The support is a morphism of monads

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A valuation on a topological space X is a consistent assignment of mass to open sets $U \subseteq X$, which can be interpreted as the likelihood that an unknown element of X is in U. An analogous statement applies to Borel measures. A similar notion is that of possibility: a consistent assignment of Boolean truth values to open sets $U \subseteq X$, which can be interpreted as indicating whether an unknown element of X may be in U. One of our goals is to show that taking the support of a valuation, or a measure, interpolates between these two pictures: the support intersects an open set U if and only if U has positive mass.

Technically, we consider the monad V of continuous valuations with values in $[0, 1]$ on the category Top of topological spaces and continuous maps, this is similar to the extended probabilistic powerdomain [Kir93; Jun04]. The τ-smooth Borel probability measures form a submonad of V. We show that taking the support induces a morphism of monads from V to the monad H of closed subsets. From the point of view of denotational semantics, our morphism $V \to H$ is a continuous map from a probabilistic powerspace to the Hoare powerspace which is compatible with the respective monad structures. Our constructions arise by developing a duality theory, in the spirit of double dualization monads, for both V and H.

The support of the valuation ν is the unique closed set $\mathrm{supp}(\nu)$ such that, for each open U, the set $\mathrm{supp}(\nu)$ intersects U if and only if $\nu(U)$ is strictly positive. From the possibilistic point of view, an open set U is possible if and only if it has positive mass. Therefore the support induces a map $\mathrm{supp} : VX \to HX$ from valuations to closed subsets. We prove that this map is continuous, natural, and that it respects the two monad structures. Moreover, this morphism naturally restricts to probability measures, where it corresponds to the usual notion of support of a measure, and also yields a morphism of monads $P \to H$.

This theory can be applied to dynamical systems and in the study of convex structures.

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The hyperspace monad

Definition 1. *Let X be a topological space and consider $A \subseteq X$. We say that a subset $C \subseteq X$ hits A if and only if $C \cap A \neq \varnothing$.*

Definition 2. *Let X be a topological space. Its hyperspace HX is the set whose points are* closed subsets of X*, including the empty set, equipped with the* lower Vietoris topology

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generated by the collections

$$\{C \subseteq X \,:\, C \text{ is closed and hits } U\},$$

for each open $U \subseteq X$.

A closed is uniquely specified by the open sets that it hits.

Proposition 3. *Let X be a topological space. There is a bijection (in fact, an isomorphism of complete lattices) between HX and Scott-continuous functionals $\varphi : O(X) \to S$ into the Sierpiński space S with the following two properties.*

(a) Strictness: $\varphi(\varnothing) = 0$.

(b) Modularity: For each $U, V \in O(X)$ we have $\varphi(U \cap V) \vee \varphi(U \cup V) = \varphi(U) \vee \varphi(V)$.

The assignment $X \mapsto HX$ generates a monad. This has been known at least since [Sch93], see also [CT97]. We call it the *hyperspace monad*. To a continuous map $f : X \to Y$ this functor assigns the continuous map $HX \to HY$ given by the closure of the image of a set under f. The unit $\sigma : X \to HX$ assigns to $x \in X$ its closure. The multiplication $\mathcal{U} : HHX \to HX$ is given by the closure of the union.

It is well-known that the algebras of this monad, at least on the category of T_0 spaces, are topological semilattices, join-semilattices for which the join is continuous [Sch93].

The valuation and measure monads

Definition 4. *Let X be a topological space. The space VX is the set of continuous $[0,1]$-valued valuations on X, equipped with the topology generated by the sets*

$$\theta(U, r) := \{\nu \,:\, \nu(U) > r\}$$

for open $U \subseteq X$ and $r \in [0,1)$.

V is a functor whose action on morphism is given, as usual, by the pushforward. This functor also generates a monad. The unit $\delta : X \to VX$ assigns $x \mapsto \delta_x$, where δ_x is the point-mass valuation at $x \in X$. The multiplication is given by lower integration [Kir93; Jun04]. This monad is similar to the *extended probabilistic powerdomain* [Jun04]. The difference is that the extended probabilistic powerdomain consists of the $[0,\infty]$-valued valuations.

Every τ-smooth Borel measure induces a continuous valuation. The converse, that every continuous valuation can be extended to a τ-smooth Borel measure, is not true. The 1-normalized valuations which extend to τ-smooth Borel measures form a submonad. We denote by P the functor that assigns to a topological space the extendable normalized valuations (equivalently the τ-smooth Borel probability-measures). The respective monad structure is inherited from V. The submonad P seems to be the most general monad of probability measures on topological spaces, and, as far as we know, is a new construction. It extends the Giry monad on Polish spaces [Gir82], the Radon monad on compact Hausdorff spaces [Świ74], and Banakh's probability monad on Tikhonov spaces [Ban95].

The support as a morphism of monads

Intuitively, the support is the set of points whose neighborhoods have positive mass.

Definition 5. *Let X be a topological space and consider $\nu \in VX$. Its support* $\operatorname{supp}(\nu)$ *is the unique closed subset of X that hits an open set $U \subseteq X$ if and only if $\nu(U) > 0$.*

By Proposition 3, the above uniquely determines a closed subset of X. If ν is extendable, its support coincides with the usual support of the respective measure. The support defines a continuous map $\mathrm{supp} : VX \to HX$ that commutes with the structure maps of V.

Theorem 6. *The support induces a morphism of monads* $\mathrm{supp} : (V, \delta, \mathcal{E}) \to (H, \sigma, \mathcal{U})$*. This restricts to a morphism of monads* $(P, \delta, \mathcal{E}) \to (H, \sigma, \mathcal{U})$*.*

Applications

Convex structures. An *abstract convex set* is a structure that generalizes the properties of convex regions of vector spaces [Fri09]. There are convex structures, namely join-semilattices, which are in general not embeddable as convex regions of vector spaces. As we prove, every V-algebra satisfies a topological analogue of the axioms of an abstract convex set. Via the morphism $\mathrm{supp} : V \to H$, every H-algebra is canonically a V-algebra. From this point of view, it is very natural that join-semilattices are abstract convex sets.

Dynamical systems. An iterated map system is the dynamical system generated by a continuous map $f : X \to X$. One is interested in the $\mathbb{N}_0$-action $(x, t) \mapsto f^t(x)$. A morphism between the systems (X, f) and (Y, g) is a continuous surjection $m : X \to Y$ that intertwines the actions. Any system (X, f) induces an $\mathbb{N}_0$-action on the space of closed subsets via direct images of f. It also induces an action on the space of probability measures on X via pushforward along f. One motivation for our work was to relate these collective dynamics by a morphism, the support. Under appropriate topological conditions, supp is a continuous surjection and therefore a morphism of dynamical systems. This is also useful for the study of chaotic systems. It follows that if X is a compact space and PX is also compact, then the *topological entropy* of (PX, Pf) is greater than or equal to the topological entropy of (HX, Hf). This means that (HX, Hf) can only be topologically chaotic if (PX, Pf) is topologically chaotic.

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