Hopf-Frobenius algebras and a new Drinfeld double

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In the monoidal categories approach to quantum theory [**?**, **?**] Hopf algebras [**?**] have a central role in the formulation of complementary observables [**?**]. In this setting, a quantum observable is represented as special commutative †-Frobenius algebra; a pair of such observables are called *strongly complementary* if the algebra part of the first and the coalgebra part of the second jointly form a Hopf algebra. In abstract form, this combination of structures has been studied under the name "interacting Frobenius algebras" [**?**] where it is shown that relatively weak commutation rules between the two Frobenius algebras produce the Hopf algebra structure. From a different starting point Bonchi et al [**?**] showed that a distributive law between two Hopf algebras yields a pair of Frobenius structures, an approach which has been generalised to provide a model of Petri nets [**?**]. Given the similarity of the two structures it is appropriate to consider both as exemplars of a common family of *Hopf-Frobenius algebras*.

In the above settings, the algebras considered are both commutative and cocommutative. However more general Hopf algebras, perhaps not even symmetric, are a ubiquitous structure in mathematical physics, finding application in gauge theory [**?**], condensed matter theory [**?**], quantum field theory [**?**] and quantum gravity [**?**]. We take the first steps towards generalising the concept of Hopf-Frobenius algebra to the non-commutative case, and opening the door to applications of categorical quantum theory in other areas of physics.

Loosely speaking, a Hopf-Frobenius algebra consists of two monoids and two comonoids such that one way of pairing a monoid with a comonoid gives two Frobenius algebras, and the other pairing yields two Hopf algebras, with the additional condition that antipodes are constructed from the Frobenius forms. This schema is illustrated in Figure **??**.

Fundamental to the concept of a Hopf-Frobenius algebra is a particular pair of morphisms called an integral and a cointegral. We show that when these morphisms are 'compatible' in a particular sense, they produce structure similar to a Hopf-Frobenius algebra. It is from this that we produce necessary and sufficient conditions to extend a Hopf algebra to a Hopf-Frobenius algebra in a symmetric monoidal category. It was previously known that in $\mathbf{FVect_k}$, the category of finite dimensional vector spaces, every Hopf algebra carries a Frobenius algebra on both its monoid [**?**] and its comonoid [**?**, **?**]; in fact we show that every Hopf algebra in $\mathbf{FVect_k}$ is Hopf-Frobenius. We are therefore able to find many examples of Hopf-Frobenius algebras that are not commutative or cocommutative. Finally, due to the fact that every Frobenius algebra is self dual, in a compact closed category we may find a natural isomorphism between the algebra and its dual. We use this isomorphism to construct a Hopf algebra on $H \otimes H$ that is isomorphic to the Drinfeld double.

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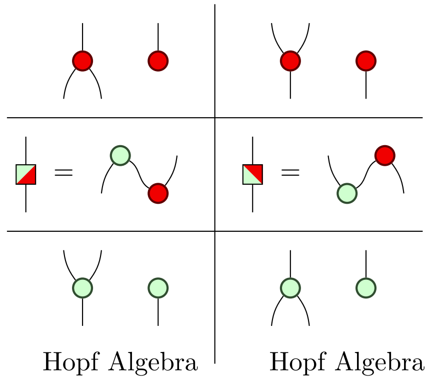

Figure 1: The elements of a Hopf-Frobenius algebra

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