Natural Model Semantics for Comonadic and Adjoint Modal Type Theory: Extended Abstract

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Précis. We delineate work in progress supporting an approach to the categorical semantics of (comonadic, adjoint, ...) modal dependent type theory based on the notion of a (weak) morphism of natural models [15]. In particular, we suggest as a model for the comonadic operator of [22] a notion of Cartesian comonad on a natural model. We also introduce the notion of a geometric morphism of natural models, and use this to interpret (dependent analogs of) the adjoint operators of [4].

1 Introduction

Dependent type theories with modal operators (that is, type operators[1]) date to at least the 2000s [3] [14], and have become a subject of intense investigation in homotopy type theory [24]. The categorical semantics of such systems remains underexplored, however.

We pursue an approach to the categorical semantics of modal dependent type theory within the framework of natural models, in particular highlighting the role of morphisms of natural models. Natural models were developed [2] as an equivalent, category-theoretic formulation of categories with families (CwFs) [8], which provide a popular categorical semantics for dependent type theory. The notion of a weak morphism of natural models is introduced in [15]. Independently, [7] introduced weak morphisms of CwFs and used the notion

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[1] In order to encompass adjoint type theory [4] [19] [11], we extend use of the term "modal" even to partially-defined type operators.

to interpret a particular modal dependent type theory. (Henceforth, we drop saying "weak".)

Intuitively, whereas ordinary type theory is interpreted in certain structured categories, such as natural models, *modal* type theory should be interpreted using diagrams of such categories, the constituent functors of which are used to interpret the (partial) modal operators. By interpreting modal type theories using morphisms of natural models, we are able to maintain (at least for the type theories treated here and, evidently, in [7]) this intuitive picture of what a model should be, abstracting away from the complexities of the type theories and their interpretations, which are, as yet, treated in a more *ad hoc* fashion.

We will suggest semantics using morphisms of natural models for two related type theories, one comonadic and one adjoint. Our adjoint type theory, **AdjTT**, gives dependent analogs of the adjoint operators in [4]. Since its interpretation is the more straightforward, **AdjTT** is prioritized. We introduce the notion of a geometric morphism of natural models, and show how to use it to interpret **AdjTT**. Our comonadic type theory, **CoTT**, pulls out the comonadic operator of [22], and also closely follows [14]. We suggest a notion of Cartesian comonad on a natural model to model **CoTT**. These are apparently the first suggestions for modeling fully dependent incarnations of these adjoint and comonadic operators.

Though discussion of examples is deferred to later work, we note that numerous ordinary models of dependent type theory do extend to models of both **CoTT** and **AdjTT**. These include geometric (so-called "cohesive") examples such as simplicial sets and the groupoid model of type theory [10], as well as examples from Kripke semantics of modal logic.[2]

Requisite notions from the theory of natural models are discussed in Section 2. Discussion of **AdjTT** and **CoTT** and their respective natural model semantics follows in Sections 3 and 4.

[2] Apropos of this last example, we also note that a motivation for the present work is the particular relevance to linguistics and philosophy. The intensional logic of Montague [13], which is of foundational importance in natural language semantics and philosophy of language, is an early example of a comonadic type theory [26]. Both **CoTT** and **AdjTT** encode a comonad in the dependently-typed setting, and so represent dependently-typed analogs of intensional logic. They thus integrate both the dependently-typed approach to natural language semantics [23] [18] and the classical intensional logic approach. Applications to natural language are developed in other work by the author [27] [28].

2 Natural Model Semantics

We will require the notions of natural model and of morphism of natural models.

2.1 Natural Models

Natural models are an equivalent, category-theoretic formulation of categories with families [8], which provide a popular categorical semantics for dependent type theory. The natural model formulation was noted independently in [1] and [9].

Definition 1. A ***natural model*** consists of

- a category $\mathbf{C}$
- a distinguished terminal object $1 \in \mathbf{C}$
- presheaves $\mathrm{Ty}, \mathrm{Tm} : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ and a natural transformation $p : \mathrm{Tm} \to \mathrm{Ty}$

such that p is representable.

To unpack this, we recall the definition of a representable natural transformation of presheaves (*cf.* [17]).

Definition 2. Given presheaves $P, Q : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ and a natural transformation $\alpha : Q \to P$, α is said to be ***representable*** if for any $C \in \mathbf{C}$ and $x \in P(C)$, there exists $p_x : D \to C$ and $y \in Q(D)$ such that the following square is a pullback.

$$
\begin{tikzcd}
\mathrm{y}\, D \arrow[r, "y"] \arrow[d, "\mathrm{y}\, p_x"'] \arrow[dr, phantom, "\lrcorner", very near start] & Q \arrow[d, "\alpha"] \\
\mathrm{y}\, C \arrow[r, "x"'] & P
\end{tikzcd}
.
$$

Below, as here, we will freely use the Yoneda lemma to conflate presheaf elements $x \in P(C)$ with the corresponding map $x : \mathrm{y}\, C \to P$. A natural model whose underlying category is $\mathbf{C}$ may simply be referred to as $\mathbf{C}$. Given a natural model $\mathbf{C}$, its constituent data will be written $\mathbf{C}, 1_{\mathbf{C}}, \mathrm{Ty}_{\mathbf{C}}$, *etc.*

A natural model presents a CwF, with $\mathbf{C}$ as the category of contexts and substitutions and Ty and Tm as the presheaves of types and terms. The representability condition on p corresponds to the comprehension axiom for a CwF.

For, given a "type" $A \in \mathrm{Ty}(\Gamma)$, representability ensures we have some $\Gamma.A \in \mathbf{C}$, $p_A : \Gamma.A \to \Gamma$, and $v_A : \mathrm{y}(\Gamma.A) \to \mathrm{Tm}$ which provide the comprehension of A. Figure 1 shows this comprehension of a type A in a natural model.

$$\begin{tikzcd} \mathrm{y}(\Gamma.A) \arrow[r, "v_A"] \arrow[d, "\mathrm{y}\, p_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Tm} \arrow[d, "p"] \\ \mathrm{y}\,\Gamma \arrow[r, "A"'] & \mathrm{Ty} \end{tikzcd}$$

Figure 1: The Comprehension of a Type A

Furthermore, given a "term" of $A \in \mathrm{Ty}(\Gamma)$, *i.e.* $a \in \mathrm{Tm}(\Gamma)$ such that $p \circ a = A$, we may form a corresponding section of p_A, which we denote $s_a : \Gamma \to \Gamma.A$, using the pullback property of $\mathrm{y}\, p_A$. The reader may find greater detail and discussion in [2].

2.2 Morphisms of Natural Models

We turn to morphisms of natural models, the theory of which is developed in [15]. Roughly, a morphism maps (some) types to types, preserving the context extension/comprehension operation. We will take morphisms as the basic semantic objects corresponding to modal type operators.

In order to define morphisms of natural models, we will use a notion of lax morphism as auxiliary. The definition of lax morphism is as follows:

Definition 3. A ***lax morphism***[3] of natural models $F : \mathbf{C} \to \mathbf{D}$ consists of:

- a functor, also denoted $F : \mathbf{C} \to \mathbf{D}$, between the underlying categories
- a natural transformation $\phi_{\mathrm{Ty}} : F_! \,\mathrm{Ty}_{\mathbf{C}} \to \mathrm{Ty}_{\mathbf{D}}$
- a natural transformation $\phi_{\mathrm{Tm}} : F_! \,\mathrm{Tm}_{\mathbf{C}} \to \mathrm{Tm}_{\mathbf{D}}$

such that the following diagram commutes:

[3] This is the definition of premorphism given by [15], except that here, distinguished terminal objects need not be preserved. The author of [15] suggested via [16] the term "lax morphism" for this modified notion.

$$\begin{tikzcd}
F_! \operatorname{Tm}_{\mathbf{C}} \arrow[r, "\phi_{\mathrm{Tm}}"] \arrow[d, "F_! p_{\mathbf{C}}"'] & \operatorname{Tm}_{\mathbf{D}} \arrow[d, "p_{\mathbf{D}}"] \\
F_! \operatorname{Ty}_{\mathbf{C}} \arrow[r, "\phi_{\mathrm{Ty}}"'] & \operatorname{Ty}_{\mathbf{D}}
\end{tikzcd}.$$

Here, $F_! : \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}} \to \mathbf{Set}^{\mathbf{D}^{\mathrm{op}}}$ is the left Kan extension functor, so we have to hand an adjunction $F_! \dashv F^*$, where F^* denotes precomposition by F.

To make more sense of the definition of lax morphism, we use the following notation:

Convention 4. Given a lax morphism $F : \mathbf{C} \to \mathbf{D}$, and a type $A \in \operatorname{Ty}(\Gamma)$ in context $\Gamma \in \mathbf{C}$, we write F/A for the composite

$$\mathrm{y}\, F\Gamma \cong F_!\, \mathrm{y}\, \Gamma \xrightarrow{F_! A} F_! \operatorname{Ty}_{\mathbf{C}} \xrightarrow{\phi_{\mathrm{Ty}}} \operatorname{Ty}_{\mathbf{D}} \quad .$$

Similarly, given a term $a \in \operatorname{Tm}(\Gamma)$, we write F/a for the composite

$$\mathrm{y}\, F\Gamma \cong F_!\, \mathrm{y}\, \Gamma \xrightarrow{F_! a} F_! \operatorname{Tm}_{\mathbf{C}} \xrightarrow{\phi_{\mathrm{Tm}}} \operatorname{Tm}_{\mathbf{D}} \quad .$$

One may think of F/A and F/a as the results of applying the morphism F to A and a. These operations are implicated in the interpretation of (respectively) formation and introduction rules for modal type operators.

A morphism is, of course, to be a lax morphism for which relevant comparison maps are isomorphisms. We set up the precise statement now.

Definition 5. A lax morphism $F : \mathbf{C} \to \mathbf{D}$ is said to preserve the (distinguished) terminal object whenever the unique morphism $! : F(1_{\mathbf{C}}) \to 1_{\mathbf{D}}$ is an isomorphism.

Lemma 6. Let $F : \mathbf{C} \to \mathbf{D}$ be a lax morphism. Then, given a type $A \in \operatorname{Ty}_{\mathbf{C}}(\Gamma)$ in context $\Gamma \in \mathbf{C}$, there is a unique comparison map $\tau_A : F(\Gamma.A) \to F\Gamma.(F/A)$ such that $Fp_A = p_{F/A} \circ \tau_A$ and $F/v_A = v_{F/A} \circ \mathrm{y}(\tau_A)$, i.e., such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{y}(F(\Gamma.A)) \arrow[dr, dashed, "\mathrm{y}(\tau_A)"] \arrow[ddr, bend right, "\mathrm{y}(Fp_A)"'] \arrow[drr, bend left, "F/v_A"] & & \\
& \mathrm{y}(F\Gamma.(F/A)) \arrow[r, "v_{F/A}"] \arrow[d, "\mathrm{y}(p_{F/A})"'] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Tm}_{\mathbf{D}} \arrow[d, "p_{\mathbf{D}}"] \\
& \mathrm{y}(F\Gamma) \arrow[r, "F/A"'] & \operatorname{Ty}_{\mathbf{D}}
\end{tikzcd}.$$

$$\frac{}{\cdot \Vdash}\ \text{Emp.} \qquad \frac{\Delta \Vdash B \text{ type}}{\Delta, u :: B \Vdash}\ \text{Ext.}^{\mathcal{D}} \qquad \frac{\Delta, u :: A, \Delta' \Vdash}{\Delta, u :: A, \Delta' \Vdash u : A}\ \text{Var.}^{\mathcal{D}}$$

$$\frac{\Delta \Vdash}{\Delta \mid \cdot \vdash}\ \text{Lock} \qquad \frac{\Delta \mid \Gamma \vdash B \text{ type}}{\Delta \mid \Gamma, x : B \vdash}\ \text{Ext.}^{\mathcal{C}} \qquad \frac{\Delta \mid \Gamma, x : A, \Gamma' \vdash}{\Delta \mid \Gamma, x : A, \Gamma' \vdash x : A}\ \text{Var.}^{\mathcal{C}}$$

Table 1: The Context Rules for **AdjTT**

Proof. Follows from the pullback property of $\mathrm{y}(F\Gamma.(F/A))$. □

Any lax morphism F is thus said to laxly preserve context extension. Furthermore, we have following definition:

Definition 7. Let $F : \mathbf{C} \to \mathbf{D}$ be a lax morphism. Then F is said to preserve context extension if, for each type $A \in \mathrm{Ty}_{\mathbf{C}}(\Gamma)$ in each context $\Gamma \in \mathbf{C}$, the comparison map $\tau_A : F(\Gamma.A) \to F(\Gamma).(F/A)$ is an isomorphism.

Finally, we are equipped to define a morphism of natural models.

Definition 8. A lax morphism $F : \mathbf{C} \to \mathbf{D}$ of natural models that preserves context extension and terminal objects is called a ***morphism of natural models***.[4]

3 Adjoint Type Theory

We now proceed to describe the adjoint dependent type theory **AdjTT** and interpret it using morphisms of natural models.

3.1 Syntax

3.2 Provenance of **AdjTT**

Whereas a type theory ordinarily describes a single category, several type theories have generalized this picture to describe at once multiple categories and adjunctions between them. These go by names such as "adjoint calculus" [5], "adjoint logic" [19], and "call-by-push-value" [11]; here, the term term "adjoint type theory" is used.

Adjoint simple type theory has recently been treated in the great generality of arbitrary diagrams of adjunctions [12]. The extension of that approach to

[4] This is the definition of weak morphism given and explored by [15], except that here, terminal objects need not be preserved strictly. The notion of morphism of CwFs given by [7] (independently of [15]) matches the present approach.

$$\frac{\Delta \Vdash B \text{ type}}{\Delta \mid \Gamma \vdash LB \text{ type}} \; L\text{-Form.} \qquad \frac{\Delta \Vdash t : B}{\Delta \mid \Gamma \vdash t^L : LB} \; L\text{-Intro.}$$

$$\frac{\Delta \mid \Gamma, x : LA \vdash B \text{ type} \qquad \Delta \mid \Gamma \vdash s : LA \qquad \Delta, u :: A \mid \Gamma \vdash t : B[u^L/x]}{\Delta \mid \Gamma \vdash (\text{let } u^L := s \text{ in } t) : B[s/x]} \; L\text{-Elim.}$$

$$\frac{\Delta \mid \Gamma, x : LA \vdash B \text{ type} \qquad \Delta \Vdash s : A \qquad \Delta, u :: A \mid \Gamma \vdash t : B[u^L/x]}{\Delta \mid \Gamma \vdash (\text{let } u^L := s^L \text{ in } t) \equiv t[s/u] : B[s^L/x]} \; L\text{-}\beta\text{-Conv.}$$

$$\frac{\Delta \mid \Gamma, x : LA \vdash B \text{ type} \qquad \Delta \mid \Gamma \vdash s : LA \qquad \Delta \mid \Gamma, x : LA \vdash t : B}{\Delta \mid \Gamma \vdash (\text{let } u^L := s \text{ in } t[u^L/x]) \equiv t[s/x] : B[s/x]} \; L\text{-}\eta\text{-Conv.}$$

Table 2: The Rules for L

dependent types is apparently the subject of intense investigation by its authors and collaborators.

Here, in **AdjTT**, we treat only the case of a single adjunction, but for dependent types. This amounts to producing dependently-typed analogs of the adjoint operators of [4], and extends the partial result of [25]. While perhaps novel, our rule for the left adjoint operator L is adapted rather straightforwardly from the comonadic operator $\flat$ of [22].

3.2.1 AdjTT

Contexts and Judgments We now delineate **AdjTT**. This type theory is thought of as describing two categories interacting via an adjunction. Accordingly, we have two kinds of types, which we call $\mathcal{D}$-types and $\mathcal{C}$-types. A variable assumption of a $\mathcal{D}$-type is denoted

$$u :: A \quad ,$$

and lists of $\mathcal{D}$-typed variables are denoted $\Delta, \Delta', \ldots$. On the other hand, a variable assumption of a $\mathcal{C}$-type is denoted

$$x : A \quad ,$$

and lists of $\mathcal{C}$-typed variables are denoted $\Gamma, \Gamma', \ldots$. Similarly, we have $\mathcal{D}$-contexts and $\mathcal{C}$-contexts. The judgment that Δ is a $\mathcal{D}$-context is denoted

$$\Delta \Vdash \quad ,$$

whereas the judgment that $\Delta \mid \Gamma$ is a $\mathcal{C}$-context is denoted

$$\Delta \mid \Gamma \vdash \quad .$$

$$\frac{\Delta \mid \cdot \vdash B \ \mathsf{type}}{\Delta \Vdash RB \ \mathsf{type}} \ R\text{-Form.}$$

$$\frac{\Delta \mid \cdot \vdash t : B}{\Delta \Vdash t^R : RB} \ R\text{-Intro.} \qquad \frac{\Delta \Vdash t : RB}{\Delta \mid \Gamma \vdash t_R : B} \ R\text{-Elim.}$$

$$\frac{\Delta \mid \cdot \vdash t : B}{\Delta \mid \Gamma \vdash (t^R)_R \equiv t : B} \ R\text{-}\beta\text{-Conv.} \qquad \frac{\Delta \Vdash t : RB}{\Delta \Vdash (t_R)^R \equiv t : B} \ R\text{-}\eta\text{-Conv.}$$

Table 3: The Rules for R

Note here that $\mathcal{C}$-contexts can include assumptions of $\mathcal{D}$-type, which are thought of as tacitly transported by the left adjoint L to $\mathcal{C}$-types. As suggested by the ordering of $\Delta \mid \Gamma$, $\mathcal{C}$-types can depend on $\mathcal{D}$-types, but not *vice versa*. Accordingly, we have typing judgments for $\mathcal{D}$-terms

$$\Delta \Vdash t : B$$

and $\mathcal{C}$-terms

$$\Delta \mid \Gamma \vdash t : B \quad .$$

A $\mathcal{D}$- (resp. $\mathcal{C}$-)term may be substituted only for a $\mathcal{D}$- (resp. $\mathcal{C}$-)variable, though in any context. The rules for contexts and variables are presented in Table 1.

The Operators L and R The calculus includes a pair of type operators L and R. Of course, L is thought of as left adjoint to R, and it takes $\mathcal{D}$-types to $\mathcal{C}$-types, while R does the opposite. The rules for L and R are presented in Tables 2 and 3.

The L-Elim. rule crystallizes the idea that $\mathcal{D}$-assumptions $u :: A$ in a $\mathcal{C}$-context are akin to $\mathcal{C}$-assumptions $x : LA$; we can "substitute" a term $s : LA$ for the assumption $u :: A$ almost as if it were an assumption $x : LA$. However, as the L-β-Conv. rule indicates, this explicit let-substitution for a $\mathcal{D}$-variable only reduces to an ordinary substitution when this $s : LA$ indeed comes directly from a $\mathcal{D}$-term. The rules R-Form. and R-Intro. are notable in that they apply only when no $\mathcal{C}$-assumptions are present. These rules thus are not guaranteed to apply after a let-substitution, nor do they commute with let-substitution.

3.3 Semantics

3.3.1 Geometric Morphisms

Our notion of model for **AdjTT** takes an appealingly simple form. We purloin a topos-theoretic term for this structure:

Definition 9. Given natural models $\mathcal{E}$ and $\mathcal{F}$, a ***geometric morphism*** $f : \mathcal{E} \to \mathcal{F}$ from $\mathcal{E}$ to $\mathcal{F}$ consists of

- a morphism of natural models $f_* : \mathcal{E} \to \mathcal{F}$ and
- a morphism of natural models $f^* : \mathcal{F} \to \mathcal{E}$

such that there is an adjunction $f^* \dashv f_*$ of the underlying functors.

Remark 10. The representable natural transformation $p_{\mathcal{E}} : \mathrm{Tm}_{\mathcal{E}} \to \mathrm{Ty}_{\mathcal{E}}$ of a natural model $\mathcal{E}$ is somewhat analogous to a classifying morphism of a (higher) toposes, meaning $\mathcal{E}$ is itself somewhat analogous to a topos. We note that the requirement that f^* be a morphism of natural models will serve a similar role in our interpretation of modal type theory to the requirement that the inverse image of a geometric morphism (of toposes) preserve finite limits in the interpretation of modal logic (cf. [20]). A geometric morphism of natural models is thus something more than "an adjunction between natural models", just as a geometric morphism of toposes is something more than an adjunction between toposes.

For the rest of Section 3, we work with a geometric morphism $f : \mathcal{E} \to \mathcal{F}$ of natural models, and, for readability, we write R for the morphism f_* and L for the morphism f^*. We now introduce some notation to aid in the interpretation of **AdjTT** in a geometric morphism.

Definition 11. Let $\Delta \in \mathcal{F}$. Then

- $RA :\equiv (R/A) \circ \mathrm{y}(\eta_\Delta) : \mathrm{y}\,\Delta \to \mathrm{Ty}_{\mathcal{F}}$, where $A : \mathrm{y}(L\Delta) \to \mathrm{Ty}_{\mathcal{E}}$,
- $Ra :\equiv (R/a) \circ \mathrm{y}(\eta_\Delta) : \mathrm{y}\,\Delta \to \mathrm{Tm}_{\mathcal{F}}$, where $a : \mathrm{y}(L\Delta) \to \mathrm{Tm}_{\mathcal{E}}$,
- $LA :\equiv (L/A) : \mathrm{y}(L\Delta) \to \mathrm{Ty}_{\mathcal{E}}$, where $A : \mathrm{y}\,\Delta \to \mathrm{Ty}_{\mathcal{F}}$, and
- $La :\equiv (L/a) : \mathrm{y}(L\Delta) \to \mathrm{Tm}_{\mathcal{E}}$, where $a : \mathrm{y}\,\Delta \to \mathrm{Tm}_{\mathcal{F}}$.

It is this new $R(-)$, not $R/(-)$, which will interpret the formation and introduction rules for the type operator R of **AdjTT**.

3.3.2 Interpretation

With our notion of geometric morphism, we now proceed to interpret **AdjTT**.

While the interpretation for the most part goes as expected, we note that, whereas a $\mathcal{D}$-context Δ is interpreted as an object $[\![\Delta]\!]$ of $\mathcal{F}$, the more complex $\mathcal{C}$-context $\Delta \mid \Gamma$ is interpreted not as an object of $\mathcal{E}$, but as an object $[\![\Delta \mid \Gamma]\!]$ of the slice category $\mathcal{E} \downarrow L[\![\Delta]\!]$. The objects of $\mathcal{E} \downarrow L[\![\Delta]\!]$ are pairs of form $(E, f : E \to L[\![\Delta]\!])$. We write α_0 and α_1 for the elements of such a pair α. Despite this complication, the interpretation of a dependent type $\Delta \mid \Gamma \vdash A$ will simply lie in $\mathrm{Ty}_{\mathcal{E}}([\![\Delta \mid \Gamma]\!]_0)$.

The partial interpretation function $[\![-]\!]$ is now delineated.[5]

Definition 12. The partial interpretation function $[\![-]\!]$ is given by recursion on raw syntax as follows:

(Emp.). $[\![\cdot]\!] = 1_{\mathcal{F}} \in \mathcal{F}$

(Ext.$^{\mathcal{D}}$). $[\![\Delta, u :: B]\!] = [\![\Delta]\!].[\![B]\!] \in \mathcal{F}$

(Var.$^{\mathcal{D}}$). $[\![\Delta, u :: A \Vdash u : A]\!] = v_{[\![A]\!]} \in \mathrm{Tm}_{\mathcal{F}}([\![\Delta]\!].[\![A]\!])$

(Lock). $[\![\Delta \mid \cdot]\!] = (L[\![\Delta]\!], \mathrm{id}_{L[\![\Delta]\!]}) \in \mathcal{E} \downarrow L[\![\Delta]\!]$

(Ext.$^{\mathcal{C}}$). $[\![\Delta \mid \Gamma, x : B]\!] = ([\![\Delta \mid \Gamma]\!]_0.[\![B]\!], [\![\Delta \mid \Gamma]\!]_1 \circ p_{[\![B]\!]}) \in \mathcal{E} \downarrow L[\![\Delta]\!]$

(Var.$^{\mathcal{C}}$). $[\![\Delta \mid \Gamma, x : A \vdash x : A]\!] = v_{[\![A]\!]} \in \mathrm{Tm}_{\mathcal{E}}([\![\Delta \mid \Gamma]\!]_0.[\![A]\!])$

(L-Form.). $[\![\Delta \mid \cdot \vdash LB]\!] = L[\![B]\!] \in \mathrm{Ty}_{\mathcal{E}}(L[\![\Delta]\!])$

(L-Intro.). $[\![\Delta \mid \cdot \vdash t^L : LB]\!] = L[\![t]\!] \in \mathrm{Tm}_{\mathcal{E}}(L[\![\Delta]\!])$

(L-Elim.). $[\![\Delta \mid \cdot \vdash (\mathsf{let}\ u^L := r\ \mathsf{in}\ t) : B[r/x]]\!] = [\![t]\!] \circ \mathrm{y}(s_{[\![r]\!]}) \in \mathrm{Tm}_{\mathcal{E}}(L[\![\Delta]\!])$

(R-Form.). $[\![\Delta \Vdash RB]\!] = R[\![B]\!] \in \mathrm{Ty}_{\mathcal{F}}([\![\Delta]\!])$

(R-Intro.). $[\![\Delta \Vdash t^R : RB]\!] = R[\![t]\!] \in \mathrm{Tm}_{\mathcal{F}}([\![\Delta]\!])$

(R-Elim.). $[\![\Delta \mid \cdot \vdash t_R : B]\!] = v_{[\![B]\!]} \circ \mathrm{y}(\varepsilon^{L[\![\Delta]\!]}_{[\![B]\!]} \circ s_{L[\![t]\!]}) \in \mathrm{Tm}_{\mathcal{E}}(L[\![\Delta]\!])$[6]

This interpretation is sound in the following senses.

[5] For readability, we omit the interpretation of judgments with fully weakened contexts.

[6] Here, $\varepsilon^{L[\![\Delta]\!]}_{[\![B]\!]} : L[\![\Delta]\!].LR[\![B]\!] \to L[\![\Delta]\!].[\![B]\!]$ is the "indexed counit" induced by the adjunction. This unwinds as $\varepsilon^{L[\![\Delta]\!]}_{[\![B]\!]} \equiv \varepsilon_{L[\![\Delta]\!].[\![B]\!]} \circ \pi_2$ where $\pi_2 : L[\![\Delta]\!].LR[\![B]\!] \to LR(L[\![\Delta]\!].[\![B]\!])$ is the morphism which, together with $p_{LR[\![B]\!]}$, exhibits $\mathrm{y}(L[\![\Delta]\!].LR[\![B]\!])$ as the pullback of $\mathrm{y}(LR(L[\![\Delta]\!].[\![B]\!]))$ along $\mathrm{y}(L\eta_{[\![\Delta]\!]})$.

Proposition 13. The following facts hold:

- When $\Delta \Vdash$, also $[\![\Delta]\!] \in \mathcal{F}$.
- When $\Delta \mid \Gamma \vdash$, also $[\![\Delta \mid \Gamma]\!] \in \mathcal{E} \downarrow L[\![\Delta]\!]$.
- When $\Delta \Vdash B$ type, also $[\![B]\!] \in \mathrm{Ty}_{\mathcal{F}}([\![\Delta]\!])$.
- When $\Delta \mid \Gamma \vdash B$ type, also $[\![B]\!] \in \mathrm{Ty}_{\mathcal{E}}([\![\Delta \mid \Gamma]\!]_0)$.
- When $\Delta \Vdash t : B$, also $[\![t]\!] \in \mathrm{Tm}_{\mathcal{F}}([\![\Delta]\!])$ and $p_{\mathcal{F}} \circ [\![t]\!] = [\![B]\!]$.
- When $\Delta \mid \Gamma \vdash t : B$, also $[\![t]\!] \in \mathrm{Tm}_{\mathcal{E}}([\![\Delta \mid \Gamma]\!]_0)$ and $p_{\mathcal{E}} \circ [\![t]\!] = [\![B]\!]$.
- When $\Delta \equiv \Delta' \Vdash$, also $[\![\Delta]\!] = [\![\Delta']\!]$ and when $(\Delta \mid \Gamma) \equiv (\Delta' \mid \Gamma') \vdash$, also $[\![\Delta \mid \Gamma]\!] = [\![\Delta' \mid \Gamma']\!]$.
- When $\Delta \Vdash B \equiv B'$ type or $\Delta \mid \Gamma \vdash B \equiv B'$ type, also $[\![B]\!] = [\![B']\!]$.
- When $\Delta \Vdash t \equiv t' : B$ or $\Delta \mid \Gamma \vdash t \equiv t' : B$, also $[\![t]\!] = [\![t']\!]$.

4 Comonadic Type Theory

Beyond adjoint dependent type theory, morphisms of natural models may also be used to model comonadic dependent type theory. We outline such a type theory, **CoTT**, and suggest a natural model semantics.

4.1 Syntax

4.1.1 Provenance of **CoTT**

The system **CoTT** simply pulls out the comonadic operator from the more complex system of [22], which was motivated by geometric models of homotopy type theory. Rules for a dependent comonadic operator were first given in [14].

4.1.2 **CoTT**

Tables 4 and 5 describe the type theory **CoTT**.

$$\frac{}{\cdot \mid \cdot \vdash} \text{Emp.}$$

$$\frac{\Delta \mid \cdot \vdash B \ \mathsf{type}}{\Delta, u :: B \mid \cdot \vdash} \text{Ext.}^\flat \qquad \frac{\Delta, u :: A, \Delta' \mid \Gamma \vdash}{\Delta, u :: A, \Delta' \mid \Gamma \vdash u : A} \text{Var.}^\flat$$

$$\frac{\Delta \mid \Gamma \vdash B \ \mathsf{type}}{\Delta \mid \Gamma, x : B \vdash} \text{Ext.} \qquad \frac{\Delta \mid \Gamma, x : A, \Gamma' \vdash}{\Delta \mid \Gamma, x : A, \Gamma' \vdash x : A} \text{Var.}$$

Table 4: The Context Rules for **CoTT**

$$\frac{\Delta \mid \cdot \vdash B \ \mathsf{type}}{\Delta \mid \Gamma \vdash \flat B \ \mathsf{type}} \flat\text{-Form.} \qquad \frac{\Delta \mid \cdot \vdash t : B}{\Delta \mid \Gamma \vdash t^\flat : \flat B} \flat\text{-Intro.}$$

$$\frac{\Delta \mid \Gamma, x : \flat A \vdash B \ \mathsf{type} \qquad \Delta \mid \Gamma \vdash s : \flat A \qquad \Delta, u :: A \mid \Gamma \vdash t : B[u^\flat/x]}{\Delta \mid \Gamma \vdash (\mathsf{let}\ u^\flat := s \ \mathsf{in}\ t) : B[s/x]} \flat\text{-Elim.}$$

$$\frac{\Delta \mid \Gamma, x : \flat A \vdash B \ \mathsf{type} \qquad \Delta \mid \cdot \vdash s : A \qquad \Delta, u :: A \mid \Gamma \vdash t : B[u^\flat/x]}{\Delta \mid \Gamma \vdash (\mathsf{let}\ u^\flat := s^\flat \ \mathsf{in}\ t) \equiv t[s/u] : B[s^\flat/x]} \flat\text{-}\beta\text{-Conv.}$$

$$\frac{\Delta \mid \Gamma, x : \flat A \vdash B \ \mathsf{type} \qquad \Delta \mid \Gamma \vdash s : \flat A \qquad \Delta \mid \Gamma, x : \flat A \vdash t : B}{\Delta \mid \Gamma \vdash \mathsf{let}\ u^\flat := s \ \mathsf{in}\ t[u^\flat/x] \equiv t[s/x] : B[s/x]} \flat\text{-}\eta\text{-Conv.}$$

Table 5: The Rules for $\flat$

4.2 Semantics

4.2.1 Cartesian Comonads

Our notion of model for **CoTT** takes an appealingly simple form:

Definition 14. Let $\flat : \mathcal{E} \to \mathcal{E}$ be an endomorphism on a natural model $\mathcal{E}$. This $\flat$ is said to be a ***Cartesian comonad*** on $\mathcal{E}$ when its underlying functor is a comonad.

Remark 15. The requirement that $\flat$ be a morphism of natural models serves a similar role in our interpretation of modal dependent type theory to the requirement of product- or finite limit-preservation in the interpretation of modal simple type theory and modal logic (cf. [6], [26]). A Cartesian comonad on a natural model is thus something more than "a comonad on a natural model".

We now introduce some notation of aid in the interpretation of **CoTT**. We write $\mathcal{E}^\flat$ for the category of coalgebras for $\flat$, $U : \mathcal{E}^\flat \to \mathcal{E}$ for the forgetful functor, and $K : \mathcal{E} \to \mathcal{E}^\flat$ for the cofree functor. As the name suggests, we have $U \dashv K$. This facilitates the following definition:

Definition 16. Let $\Delta \in \mathcal{E}^\flat$. Then

- $\flat A :\equiv (\flat/A) \circ \mathrm{y}(U\eta_\Delta) : \mathrm{y}(U\Delta) \to \mathrm{Ty}$, where $A : \mathrm{y}(U\Delta) \to \mathrm{Ty}$, and
- $\flat a :\equiv (\flat/a) \circ \mathrm{y}(U\eta_\Delta) : \mathrm{y}(U\Delta) \to \mathrm{Tm}$, where $a : \mathrm{y}(U\Delta) \to \mathrm{Tm}$.

These defined operations interpret the rules $\flat$-Form. and $\flat$-Intro., respectively. The interpretation of **CoTT**, omitted due to space constraints, proceeds along comparable lines to that for **AdjTT**.

4.2.2 Comparison to Other Work

The lecture [21] describes independent work towards modeling a comonadic dependent type theory, in which the comonad is required to be idempotent (that is, $\flat\flat A \simeq \flat A$, for any type A). That work aims to include comonadic operators in the context of model category theory, and makes the observation that such operators do not in general take fibrations to fibrations. This raises the prospect that in some models important in homotopy type theory, $\flat(\Delta.A)$ will not in general be isomorphic to $(\flat\Delta).(\flat/A)$: the former may not be a fibration, whereas $(\flat\Delta).(\flat/A)$, as a comprehension, will be. Thus, the requirement that $\flat$ be a morphism of natural models (as opposed to, say, a lax morphism) may be too strong for some models.

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References

[1] Steve Awodey. Models of type theory. Institute for Advanced Study, Princeton. Notes from lecture on 5th October, 2012.

[2] Steve Awodey. Natural models of homotopy type theory. *Mathematical Structures in Computer Science*, 28(2):241–286, 2018.

[3] Steve Awodey and Andrej Bauer. Propositions as [types]. *Journal of Logic and Computation*, 14(4):447–471, 2004.

[4] Nick Benton. A mixed linear and non-linear logic: Proofs, terms and models. In *International Workshop on Computer Science Logic*, pages 121–135. Springer, 1994.

[5] Nick Benton and Philip Wadler. Linear logic, monads and the lambda calculus. In *Proceedings 11th Annual IEEE Symposium on Logic in Computer Science*, pages 420–431. IEEE, 1996.

[6] Gavin Bierman and Valeria de Paiva. On an intuitionistic modal logic. *Studia Logica*, 65(3):383–416, 2000.

[7] Ranald Clouston, Bassel Mannaa, Rasmus Ejlers Møgelberg, Andrew Pitts, and Bas Spitters. Modal dependent type theory and dependent right adjoints. *arXiv preprint, arXiv:1804.05236*, 2018.

[8] Peter Dybjer. Internal type theory. In *International Workshop on Types for Proofs and Programs*, pages 120–134. Springer, 1995.

[9] Marcelo Fiore. Discrete generalised polynomial functors. In *9th International Colloquium on Automata, Languages and Programming*. Slides, 2012.

[10] Martin Hofmann and Thomas Streicher. The groupoid interpretation of type theory. *Twenty-five Years of Constructive Type Theory (Venice, 1995)*, 36: 83–111, 1998.

[11] Paul Blain Levy. Call-by-push-value: A subsuming paradigm. In *International Conference on Typed Lambda Calculi and Applications*, pages 228–243. Springer, 1999.

[12] Daniel R Licata, Michael Shulman, and Mitchell Riley. A fibrational framework for substructural and modal logics. In *2nd International Conference on Formal Structures for Computation and Deduction*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2017.

[13] Richard Montague. The proper treatment of quantification in ordinary English. In K. Hintikka, J. Moravcsik, and Suppes P., editors, *Approaches to Natural Language*, pages 221–242. D. Reidel, Dordrecht, 1973.

[14] Aleksandar Nanevski, Frank Pfenning, and Brigitte Pientka. Contextual modal type theory. *ACM Transactions on Computational Logic*, 9(3):23, 2008.

[15] Clive Newstead. Algebraic models of dependent type theory. Ph.D. thesis, Carnegie Mellon University, Pittsburgh, 2018.

[16] Clive Newstead. Personal Communication, 2019.

[17] The nLab. Representable morphism. https://ncatlab.org/nlab/show/representable+morphism. Accessed: May 3, 2019.

[18] Aarne Ranta. *Type-Theoretical Grammar*. Oxford University Press, 1994.

[19] Jason Reed. A judgmental deconstruction of modal logic. Unpublished manuscript, 2009.

[20] Gonzalo Reyes and Houman Zolfaghari. Topos-theoretic approaches to modality. In A. Carboni, M. Pedicchio, and G. Rosolini, editors, *Category Theory, Proceedings of the International Conference Held in Como, 1990*, volume 1488 of *Lecture Notes in Mathematics*, pages 359–378. 1991.

[21] Michael Shulman. Geometric HoTT and comonadic modalities. In *Geometry in Modal Homotopy Type Theory*, Carnegie Mellon University, Pittsburgh. Lecture, 2019.

[22] Michael Shulman. Brouwer's fixed-point theorem in real-cohesive homotopy type theory. *Mathematical Structures in Computer Science*, 28(6): 856–941, 2018.

[23] Göran Sundholm. Constructive generalized quantifiers. *Synthese*, 79(1): 1–12, 1989.

[24] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. https://homotopytypetheory.org/book, Institute for Advanced Study, Princeton, 2013.

[25] Matthijs Vákár. In search of effectful dependent types. Ph.D. thesis, Oxford University, Oxford, 2017.

[26] Colin Zwanziger. Montague's intensional logic as comonadic type theory. M.S. thesis, Carnegie Mellon University, Pittsburgh, 2017.

[27] Colin Zwanziger. Propositional attitude operators in homotopy type theory. In *Fifth Workshop on Natural Language and Computer Science*, Oxford, Extended Abstract, 2018.

[28] Colin Zwanziger. Dependently-typed Montague semantics in the proof assistant Agda-flat. In *Proceedings of the 16th Meeting on Mathematics of Language*, Toronto, Forthcoming.