THE COLLEGES OF OXFORD UNIVERSITY
MATHEMATICS, JOINT SCHOOLS, AND COMPUTER SCIENCE

SUNDAY 10 DECEMBER 2006

Time allowed: $2\frac{1}{2}$ hours

For candidates applying for Mathematics, Mathematics & Statistics, Computer Science, Mathematics & Computer Science, or Mathematics & Philosophy

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Write your name, college (where you are sitting the test), and your proposed course (from the list above) in BLOCK CAPITALS

NAME:

COLLEGE:

COURSE:

NOTE: This paper contains 7 questions, of which you should attempt 5. Throughout the paper are directions as to which questions are appropriate for your course.

{ **Mathematics**, **Maths & Philosophy**, **Maths & Statistics** } candidates should attempt **Questions 1, 2, 3, 4, 5**.

Maths & Computer Science candidates should attempt **Questions 1, 2, 3, 5, 6.**

Computer Science candidates should attempt **Questions 1, 2, 5, 6, 7.**

No credit can be gained by attempting extra questions.

Question 1 is a multiple choice question with ten parts, for which marks are given solely for the correct answers, though you may use the space between parts for rough work. Answer Question 1 on the grid on Page 2. Each part is worth 4 marks.

Answers to Questions 2–7 should be written in the space provided, continuing onto the blank pages at the end of this booklet if necessary. Each of Questions 2–7 is worth 15 marks.

ONLY ANSWERS WRITTEN IN THIS BOOKLET WILL BE MARKED.
DO NOT INCLUDE EXTRA SHEETS OR ROUGH WORK.

THE USE OF CALCULATORS OR FORMULA SHEETS IS PROHIBITED.

1. For **ALL APPLICANTS**.

For each part of the question on pages 3–7 you will be given four possible answers, just one of which is correct. Indicate for each part **A**–**J** which answer (a), (b), (c), or (d) you think is correct with a tick (✓) in the corresponding column in the table below. Please show any rough working in the space provided between the parts.

	(a)	(b)	(c)	(d)
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				

A. Which of the following numbers is largest?

(a) $\left(\left(2^3\right)^2\right)^3$, (b) $\left(2^3\right)^{\left(2^3\right)}$, (c) $2^{\left(\left(3^2\right)^3\right)}$, (d) $2^{\left(3^{\left(2^3\right)}\right)}$.

B. The equation

$$\left(2+x-x^2\right)^2=16$$

has:

(a) no real roots,
(b) one real root,
(c) two real roots,
(d) three real roots.

C. The function f is defined for whole positive numbers and satisfies $f(1) = 1$ and also the rules

$$f(2n) = f(n),$$
$$f(2n+1) = f(n) + 1,$$

for all values of n. It follows that $f(9)$ equals

(a) 1, (b) 2, (c) 3, (d) 4.

D. The function

$$y = x^2 \ln x$$

defined for $x > 0$ satisfies

(a) $x\,(\mathrm{d}y/\mathrm{d}x) = 2y + x^2$, (and only this part),
(b) $\mathrm{d}y/\mathrm{d}x > 0$ for all x, (and only this part),
(c) $\mathrm{d}^2y/\mathrm{d}x^2 \neq 0$ for all x, (and only this part),
(d) all of the above.

E. The cubic

$$x^3 + ax + b$$

has both $(x-1)$ and $(x-2)$ as factors. Then

(a) $a = -7$ and $b = 6$,
(b) $a = -3$ and $b = 2$,
(c) $a = 0$ and $b = -2$,
(d) $a = 5$ and $b = 4$.

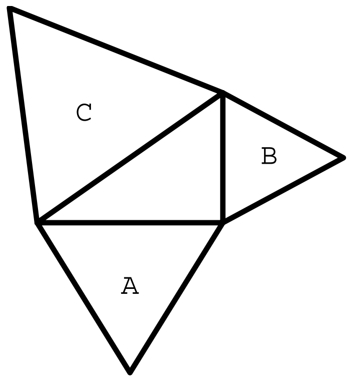
F. The inequality

$$\frac{x^2+1}{x^2-1} < 1$$

is true:

(a) for no values of x,
(b) whenever $-1 < x < 1$,
(c) whenever $x > 1$,
(d) for all values of x.

G. Three equilateral triangles with areas A, B, C are drawn on the sides of a right-angled triangle as in the diagram below.



These areas are related by the equation:

(a) $A^{1/2} + B^{1/2} = C^{1/2}$,
(b) $A + B = C$,
(c) $A^{3/2} + B^{3/2} = C^{3/2}$,
(d) $A^2 + B^2 = C^2$.

H. How many solutions does the equation

$$2 = \sin x + \sin^2 x + \sin^3 x + \sin^4 x + \ldots$$

have in the range $0 \leqslant x < 2\pi$?

(a) 0, (b) 1, (c) 2, (d) 3.

I. The equation

$$|x| + |x - 1| = 0$$

has

(a) no solutions,
(b) one solution,
(c) two solutions,
(d) three solutions.

J. The two circles with equations

$$x^2 + y^2 = 1, \qquad (x-a)^2 + (y-b)^2 = r^2$$

(where $r > 0$) do *not* intersect if

(a) $\sqrt{a^2+b^2} + r < 1$, (and only this part),
(b) $\sqrt{a^2+b^2} + 1 < r$, (and only this part),
(c) $\sqrt{a^2+b^2} - r > 1$, (and only this part),
(d) all of the above.

2. For **ALL APPLICANTS.**

The real numbers x and y satisfy the equation

$$x^2 + xy + y^2 = 1. \tag{1}$$

(i) If $y = 1$ then find the possible values of x.

(ii) For what values of y are there two possible (real) values of x?

(iii) Find the largest possible value of y which satisfies (1) and the corresponding value of x.

(iv) Show that, for all values of θ, the numbers

$$\begin{aligned} x &= \frac{1}{\sqrt{3}}\cos\theta + \sin\theta, \\ y &= \frac{1}{\sqrt{3}}\cos\theta - \sin\theta, \end{aligned}$$

satisfy the equation (1).

3.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\\ \textbf{MATHEMATICS \& COMPUTER SCIENCE}\end{array}\right\}$ **ONLY.**

Computer Science **applicants should turn to page 14.**

Let

$$f(x) = x^3 - 3x^2 + 2x.$$

(i) On the axes below, sketch the curve $y = f(x)$ for the range $-1 < x < 3$, carefully labelling any turning points.

(ii) The equation $f(x) = k$ has exactly one positive solution and exactly one negative solution. Find k.

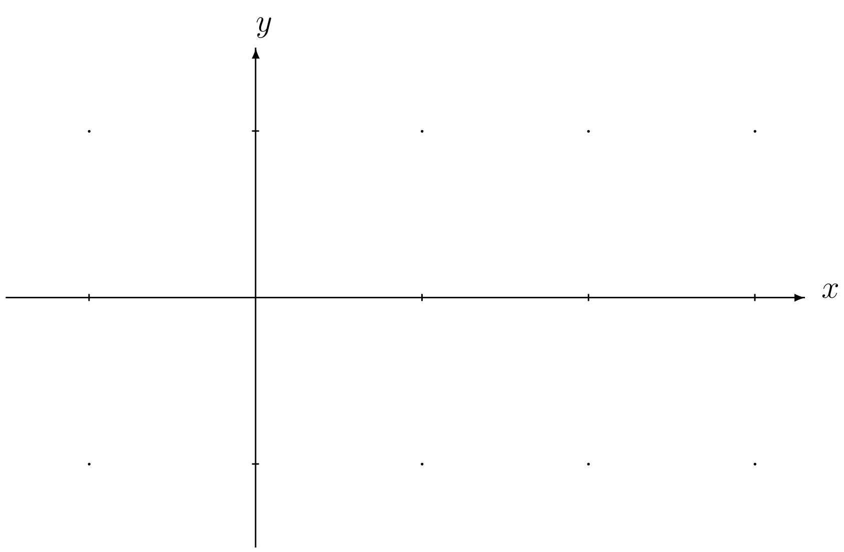
For x in the range $0 \leqslant x \leqslant 2$ the functions $g(x)$ and $h(x)$ are defined by

$$\begin{aligned} g(x) &= \int_0^x f(t)\, \mathrm{d}t \\ h(x) &= \int_0^x |f(t)|\, \mathrm{d}t. \end{aligned}$$

(iii) Find the value X_1 of x in the range $0 \leqslant x \leqslant 2$ for which $g(x)$ is greatest.

Calculate $g(X_1)$.

(iv) Find the value X_2 of x in the range $0 \leqslant x \leqslant 2$ for which $h(x)$ is greatest. [You are not asked to calculate $h(X_2)$.]

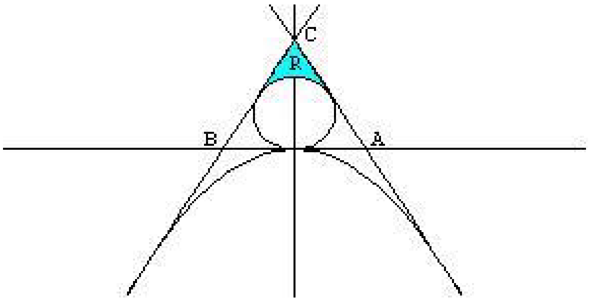


4.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\end{array}\right\}$ **ONLY.**

Mathematics & Computer Science **and** *Computer Science* **applicants should turn to page 14.**

In the diagram below are drawn the circle $x^2 + (y-1)^2 = 1$ and the parabola $y = -\frac{1}{4}x^2$.



(i) Find the equation of the tangent to the parabola $y = -\frac{1}{4}x^2$ at the point $\left(t, -\frac{1}{4}t^2\right)$.

(ii) Show that, when $t = 2\sqrt{3}$, this tangent is also a tangent to the circle $x^2+(y-1)^2 = 1$.

By symmetry the tangent to the parabola at $\left(-2\sqrt{3}, -3\right)$ is also a tangent to the circle. These tangents are also shown on the diagram above.

(iii) The first and second tangents meet the x-axis at A and B respectively; the two tangents intersect at C. Find the angle CAB.

Deduce that the area of the shaded region R, bounded by the circle and the two tangents, is

$$\sqrt{3} - \frac{\pi}{3}.$$

5. For **ALL APPLICANTS.**

For any four numbers $a, b, c, d,$ the symbol

$$\begin{array}{c|c} a & b \\ \hline c & d \end{array}$$

represents a number that depends on a, b, c and d and has the following properties:

$$\begin{array}{c|c} s\times a & s\times b \\ \hline c & d \end{array} \;=\; s\times \begin{array}{c|c} a & b \\ \hline c & d \end{array} \qquad (1)$$

$$\begin{array}{c|c} a & b \\ \hline c & d \end{array} \;=\; \begin{array}{c|c} d & b \\ \hline c & a \end{array} \qquad (2)$$

$$\begin{array}{c|c} a & b \\ \hline c & d \end{array} \;=\; \begin{array}{c|c} a & c \\ \hline b & d \end{array} \qquad (3)$$

$$\begin{array}{c|c} a+x & b+y \\ \hline c & d \end{array} \;=\; \begin{array}{c|c} a & b \\ \hline c & d \end{array} + \begin{array}{c|c} x & y \\ \hline c & d \end{array} \qquad (4)$$

You may assume nothing else about $\begin{array}{c|c} a & b \\ \hline c & d \end{array}$.

(i) Use property (1) to show that $\begin{array}{c|c} 0 & 0 \\ \hline c & d \end{array} = 0.$

(ii) Hence use properties (2) and (3) to show that each of

$$\begin{array}{c|c} a & 0 \\ \hline c & 0 \end{array}, \quad \begin{array}{c|c} 0 & b \\ \hline 0 & d \end{array}, \quad \begin{array}{c|c} a & b \\ \hline 0 & 0 \end{array}$$

is also zero.

(iii) Show that

$$\begin{array}{c|c} a & b \\ \hline s\times c & s\times d \end{array} = s\times \begin{array}{c|c} a & b \\ \hline c & d \end{array}.$$

(iv) Show that

$$\begin{array}{c|c} a & b \\ \hline c+x & d+y \end{array} = \begin{array}{c|c} a & b \\ \hline c & d \end{array} + \begin{array}{c|c} a & b \\ \hline x & y \end{array}$$

6. For **APPLICANTS IN** $\left\{ \begin{array}{l} \textbf{MATHS \& COMPUTER SCIENCE} \\ \textbf{COMPUTER SCIENCE} \end{array} \right\}$ **ONLY.**

Portia has three boxes made from gold, silver and lead. She has placed a prize in one of these boxes and challenges a friend, Bassanio, to find the prize.

She explains that on each box there is a message which may be true or false. On the basis of these messages Bassanio should be able to choose the box with the prize in.

(i) Initially suppose that there are only two boxes, gold and silver, one of which contains the prize. The messages read:

Gold: The prize is not in here.

Silver: Exactly one of these messages is true.

Which box contains the prize? Explain your answer. [Hint: consider separately the cases where the message on the silver box is true or false.]

(ii) Now suppose that there are all three boxes and that Portia has left the following messages on them:

Gold: The prize is in here.

Silver: The prize is in here.

Lead: At least two of these messages are false.

Which box should Bassanio choose? Explain your answer. [Hint: show that the message on the lead box cannot be false.]

(iii) In this version of the challenge, Portia puts a dagger into one of the boxes. Bassanio must choose a box that does *not* contain the dagger. The messages on the boxes now read as follows:

Gold: The dagger is in this box.

Silver: The dagger is not in this box.

Lead: At most one of these messages is true.

Which box should Bassanio choose? Explain your answer.

7. For **APPLICANTS IN COMPUTER SCIENCE ONLY.**

The game of ABC involves creating "words" formed from the letters **A, B, C**, according to certain rules. All forms of the game start with the word **AB.**

In the simple form of the game, there is a single rule:

$$\text{If the current word is of the form } \mathbf{A}x \text{, for some word } x \text{,}$$
$$\text{then it may be replaced with the word } \mathbf{A}xx. \tag{1}$$

For example the word **ABC** could be replaced by **ABCBC.**

(i) Show how to produce the word **ABBBB.**

(ii) Describe precisely all the words that can be produced in the simple form of the game. You may want to write $\mathbf{B}^n$ as a shorthand for the word formed from n copies of **B**; e.g. $\mathbf{AB}^4 = \mathbf{ABBBB}$.

In the intermediate form of the game, there is a second rule:

$$\text{If the current word is of the form } x\mathbf{BBB} \text{, for some word } x \text{,}$$
$$\text{then it may be replaced by the word } \mathbf{C}x. \tag{2}$$

(iii) Show how to produce the word **CCABB.**

(iv) Describe precisely all the words that can be produced in the intermediate form of the game. Explain your answer.

In the advanced form of the game, there is a third rule:

$$\text{If the current word is of the form } x\mathbf{B} \text{, for some word } x \text{,}$$
$$\text{then it may be replaced by the word } x. \tag{3}$$

(v) Describe precisely all the words that can be produced in the advanced form of the game. Explain your answer.

End of Last Question