String diagrams and the algebra of entanglement

Amar Hadzihasanovic

University of Oxford

Oxford, 15 October 2014

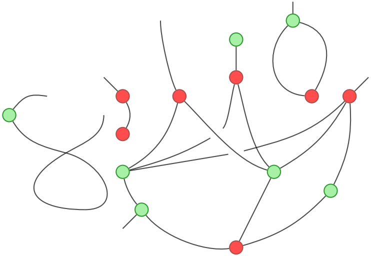
My favourite thing about ZX

My favourite thing about ZX

It's undirected

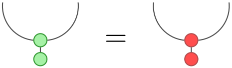
My favourite thing about ZX

It's undirected
and *at least* as symmetrical as it looks.



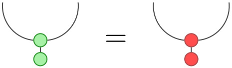
Where does it come from?

- **Algebraic** perspective: two *commutative* Frobenius algebras that happen to satisfy



Where does it come from?

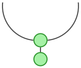
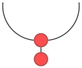
- **Algebraic** perspective: two *commutative* Frobenius algebras that happen to satisfy



- **Geometric** perspective: fixed basis ($\rightsquigarrow$ one "right" duality), and all vertices are symmetrical with respect to it

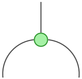
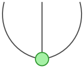
Where does it come from?

- **Algebraic** perspective: two *commutative* Frobenius algebras that happen to satisfy

 $=$ 

- **Geometric** perspective: fixed basis ($\rightsquigarrow$ one "right" duality), and all vertices are symmetrical with respect to it

  the ambient category is already undirected...

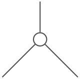
 $\mapsto \; 000 + 111 \; \leftarrow\!\shortmid$ 

$|0\rangle\langle 00| + |1\rangle\langle 11|$

$|000\rangle + |111\rangle$

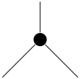
Here's some more things that are symmetrical.

GHZ



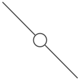
$000 - 111$

W



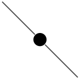
$001 + 010 + 100$

Z



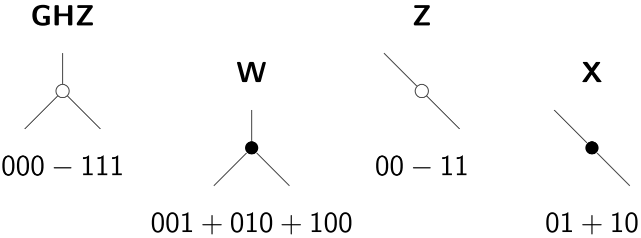
$00 - 11$

X

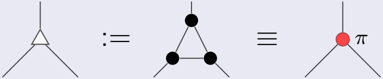


$01 + 10$

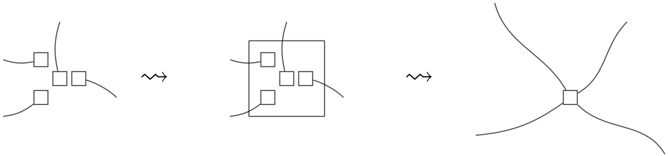
Here's some more things that are symmetrical.



Also,

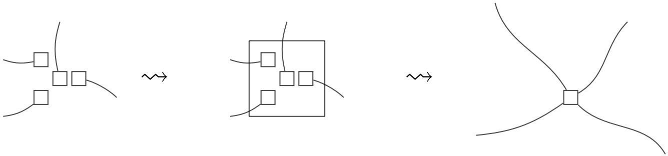


Entanglement for communication



An entangled state can be used to generate shared knowledge

Entanglement for communication



An entangled state can be used to generate shared knowledge

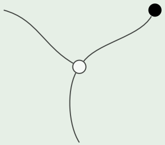
- What communication can (in theory) be achieved does **not** depend on *local, invertible* operations performed by the users

And this is what **SLOCC classification** is about

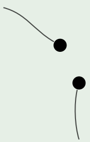
Three users connected, one does not cooperate...

Three users connected, one does not cooperate...

GHZ

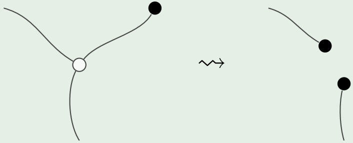


$\rightsquigarrow$

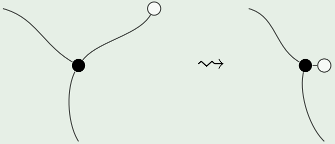


Three users connected, one does not cooperate...

GHZ



W



And for $n \geq 4$...

And for $n \geq 4$...

...a continuous infinity of SLOCC classes.

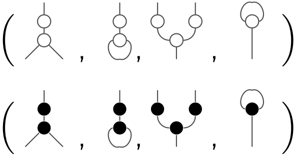
Learning from undirectedness

A *tripartite state* is the same as a binary operation

Learning from undirectedness

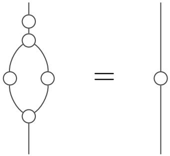
A *tripartite state* is the same as a binary operation

- We can associate *commutative Frobenius algebras* to the GHZ and W states:



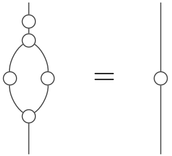
Not everybody can be special.

- The GHZ Frobenius algebra is **special**:

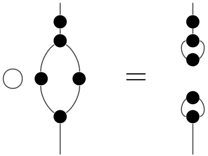


Not everybody can be special.

- The GHZ Frobenius algebra is **special**:



- The W Frobenius algebra is **anti-special**:



Towards a compositional classification?

Bob and Aleks have shown that

1. to all symmetric, connected 3-qubit states, it is possible to associate a commutative Frobenius algebra

Towards a compositional classification?

Bob and Aleks have shown that

1. to all symmetric, connected 3-qubit states, it is possible to associate a commutative Frobenius algebra
2. this commutative Frobenius algebra is
   - **special** iff the state is SLOCC-equivalent to GHZ,
   - **anti-special** iff the state is SLOCC-equivalent to W

Towards a compositional classification?

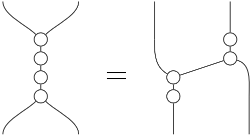
Bob and Aleks have shown that

1. to all symmetric, connected 3-qubit states, it is possible to associate a commutative Frobenius algebra
2. this commutative Frobenius algebra is
   - **special** iff the state is SLOCC-equivalent to GHZ,
   - **anti-special** iff the state is SLOCC-equivalent to W

Through the diagrammatic calculus: GHZ and W as building blocks for higher classes?

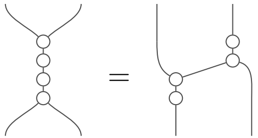
Affinity...

The GHZ monoid is still a Frobenius algebra with its **adjoint**:



Affinity...

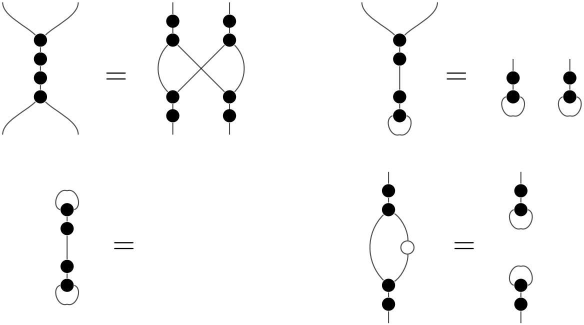
The GHZ monoid is still a Frobenius algebra with its **adjoint**:



(this actually implies specialness)

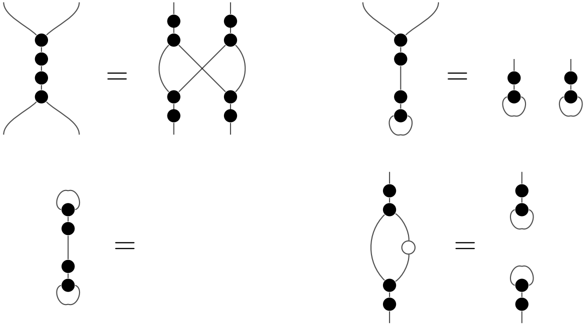
...and disagreement

The W monoid, however, is a **Hopf algebra** with its adjoint...



...and disagreement

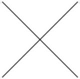
The W monoid, however, is a **Hopf algebra** with its adjoint...



...but *not* with the usual symmetric braiding -

Spes unica!

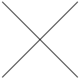
- we need the following crossing instead:



$$\mapsto \quad |00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| - |11\rangle\langle 11|$$

Spes unica!

- we need the following crossing instead:

 $$\mapsto \quad |00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| - |11\rangle\langle 11|$$

We will draw the symmetric braiding as

 , 

Forget about qubits for a while.

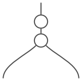
We can put the GHZ, W monoids on top of more general R-modules on two generators

(let us say, $1 \equiv |0\rangle$ and $\gamma \equiv |1\rangle$)

Forget about qubits for a while.

We can put the GHZ, W monoids on top of more general R-modules on two generators

(let us say, $1 \equiv |0\rangle$ and $\gamma \equiv |1\rangle$)

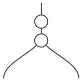


$$1 \otimes 1 \mapsto 1, \quad 1 \otimes \gamma \mapsto 0, \quad \gamma \otimes 1 \mapsto 0, \quad \gamma \otimes \gamma \mapsto \gamma$$

Forget about qubits for a while.

We can put the GHZ, W monoids on top of more general R-modules on two generators

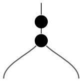
(let us say, $1 \equiv |0\rangle$ and $\gamma \equiv |1\rangle$)



$$1 \otimes 1 \mapsto 1, \quad 1 \otimes \gamma \mapsto 0, \quad \gamma \otimes 1 \mapsto 0, \quad \gamma \otimes \gamma \mapsto \gamma$$

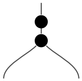
a **componentwise** product

The W algebra, concretely



$$1 \otimes 1 \mapsto 1, \quad 1 \otimes \gamma \mapsto \gamma, \quad \gamma \otimes 1 \mapsto \gamma, \quad \gamma \otimes \gamma \mapsto 0$$

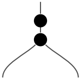
The W algebra, concretely



$$1 \otimes 1 \mapsto 1, \quad 1 \otimes \gamma \mapsto \gamma, \quad \gamma \otimes 1 \mapsto \gamma, \quad \gamma \otimes \gamma \mapsto 0$$

R-algebra with one nilpotent generator

The W algebra, concretely



$$1 \otimes 1 \mapsto 1, \quad 1 \otimes \gamma \mapsto \gamma, \quad \gamma \otimes 1 \mapsto \gamma, \quad \gamma \otimes \gamma \mapsto 0$$

R-algebra with one nilpotent generator

(when $R = \mathbb{R}$, these are called **dual numbers**)

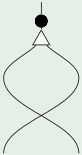
Commute, anti-commute...

Placing crossings, or **braidings**, before a monoid induces an anticommutation, or a **commutation** relation, respectively, between generators of an algebra

Commute, anti-commute...

Placing crossings, or **braidings**, before a monoid induces an anticommutation, or a **commutation** relation, respectively, between generators of an algebra

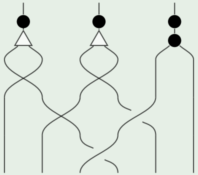
Complex multiplication



$$1 \otimes 1 \mapsto 1, \quad 1 \otimes \gamma \mapsto \gamma, \quad \gamma \otimes 1 \mapsto \gamma, \quad \gamma \otimes \gamma \mapsto -1$$

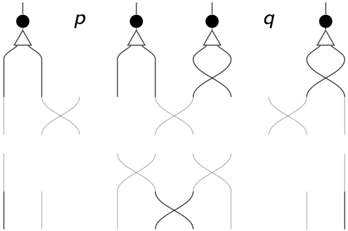
Another example

Dual quaternions



(can be used to represent rigid motions in 3D space)

More in general,



is multiplication in $\mathrm{Cl}_{p,q}(\mathbb{R})$, the real Clifford algebra with signature (p, q)

A case study

Undirectedness / map-state duality (geometry)

A case study

Undirectedness / map-state duality (geometry)

$\searrow$ The GHZ, W monoids (**algebra**)

A case study

Undirectedness / map-state duality (geometry)

$\searrow$ The GHZ, W monoids (**algebra**)

Introducing crossings (geometry) $\swarrow$

A case study

Undirectedness / map-state duality (geometry)

$\searrow$ The GHZ, W monoids (**algebra**)

Introducing crossings (geometry) $\swarrow$

$\searrow$ (Anti)commutation relations (**algebra**)

A case study

Undirectedness / map-state duality (geometry)

$\searrow$ The GHZ, W monoids (**algebra**)

Introducing crossings (geometry) $\swarrow$

$\searrow$ (Anti)commutation relations (**algebra**)

CQM and string diagrams connect the **classification of multipartite entanglement** and geometric algebra!

The ZW calculus

A **complete*** **graphical axiomatisation** of the relations between the GHZ and W vertices

The ZW calculus

A **complete*** **graphical axiomatisation** of the relations between the GHZ and W vertices

* for the category of abelian groups generated by $\mathbb{Z} \oplus \mathbb{Z}$ under tensoring†

The ZW calculus

A **complete*** **graphical axiomatisation** of the relations between the GHZ and W vertices

* for the category of abelian groups generated by $\mathbb{Z} \oplus \mathbb{Z}$ under tensoring†

† "qubits with integer coefficients"

The ZW calculus

A **complete*** **graphical axiomatisation** of the relations between the GHZ and W vertices

* for the category of abelian groups generated by $\mathbb{Z} \oplus \mathbb{Z}$ under tensoring†

† "qubits with integer coefficients"

Upcoming paper

AH, *Entanglement and (quantum) algebra with undirected string diagrams*

The ZW calculus

A **complete*** **graphical axiomatisation** of the relations between the GHZ and W vertices

* for the category of abelian groups generated by $\mathbb{Z} \oplus \mathbb{Z}$ under tensoring†

† "qubits with integer coefficients"

Upcoming paper

AH, *Entanglement and (quantum) algebra with undirected string diagrams*

(come ask me if you're interested!)