by $\mathbb{A}$. To prove the proposition we first find some useful properties of relations preserved by $*$.

CLAIM 1. If D is a subdirect product of $\mathbb{A}_1$ and $\mathbb{A}_2$ where $A_1, A_2 \in \{A, \{1,2,4\}, \{1,3,4\}\}$, then D is either $A_1 \times A_2$, or the graph of an one-to-one mapping in the case when $|A_1| = |A_2| = 3$, or the identity relation, or the relation $\varrho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$.

This claim can be checked by straightforward calculation.

CLAIM 2. If D is a subdirect product of $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$ and for any distinct $i, j \in \{1,2,3\}$, $\mathrm{pr}_{\{i,j\}} D = A_i \times A_j$, then $D = \mathbb{A}_1 \times \mathbb{A}_2 \times \mathbb{A}_3$.

If all A_1, A_2, A_3 are 3-element then the result follows from Lemma 32. So, suppose that $A_3 = A$. Notice first, that for any $a, b \in \mathbb{A}_i$ $(a, b \in \mathbb{A}_i \times \mathbb{A}_j)$, $i, j \in \{1,2,3\}$, there exists n such that the equation $(\ldots(a * x_1) * x_2 \ldots) * x_n = b$ is solvable. Let $B_a \subseteq \mathbb{A}_1 \times \mathbb{A}_2$ denote the set of all (b, c) such that $(b, c, a) \in D$. Then, for any $a \in \mathbb{A}$, B_a is a subdirect product of $\mathbb{A}_1, \mathbb{A}_2$ and therefore is of the form described in Claim 1.

Suppose first that all B_1, B_2, B_3, B_4 are graphs of one-to-one mappings. If $\mathbb{A}_1 = \mathbb{A}_2 = \mathbb{A}$ then they must be graphs of automorphisms, that is either ϱ, or the identity relation. Then $|B_1| = |B_2| = |B_3| = |B_4| = 4$ and $|B_1 \cup B_2 \cup B_3 \cup B_4| = |A| = 16$, which means in particular that B_1, B_2, B_3, B_4 are pairwise distinct automorphisms, which is impossible. In the case $|A_1| = |A_2| = 3$ we can choose $a, b \in A$ such that $a * b = a$ and $B_a \neq B_b$. Then $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} * \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in B_a$ for all $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in B_a$, $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in B_b$. It can be easily checked, that this is impossible.

Suppose then that $B_a = A_1 \times A_2$, for some $a \in A$. For any b such that $a * b = b$, we have $\begin{pmatrix} b_1 \\ b_2 \\ b \end{pmatrix} \in D$ for certain $b_1 \in A_1, b_2 \in A_2$. Therefore, for any $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \in A_1 \times A_2$, there exist $\begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix}, \ldots, \begin{pmatrix} a_{n1} \\ a_{n2} \end{pmatrix} \in A_1 \times A_2$ such that

$$\left(\cdots\left(\begin{pmatrix} b_1 \\ b_2 \\ b \end{pmatrix} * \begin{pmatrix} a_{11} \\ a_{12} \\ a \end{pmatrix}\right) * \begin{pmatrix} a_{21} \\ a_{22} \\ a \end{pmatrix} \cdots\right) * \begin{pmatrix} a_{n1} \\ a_{n2} \\ a \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ b \end{pmatrix}.$$

This means that $B_b = A_1 \times A_2$ and, since $\mathbb{A}$ is strongly connected, it holds for any $b \in A$.

CLAIM 3. If D a subdirect product of $\mathbb{A}_1, \ldots, \mathbb{A}_n$ where $A_1, \ldots, A_n \in \{A, \{1,2,4\}, \{1,3,4\}\}$, then D is an almost trivial relation.

The proof of the claim is quite similar to the proofs of Lemma 33 and Corollary 35.

Now the proof can be completed in the same way as the proof of the previous proposition. □

5 Concluding remarks

As was noted above, the aims of this paper were two-fold. The first was to find new tractable classes of the CSP, the second was to make progress towards a complete classification of relation types with respect to tractability.

Pursuing the first aim we have found a new class of tractable relations, namely, those closed under a binary conservative commutative operation.

The next step in this direction could be to relax some of these conditions. Omitting arity restrictions we get the class of conservative algebras, that is algebras such that every subset of their universe is a subalgebra. This class of algebras corresponds to the class of constraint satisfaction problem in which every subset of the domain can appear as a constraint relation.

Alternatively, omitting the restriction of conservativity we obtain the very rich class of commutative groupoids. The first natural generalisation of the class of conservative groupoids inside commutative groupoids is the so-called *2-semilattices*. A groupoid is called a 2-semilattice if it satisfies all semilattice identities depending on at most 2 variables. In fact, a groupoid is a 2-semilattice whenever is satisfies only 3 identities: $x*x = x$, $x*y = y*x$, $(x*y)*y = x*y$. Every 2-semilattice has a corresponding digraph and we expect that the argument for 2-semilattices will be quite similar to the one for conservative groupoids.

To achieve the second aim we considered the classification with respect to tractability of the maximal possible classes of relations. These classes correspond to minimal clones. In the case of a two-element domain such a classification is enough to classify all classes of relations. However, the problem is much harder for larger domains, and a classification of minimal clones does not give a classification of all relation classes, although it does provide some information about possible tractable classes.

We pointed out that the only minimal clones whose complexity has not been determined are the binary minimal clones. We were able to give a complete classification of the complexity of minimal clones on a four-element set, by making use of the explicit description of binary minimal clones on a 4-element set given in [23]. This extends the results of [13], which considered minimal clones on a three-element set.

Unfortunately, a complete description of binary minimal clones is still unknown for larger sets. In [23] it was proved that there are several sets of identities such that any minimal binary groupoid must satisfy all identities from one of these sets. It may be possible, by carefully studying these identities to recognise tractable minimal groupoids on an arbitrary set.

References

[1] A.A. Bulatov, A.A. Krokhin, and P.G. Jeavons. Classifying complexity of constraints using finite algebras. Submitted to SIAM Journal of Computing.

[2] A.A. Bulatov, A.A. Krokhin, and P.G. Jeavons. Constraint satisfaction problems and finite algebras. In *Proceedings 27th International Colloquium on Automata, Languages and Programming—ICALP'00*, volume 1853 of *Lecture Notes in Computer Science*, pages 272–282. Springer-Verlag, 2000.

[3] D. Cohen, P. Jeavons, P. Jonsson, and M. Koubarakis. Building tractable disjunctive constraints. *Journal of the ACM*, 47:826–853, 2000.

[4] M.C. Cooper, D.A. Cohen, and P.G. Jeavons. Characterising tractable constraints. *Artificial Intelligence*, 65:347–361, 1994.

[5] T. Feder and M.Y. Vardi. The computational structure of monotone monadic SNP and constraint satisfaction: A study through Datalog and group theory. *SIAM Journal of Computing*, 28:57–104, 1998.

[6] M. Garey and D.S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* Freeman, San Francisco, CA., 1979.

[7] P.G. Jeavons. On the algebraic structure of combinatorial problems. *Theoretical Computer Science*, 200:185–204, 1998.

[8] P.G. Jeavons, D.A. Cohen, and M.C. Cooper. Constraints, consistency and closure. *Artificial Intelligence*, 101(1-2):251–265, 1998.

[9] P.G. Jeavons, D.A. Cohen, and M. Gyssens. A unifying framework for tractable constraints. In *Proceedings 1st International Conference on Constraint Programming—CP'95 (Cassis, France, September 1995)*, volume 976 of *Lecture Notes in Computer Science*, pages 276–291. Springer-Verlag, 1995.

[10] P.G. Jeavons, D.A. Cohen, and M. Gyssens. Closure properties of constraints. *Journal of the ACM*, 44:527–548, 1997.

[11] P.G. Jeavons and M.C. Cooper. Tractable constraints on ordered domains. *Artificial Intelligence*, 79(2):327–339, 1995.

[12] L. Kirousis. Fast parallel constraint satisfaction. *Artificial Intelligence*, 64:147–160, 1993.

[13] A.A. Krokhin and P.G. Jeavons. The complexity of maximal constraint languages. To appear in Proceedings of STOC'01.

[14] A.K. Mackworth. Consistency in networks of relations. *Artificial Intelligence*, 8:99–118, 1977.

[15] R.N. McKenzie, G.F. McNulty, and W.F. Taylor. *Algebras, Lattices and Varieties*, volume I. Wadsworth and Brooks, California, 1987.

[16] U. Montanari. Networks of constraints: Fundamental properties and applications to picture processing. *Information Sciences*, 7:95–132, 1974.

[17] J.K. Pearson and P.G. Jeavons. A survey of tractable constraint satisfaction problems. Technical Report CSD-TR-97-15, Royal Holloway, University of London, July 1997.

[18] R. Pöschel and L.A. Kalužnin. *Funktionen- und Relationenalgebren.* DVW, Berlin, 1979.

[19] E.L. Post. *The two-valued iterative systems of mathematical logic*, volume 5 of *Annals Mathematical Studies.* Princeton University Press, 1941.

[20] R.W. Quackenbush. A survey of minimal clones. *Aequationes Math.*, 50(1–2):3–16, 1995.

[21] I.G. Rosenberg. Minimal clones I: the five types. In *Lectures in Universal Algebra (Proc. Conf. Szeged 1983)*, volume 43 of *Colloq. Math. Soc. Janos Bolyai*, pages 405–427. North-Holland, 1986.

[22] T.J. Schaefer. The complexity of satisfiability problems. In *Proceedings 10th ACM Symposium on Theory of Computing (STOC'78)*, pages 216–226, 1978.

[23] B. Szczepara. *Minimal clones generated by groupoids.* PhD thesis, Université de Móntreal, 1996.

[24] A. Szendrei. *Clones in Universal Algebra*, volume 99 of *Seminaires de Mathematiques Superieures.* University of Montreal, 1986.

[25] P. van Beek and R. Dechter. On the minimality and decomposability of row-convex constraint networks. *Journal of the ACM*, 42:543–561, 1995.

[26] P. van Hentenryck, Y. Deville, and C-M. Teng. A generic arc-consistency algorithm and its specializations. *Artificial Intelligence*, 57:291–321, 1992.