AN ALGORITHM FOR THE DETECTION OF PEAKS AND TROUGHS IN PHYSIOLOGICAL SIGNALS

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Summary

This paper offers a simple mathematical definition of peaks and troughs in a physiological signal. An algorithm is then given for their detection; a proof of its correctness is also provided. Some results are presented with actual physiological data.

Contents

1 Introduction

Medical research and modern clinical practice increasingly involve the recording of continuous physiological signals [Joh96]. Many of the signals are inherently cyclic: for example, arterial blood pressure, abdominal/chest movement, respiratory flow, and the electrocardiogram. One of the first steps in the processing of cyclic data is usually the identification of the peaks and troughs: for example, systolic and diastolic blood pressures, peak inspiration and end-expiration, and the various waves of the electrocardiogram.

This paper offers a mathematical definition of a 'peak' and a 'trough' in a physiological signal and this is illustrated by reference to actual data. For convenience we adopt the notational conventions of the Z specification language because they are fully documented elsewhere [Spi88]; we provide a brief glossary of Z symbols in the appendix. Finally, an algorithm is given for the detection of peaks and troughs and it is shown to be correct.

2 Formalism

Since all measurements are made with finite precision, any physiological measurement can, with suitable scaling, be represented by an integer. For simplicity therefore, we regard a physiological *signal* as a sequence of integers.

$$\mathrm{Signal} \;\widehat{=}\; \mathsf{seq}\,\mathbb{Z} \tag{1}$$

Let q be an arbitrary signal and let indices i and j refer to arbitrary elements (samples) in q. A peak in q is a maximal element that 'dominates' its surroundings by some positive, pre-determined threshold (δ). We say that element j *dominates* element i precisely when $q[j]$ exceeds $q[i]$ by δ and the signal between i and j is bounded below by $q[i]$ and above by $q[j]$. Let $i \underset{q}{\overset{\delta}{\prec}} j$ mean that in signal q element i is dominated by a subsequent element j. Correspondingly, let $j \underset{q}{\overset{\delta}{\succ}} i$ mean that element i is dominated by a preceding element j.

$$\_ \prec \_\,,\quad \_ \succ \_\,;\ \mathbb{N}_1 \to \mathrm{Signal} \to \mathbb{N} \leftrightarrow \mathbb{N}$$

$$\begin{aligned}
&\forall \delta : \mathbb{N}_1;\ q : \mathrm{Signal};\ i, j : \mathbb{N} \bullet \\
&\quad i \underset{q}{\overset{\delta}{\prec}} j \quad \Leftrightarrow \quad 1 \le i \le j \le \#q \quad \wedge \quad q[i] + \delta \ \le\ q[j] \quad \wedge \quad q(\!|i \mathinner{\ldotp\ldotp} j|\!) \ \subseteq\ (q[i] \mathinner{\ldotp\ldotp} q[j]) \\
&\quad j \underset{q}{\overset{\delta}{\succ}} i \quad \Leftrightarrow \quad 1 \le j \le i \le \#q \quad \wedge \quad q[i] + \delta \ \le\ q[j] \quad \wedge \quad q(\!|j \mathinner{\ldotp\ldotp} i|\!) \ \subseteq\ (q[i] \mathinner{\ldotp\ldotp} q[j])
\end{aligned} \tag{2}$$

A *peak* element of q is any element that dominates both a preceding element and a subsequent element. Correspondingly, a *trough* element is any element that is dominated both by a preceding element and by a subsequent element. Let $\pi_\delta(q)$ and $\tau_\delta(q)$ denote, respectively, the peak and the trough elements of q.

$$\pi, \tau \ ; \ \mathrm{N}_1 \rightarrow \mathrm{Signal} \rightarrow \mathrm{P}\,\mathrm{N} \tag{3}$$

$$\begin{aligned} &\forall \delta : \mathrm{N}_1; \ q : \mathrm{Signal} \bullet \\ &\qquad \pi_\delta(q) \ = \ \mathsf{ran}\,(\underset{q}{\overset{\delta}{\prec}}) \ \cap \ \mathsf{dom}\,(\underset{q}{\overset{\delta}{\succ}}) \\ &\qquad \tau_\delta(q) \ = \ \mathsf{ran}\,(\underset{q}{\overset{\delta}{\succ}}) \ \cap \ \mathsf{dom}\,(\underset{q}{\overset{\delta}{\prec}}) \end{aligned}$$

2.1 Some Lemmas

Notice first that domination is a strict partial order (irreflexive, antisymmetric, and transitive).

Lemma 1 *Irreflexivity*

$$\delta : \mathrm{N}_1; \ q : \mathrm{Signal} \ \vdash \ (\underset{q}{\overset{\delta}{\succ}}) \cap \mathsf{id} \ = \{\} \quad \wedge \quad (\underset{q}{\overset{\delta}{\prec}}) \cap \mathsf{id} \ = \{\}$$

Proof
This follows from the respective definitions (2) since the threshold (δ) is strictly greater than zero. □

Lemma 2 *Antisymmetry*

$$\delta : \mathrm{N}_1; \ q : \mathrm{Signal} \ \vdash \ (\underset{q}{\overset{\delta}{\succ}}) \cap (\underset{q}{\overset{\delta}{\succ}})^{-1} = \{\} \quad \wedge \quad (\underset{q}{\overset{\delta}{\prec}}) \cap (\underset{q}{\overset{\delta}{\prec}})^{-1} = \{\}$$

Proof
This also follows from the respective definitions (2) since the threshold (δ) is strictly greater than zero. □

Lemma 3 *Transitivity*

$$\delta : \mathrm{N}_1; \ q : \mathrm{Signal} \ \vdash \ (\underset{q}{\overset{\delta}{\succ}}) \fatsemi (\underset{q}{\overset{\delta}{\succ}}) \ \subseteq \ (\underset{q}{\overset{\delta}{\succ}}) \quad \wedge \quad (\underset{q}{\overset{\delta}{\prec}}) \fatsemi (\underset{q}{\overset{\delta}{\prec}}) \ \subseteq \ (\underset{q}{\overset{\delta}{\prec}})$$

Proof
Choose arbitrary i, j, and k such that $i \underset{q}{\overset{\delta}{\prec}} j$ and $j \underset{q}{\overset{\delta}{\prec}} k$. Now it follows immediately from the definition of $\underset{q}{\overset{\delta}{\prec}}$ and from the transitivity of $\leq$ that $i \leq k$. Furthermore, we have that $q[i] + \delta \leq q[k]$ since

$$\begin{aligned} q[i] + \delta \ &\leq \ q[j] && \textit{Definition of } i \underset{q}{\overset{\delta}{\prec}} j. \\ &< \ q[j] + \delta && \textit{Since } \delta > 0. \\ &\leq \ q[k] && \textit{Definition of } j \underset{q}{\overset{\delta}{\prec}} k. \end{aligned}$$

Finally, we also have that $q(\!|i \mathrel{..} k|\!) \subseteq (q[i] \mathrel{..} q[k])$ *since*

$$\begin{array}{rcll}
(i \mathrel{..} k) & = & (i \mathrel{..} j) \cup (j \mathrel{..} k) & \textit{Since } i \le j \le k. \\
q(\!|i \mathrel{..} k|\!) & = & q(\!|i \mathrel{..} j|\!) \cup q(\!|j \mathrel{..} k|\!) & \textit{Set theory.} \\
 & \subseteq & (q[i] \mathrel{..} q[j]) \cup (q[j] \mathrel{..} q[k]) & \textit{Since } i \overset{\delta}{\underset{q}{\prec}} j \textit{ and } j \overset{\delta}{\underset{q}{\prec}} k. \\
 & = & (q[i] \mathrel{..} q[k]) & \textit{Since } q[i] < q[j] < q[k].
\end{array}$$

Therefore $i \overset{\delta}{\underset{q}{\prec}} k$*. Since the choice of* i*,* j*, and* k *was arbitrary this holds for all such* i*,* j*, and* k*. Therefore* $\overset{\delta}{\underset{q}{\prec}}$ *is transitive. (The proof for* $\overset{\delta}{\underset{q}{\succ}}$ *is similar.)* □

Notice that no element can simultaneously dominate to the left and be dominated from the left. Similarly, no element can simultaneously dominate to the right and be dominated from the right.

Lemma 4 *Exclusivity*

$$\delta : \mathrm{N}_1;\; q : \mathrm{Signal} \vdash \quad \mathsf{ran}\,(\overset{\delta}{\underset{q}{\succ}}) \cap \mathsf{ran}\,(\overset{\delta}{\underset{q}{\prec}}) = \{\} \quad \wedge \quad \mathsf{dom}\,(\overset{\delta}{\underset{q}{\succ}}) \cap \mathsf{dom}\,(\overset{\delta}{\underset{q}{\prec}}) = \{\}$$

Proof

Proof is by contradiction: choose arbitrary i*,* j*, and* k *such that* $i \overset{\delta}{\underset{q}{\succ}} k$ *and* $j \overset{\delta}{\underset{q}{\prec}} k$*. Consider first the case that* $i \le j$*. Now,*

$$\begin{array}{rcll}
j & \le & k & \textit{Since } j \overset{\delta}{\underset{q}{\prec}} k. \\
j & \in & (i \mathrel{..} k) & \textit{Since } i \le j. \\
q[j] & \in & q(\!|i \mathrel{..} k|\!) & \textit{Set theory.} \\
 & \subseteq & (q[k] \mathrel{..} q[i]) & \textit{Since } i \overset{\delta}{\underset{q}{\succ}} k. \\
q[j] & \in & (q[k] \mathrel{..} q[i]) & \textit{Set theory.}
\end{array}$$

Therefore $q[j] \ge q[k]$*. However, from the definition of* $j \overset{\delta}{\underset{q}{\prec}} k$ *we have that* $q[j] + \delta \le q[k]$*, and so* $q[j] < q[k]$ *simultaneously! A similar contradiction is derived in the alternative case that* $i > j$*. Therefore, no such* i*,* j*, and* k *can exist. Thus we have that* $\mathsf{ran}\,(\overset{\delta}{\underset{q}{\succ}}) \cap \mathsf{ran}\,(\overset{\delta}{\underset{q}{\prec}}) = \{\}$*. The proof of the related property* $\mathsf{dom}\,(\overset{\delta}{\underset{q}{\succ}}) \cap \mathsf{dom}\,(\overset{\delta}{\underset{q}{\prec}}) = \{\}$ *is similar.* □

It therefore follows that no element can be both a peak and a trough.

Lemma 5 *Uniqueness*

$$\delta : \mathrm{N}_1;\; q : \mathrm{Signal} \vdash \quad \pi_\delta(q) \cap \tau_\delta(q) = \{\}$$

Proof

The result follows immediately from the definition of π_δ*, the definition of* τ_δ*, and Lemma 4.*

□

Consider a mirror function μ that reflects the indices of a sequence.

$$\begin{aligned} &\mu : \mathrm{Signal} \to \mathrm{N} \nrightarrow \mathrm{N} \\ &\forall q : \mathrm{Signal} \bullet \\ &\qquad \mu(q) \;=\; \lambda i : \mathsf{dom}\, q \bullet \; \#q - i + 1 \end{aligned} \tag{4}$$

Thus $\mu(q) \mathbin{;} q$ is the reverse of sequence q and $\mu(q) \mathbin{;} \mu(q)$ is the identity function on the domain of q.

It follows that the peaks (respectively troughs) of a reversed signal are the mirror image of the peaks (respectively troughs) of the original signal.

Lemma 6 *Reversibility*

$$\delta : \mathrm{N}_1;\ q : \mathrm{Signal} \vdash \quad \mu(q)(\!|\pi_\delta(q)|\!) = \pi_\delta(\mu(q) \mathbin{;} q) \quad \wedge \quad \mu(q)(\!|\tau_\delta(q)|\!) = \tau_\delta(\mu(q) \mathbin{;} q)$$

Proof
This result follows from the symmetry of the definitions of π_δ and τ_δ. $\square$

Similarly, consider an inversion function η that negates integers.

$$\begin{aligned} &\eta : \mathrm{Z} \to \mathrm{Z} \\ &\forall z : \mathrm{Z} \bullet \\ &\qquad \eta(z) \;=\; -z \end{aligned} \tag{5}$$

Thus $q \mathbin{;} \eta$ is the inversion of sequence q.

It follows that the peaks (respectively troughs) of an inverted signal are the troughs (respectively peaks) of the original signal.

Lemma 7 *Inversion*

$$\delta : \mathrm{N}_1;\ q : \mathrm{Signal} \vdash \quad \eta(\!|\pi_\delta(q)|\!) = \tau_\delta(q \mathbin{;} \eta) \quad \wedge \quad \eta(\!|\tau_\delta(q)|\!) = \pi_\delta(q \mathbin{;} \eta)$$

Proof
This result follows from the symmetry of the definitions of π_δ and τ_δ. $\square$

The left and right domination relations enlarge monotonically as the threshold is lowered.

Lemma 8 *Antimonotonicity of domination.*

$$\delta, \epsilon : \mathrm{N}_1;\ q : \mathrm{Signal} \mid \delta \leq \epsilon \vdash \quad (\tfrac{\delta}{q}\succ) \supseteq (\tfrac{\epsilon}{q}\succ) \quad \wedge \quad (\tfrac{\delta}{q}\prec) \supseteq (\tfrac{\epsilon}{q}\prec)$$

Proof
The result follows immediately from the definitions of the domination relations. $\square$

Similarly the set of peak (respectively trough) elements enlarges monotonically as the threshold is lowered.

Lemma 9 *Antimonotonicity of peaks and troughs.*

$$\delta, \epsilon : \mathrm{N}_1;\ q : \text{Signal} \ \mid\ \delta \leq \epsilon \ \vdash\ \ \pi_\delta(q) \supseteq \pi_\epsilon(q) \ \ \wedge\ \ \tau_\delta(q) \supseteq \tau_\epsilon(q)$$

Proof
The result follows from Lemma 8. $\square$

The left and right domination relations enlarge monotonically as the signal is extended.

Lemma 10 *Monotonicity of domination.*

$$\delta : \mathrm{N}_1;\ p, q : \text{Signal} \ \mid\ p \leq q \ \vdash\ \ (\overset{\delta}{\underset{p}{\succ}}) \subseteq (\overset{\delta}{\underset{q}{\succ}}) \ \ \wedge\ \ (\overset{\delta}{\underset{p}{\prec}}) \subseteq (\overset{\delta}{\underset{q}{\prec}})$$

Proof
The result follows immediately from the definitions of the domination relations. $\square$

The peaks (respectively troughs) of a signal are a superset of the peaks (respectively troughs) of any prefix of the signal.

Lemma 11 *Monotonicity of peaks and troughs.*

$$\delta : \mathrm{N}_1;\ p, q : \text{Signal} \ \mid\ p \leq q \ \vdash\ \ \pi_\delta(p) \subseteq \pi_\delta(q) \ \ \wedge\ \ \tau_\delta(p) \subseteq \tau_\delta(q)$$

Proof
The result follows from Lemma 10. $\square$

If a signal is extended then any new peak thus created lies after any peak or trough in the original signal.

Lemma 12 *Peak creation.*

$$\delta : \mathrm{N}_1;\ p, q : \text{Signal}; i, j : \mathrm{N} \ \mid\ p \leq q \ \wedge\ i \in (\pi_\delta(p) \cup \tau_\delta(p)) \ \wedge\ j \in (\pi_\delta(q) - \pi_\delta(p)) \ \vdash\ \ i < j$$

Proof
Proof is by contradiction. Consider first that case that $i \in \pi_\delta(p)$. Suppose that $j \leq i$. Since j is a peak in q it follows by definition that j must left dominate some element of q. Let this element be h. Now $\#p < h$ otherwise j would also be a peak in p. Let k be an element in p (and, by Lemma 10, in q) that i left dominates. Therefore

$$j \leq i \leq k \leq h \ \ \wedge\ \ j \overset{\delta}{\underset{q}{\succ}} h \ \ \wedge\ \ i \overset{\delta}{\underset{q}{\succ}} k$$

Now, every member of the set $q(\!|j \mathinner{..} k|\!)$ must be bounded above by $q[j]$ since $j \overset{\delta}{\underset{q}{\succ}} h$. Let m be a minimal element in the interval $j \mathinner{..} k$. It follows that $q[m] \leq q[k]$ and so $j \overset{\delta}{\underset{p}{\succ}} m$ and so paradoxically j is a peak in p. Therefore, the premise that $j \leq i$ must be false. The proof for the case that $i \in \tau_\delta(p)$ is similar. $\square$

Similarly, if a signal is extended then any new trough thus created also lies after any peak or trough in the original signal.

Lemma 13 *Trough creation.*

$$\delta : \mathrm{N}_1;\ p, q : \mathrm{Signal}; i, j : \mathrm{N} \mid p \leq q \ \wedge\ i \in (\pi_\delta(p) \cup \tau_\delta(p)) \ \wedge\ j \in (\tau_\delta(q) - \tau_\delta(p)) \ \vdash\ i < j$$

Proof
The proof is similar to that of Lemma 12. □

Neither the first nor the last element of a signal can be a peak or trough.

Lemma 14 *Containment*

$$\delta : \mathrm{N}_1;\ q : \mathrm{Signal} \ \vdash\ \pi_\delta(q) \subseteq (2 \mathinner{..} (\#q - 1)) \ \wedge\ \tau_\delta(q) \subseteq (2 \mathinner{..} (\#q - 1))$$

Proof
The result follows immediately from the definition of π_δ and τ_δ. □

3 An Algorithm

Table 1 shows a procedure (PT) that computes the peak (P) and trough (T) elements of the signal (Q) given as the first argument. Local variable i is used as an index into Q, counting from the first element ($Q[1]$) to the last element ($Q[\#Q]$). Within the while loop, variable d indicates the direction of the signal:

$d = \uparrow$	...	proceeding from a trough to a peak.
$d = \downarrow$	...	proceeding from a peak to a trough.
$d = ?$	...	direction unknown.

Variable a records the index of the highest signal level since the last trough. Correspondingly, variable b records the index of the lowest signal level since the last peak. Variable S records the indices (there may be more than one) of the highest signal level since the last trough if the signal is rising ($d = \uparrow$) or those of the lowest signal level if the signal is falling ($d = \downarrow$). The threshold δ is assumed to be a global constant.

3.1 Analysis

Note that the guard is false if Q is the empty sequence and the procedure terminates immediately and correctly with $P = \{\}$ and $T = \{\}$ (Lemma 14). In the case that Q is non-empty

Table 1: Procedure *PT* for computing peaks and troughs.

```
procedure PT (value Q : Signal; result P, T : P N) ≙
    var S : P N; a, b, i : N; d : {?, ↑, ↓} •
    P, T, S, a, b, i, d := {}, {}, {}, 1, 1, 0, ?;
    do i ≠ #Q →
        i := i + 1;
        if   d = ? → if  Q[a] ≥ Q[i] + δ → d := ↓
                     []  Q[i] ≥ Q[b] + δ → d := ↑
                     fi ;
                     if  Q[a] < Q[i] → a := i
                     []  Q[i] < Q[b] → b := i
                     fi ;
                     S := {i}
        []   d = ↑ → if  Q[a] < Q[i]      → S, a := {i}, i
                     []  Q[a] = Q[i]      → S := S ∪ {i}
                     []  Q[a] ≥ Q[i] + δ  → P, S, b, d := P ∪ S, {i}, i, ↓
                     fi
        []   d = ↓ → if  Q[i] < Q[b]      → S, b := {i}, i
                     []  Q[i] = Q[b]      → S := S ∪ {i}
                     []  Q[i] ≥ Q[b] + δ  → T, S, a, d := T ∪ S, {i}, i, ↑
                     fi
        fi
    od
```

we have the following variant and invariant.

<u>Variant</u>

$\#Q - i$

<u>Invariant</u>

$$
\begin{array}{lll}
P = \pi_\delta(R) & & \\
T = \tau_\delta(R) & & \\
1 \leq a, b \leq \#Q \ \wedge \ 0 \leq i \leq \#Q & & \\
d = ? \ \Rightarrow & a = b = 1 \ \vee \ a, b \leq i & \ldots \text{Indices } a \text{ and } b \text{ are bounded.} \\
& \forall j : 1 \mathinner{\ldotp\ldotp} i \bullet \ Q[b] \leq Q[j] \leq Q[a] & \ldots Q[a] \text{ and } Q[b] \text{ are bounds.} \\
& Q[a] < Q[b] + \delta & \ldots \text{Bounds closer than the threshold.} \\
d = \uparrow \Rightarrow & b \leq a \leq i \ \wedge \ \forall j : b \mathinner{\ldotp\ldotp} i \bullet \ Q[j] \leq Q[a] & \ldots \text{Element } a \text{ is maximal.} \\
& S = \{\forall j : b \mathinner{\ldotp\ldotp} i \bullet \ Q[j] = Q[a]\} & \ldots \text{The maximal elements after } b \text{ are } S. \\
& \forall j : S \bullet \ b \overset{\delta}{\underset{R}{\prec}} j & \ldots \text{Every element of } S \text{ right dominates } b. \\
& S \cap \mathsf{dom}\,(\overset{\delta}{\underset{R}{\succ}}) = \{\} & \ldots \text{No element of } S \text{ left dominates in } R. \\
& \{j : \mathrm{N} \ | \ \forall k : T \bullet k < j\} \cap \mathsf{ran}\,(\overset{\delta}{\underset{R}{\prec}}) = \{\} & \ldots \text{Nil left-dominated after last trough.} \\
d = \downarrow \Rightarrow & a \leq b \leq i \ \wedge \ \forall j : a \mathinner{\ldotp\ldotp} i \bullet \ Q[j] \leq Q[b] & \ldots \text{Element } b \text{ is minimal.} \\
& S = \{\forall j : a \mathinner{\ldotp\ldotp} i \bullet \ Q[j] = Q[b]\} & \ldots \text{The minimal elements after } a \text{ are } S. \\
& \forall j : S \bullet \ a \overset{\delta}{\underset{R}{\succ}} j & \ldots \text{Element } a \text{ left dominates all of } S. \\
& S \cap \mathsf{dom}\,(\overset{\delta}{\underset{R}{\prec}}) = \{\} & \ldots \text{None of } S \text{ is right-dominated in } R. \\
& \{j : \mathrm{N} \ | \ \forall k : P \bullet k < j\} \cap \mathsf{ran}\,(\overset{\delta}{\underset{R}{\succ}}) = \{\} & \ldots \text{Nil right dominates after last peak.} \\
\underline{\textit{where}}\ R \ \widehat{=} \ (1 \mathinner{\ldotp\ldotp} i) \lhd Q & &
\end{array}
$$

Notice that the first line of the body of procedure PT establishes the invariant: if $i = 0$ then $R = <>$ and so $\pi_\delta(R) = \{\}$ and $\tau_\delta(R) = \{\}$. Furthermore, the implication given $d = ?$ is satisfied because $a = b = 1$, universal quantification of j over the empty set (1..0) is trivially true, $Q[a] = Q[b]$, and by definition $\delta > 0$.

In each case, the invariant is maintained by execution of the body of the loop given that the guard is false (i.e. $i \neq \#Q$). If initially $d = ?$ then it is clear that the program code maintains a and b as maximal and minimal elements, respectively. If after execution of the body of the loop we still have that $d = ?$ then clearly $Q[a] < Q[b] + \delta$ still holds. If, however, the value of d is changed from ? to $\uparrow$ then the implication of $d = \uparrow$ in the invariant is satisfied because $Q[i]$ is the first element to push the upper bound ($Q[a]$) as much as δ above the lower bound ($Q[b]$). This means that now $a = i$ and i is the unique maximum element in R. It follows from the definition of domination that every element of S, where $S = \{i\}$, must right dominate the minimal element b. Furthermore, since i is the last element in R, no element of S left dominates in R (Lemma 14). Also, no element of R is left-dominated because only the last element is δ or more above the lower bound. This means that R contains no peak or trough and so P and T remain empty. A similar argument applies to the case that the value of d changes from ? to $\downarrow$.

If initially $d = \uparrow$ then no new troughs are created by addition of an extra element to R. This is because any new trough would have to occur later in the sequence than any other trough (Lemma 13) yet no element after the last trough is left-dominated (invariant) except possibly the new last element ($Q[i]$). Set T is therefore unchanged. Similarly, if the new element $Q[i]$ is less than δ below the previous maximum $Q[a]$, no new peak is created because $Q[i]$ is not left-dominated by anything since not even the maximal element $Q[a]$ is large enough to do so. On the other hand, if the new element $Q[i]$ is δ or more below the previous maximum $Q[a]$, all members of S become new peak elements since everything in S both right dominates b (invariant) and left dominates i. In the later case the value of d is changed to $\downarrow$. It is easy to check that each of the minor clauses in the invariant are maintained by the program code in either case. (A similar argument applies to the case that initially $d = \downarrow$.)

4 An Example

Figure 1 shows part of a recording of neonatal chest movements exhibiting intermittent apnoea. The sensor measurements are on an integral scale 0 to 255, and the sampling frequency was 20Hz. The peak and trough elements were identified using procedure PT with a threshold (δ) of 20; they are marked, respectively, by vertical lines above and below the tracing. The thick lines represent clusters of adjacent peak elements or adjacent trough elements. (Procedure PT is easily modified to mark only the first and last such elements if required.) Notice that some of the smaller peaks and troughs are missed.

Figure 2 shows the same recording as Figure 1 but marked with a threshold of 2 rather than 20. Notice that all significant peaks and troughs are now identified. Notice also that the peak and trough elements are a superset of those identified with $\delta = 20$ (Lemma 9).

In conclusion, we have delineated a concept of peaks and troughs in this report and shown that they are easily computed by an efficient (linear time) algorithm. Moreover, the algorithm is easily adapted to the detection of any other signal feature provided that a suitable function is available that measures the extent to which the feature of interest is present at any given position in the signal. This therefore holds promise for other applications too (e.g. QRS complex detection in electrocardiograms [Fri90]).

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I am very grateful to Tony Hoare for his helpful comments in the preparation of this paper and to David Andrews and Paul Johnson of the Maternal Infant Healthcare and Telemonitoring

Figure 1: Neonatal chest movement recording showing peaks and troughs as detected with threshold $\delta = 20$.

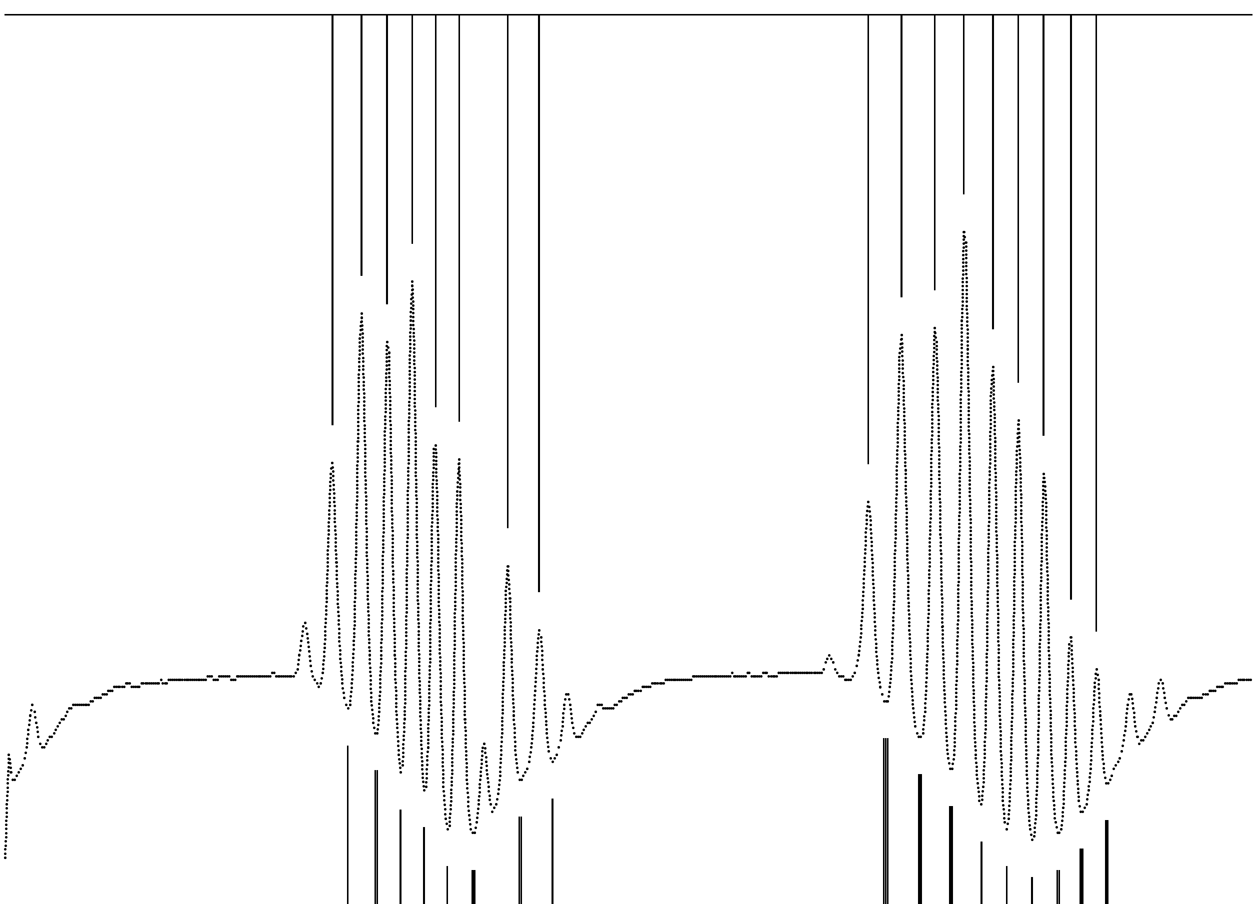
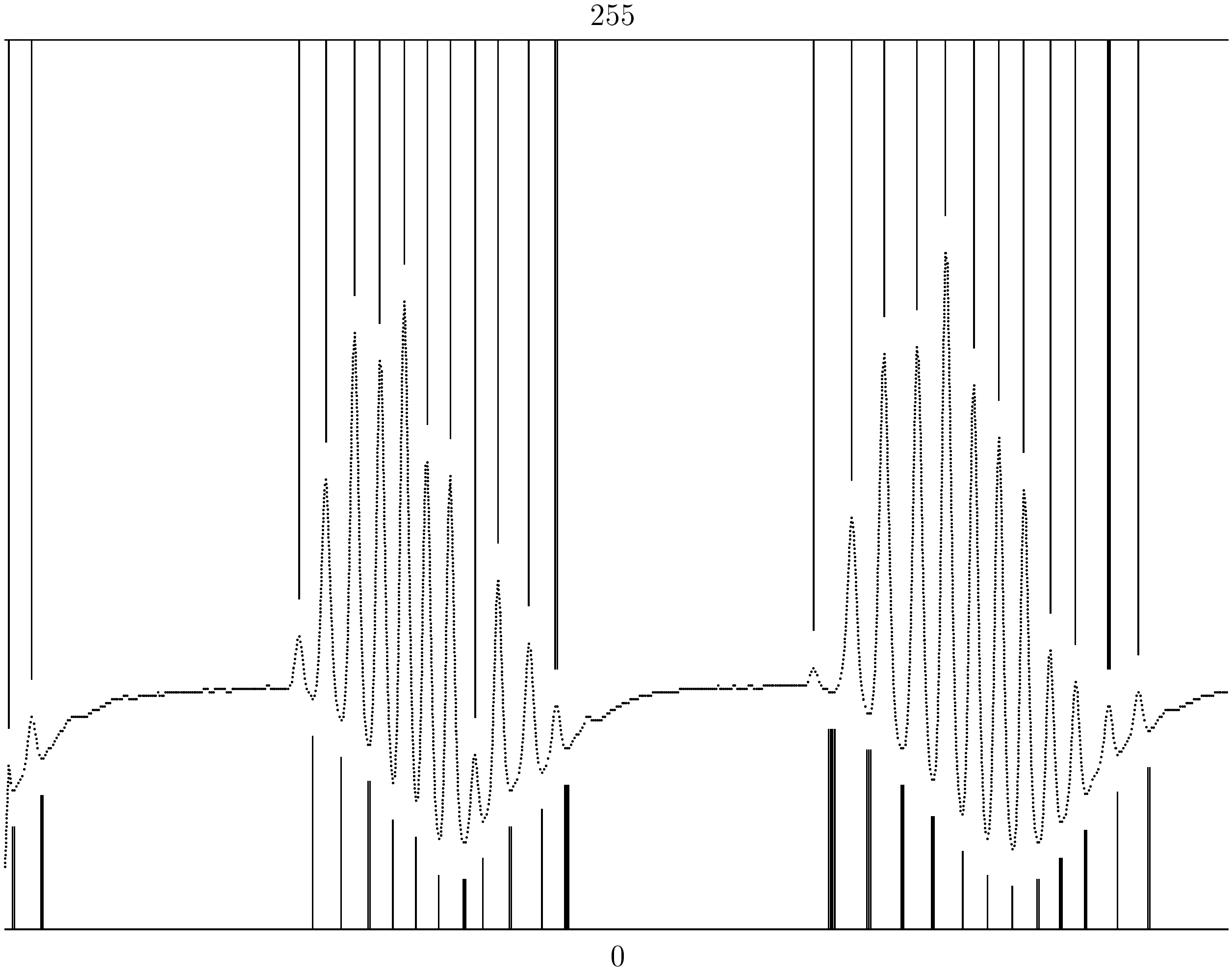


Figure 2: Neonatal chest movement recording showing peaks and troughs as detected with threshold $\delta = 2$.



Centre for their encouragement and for supplying physiological data.

Glossary

The following glossary should enable anyone unfamiliar with the Z specification language to read the formal sections of this paper.

$\mathbb{N}_1$	The set of all non-zero natural numbers ($= \mathbb{N} - \{0\}$).
$m \mathinner{\ldotp\ldotp} n$	The set of natural numbers from m up to n inclusive.
$S \leftrightarrow T$	The set of all relations between S and T.
$S \rightarrow T$	The set of all total functions from S to T.
$S \nrightarrow T$	The set of all partial functions from S to T.
$\{x : T \mid P \bullet t\}$	The set of all terms t over variable(s) x drawn from T such that predicate P holds (if present). Term t may be omitted if identical to x.
$\lambda x : T \mid P \bullet t$	The function mapping to terms t the variable(s) x drawn from T such that predicate P holds (if present). This is identical to the set $\{x : T \mid P \bullet (x, t)\}$.
$\mathsf{seq}\, S$	The set of all finite sequences of elements drawn from S. (In 'Z' a sequence is regarded as a function from an initial segment of the non-zero natural numbers to the set of all possible elements.)
$q[n]$	The n^{th} element ($= q(n)$) of sequence q.
$r \mathbin{⨾} s$	The forward composition of relations (or functions) r and s.
$r(\lvert S \rvert)$	The image of set S through relation (or function) r.

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