Quantum groupoids and logical dualities

(work in progress)

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Categories, Logic and Foundations of Physics

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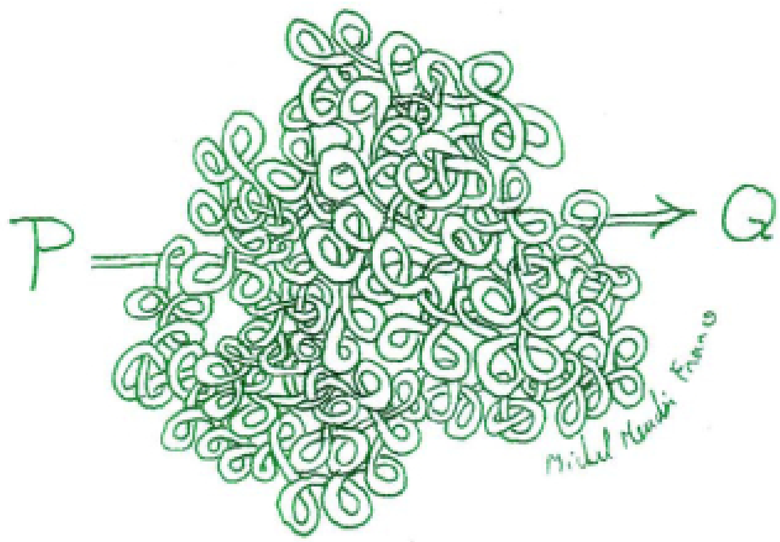
Proof-knots



Aim: formulate an **algebra** of these logical knots

Starting point: game semantics

Every proof of formula A initiates a dialogue where

Proponent tries to convince Opponent

Opponent tries to refute Proponent

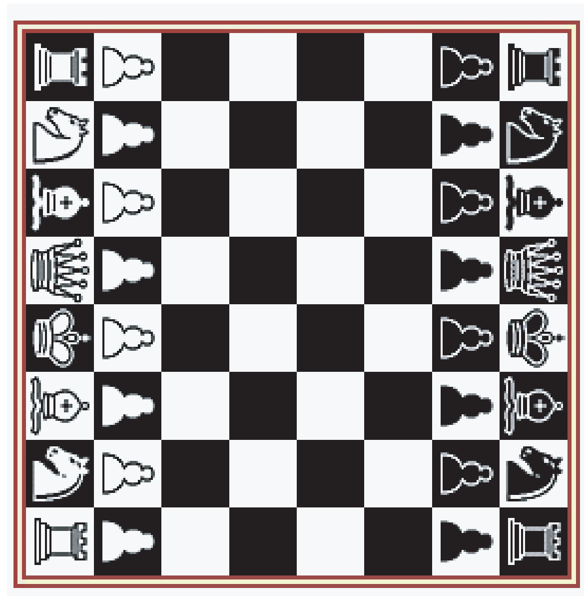
An interactive approach to logic and programming languages

Duality

Proponent
Program

plays the game

A



Opponent
Environment

plays the game

$\neg A$

Negation permutes the rôles of Proponent and Opponent

Duality

Opponent
Environment

plays the game

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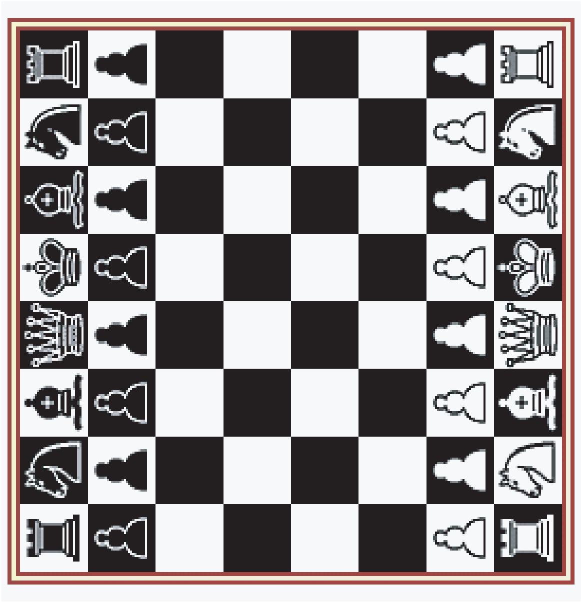


Proponent
Program

plays the game

A

Negation permutes the rôles of Opponent and Proponent

A brief history of games and categories

1977	André Joyal	A category of games and strategies
1986	Jean-Yves Girard	Linear logic
1992	Andreas Blass	A semantics of linear logic
1994	Samson Abramsky Radha Jagadeesan Pasquale Malacaria	A category of history-free strategies
1994	Martin Hyland Luke Ong	A category of innocent strategies

A disturbing gap between game semantics and linear logic

Part 1

The topological nature of negation

At the interface between topology and algebra

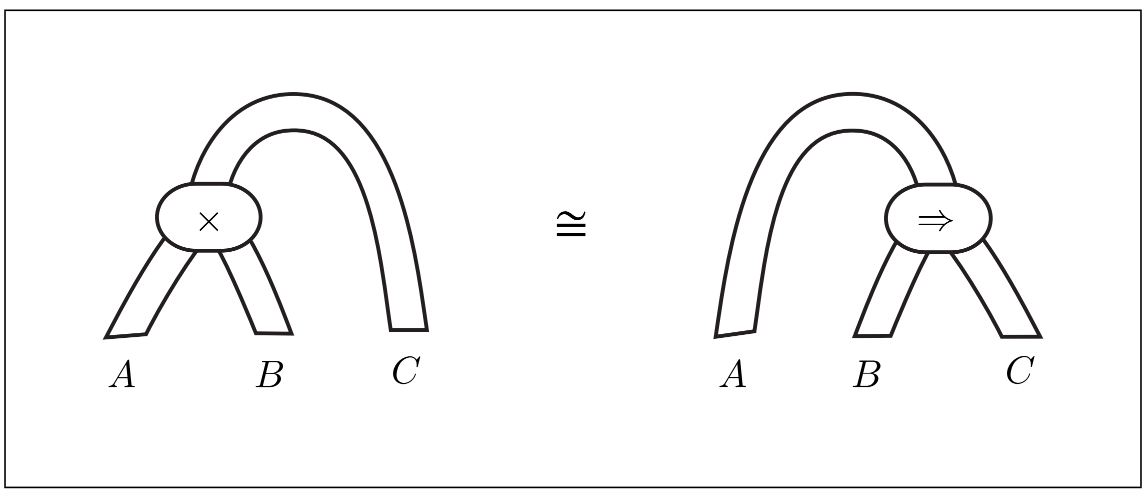
Cartesian closed categories

A **cartesian** category $\mathcal{C}$ is **closed** when there exists a functor

$$\Rightarrow \quad : \quad \mathcal{C}^{op} \times \mathcal{C} \quad \longrightarrow \quad \mathcal{C}$$

and a natural bijection

$$\varphi_{A,B,C} \quad : \quad \mathcal{C}(A \times B \, , \, C) \;\cong\; \mathcal{C}(A \, , \, B \Rightarrow C)$$



The free cartesian closed category

The objects of the category **free-ccc**$(\mathcal{C})$ are the formulas

$$A, B \quad ::= \quad X \quad | \quad A \times B \quad | \quad A \Rightarrow B \quad | \quad 1$$

where X is an object of the category $\mathcal{C}$.

The morphisms are the simply-typed λ-terms, modulo $\beta\eta$-conversion.

The simply-typed λ-calculus

Variable

$$\frac{}{x : X \vdash x : X}$$

Abstraction

$$\frac{\Gamma, x : A \vdash P : B}{\Gamma \vdash \lambda x.P : A \Rightarrow B}$$

Application

$$\frac{\Gamma \vdash P : A \Rightarrow B \qquad \Delta \vdash Q : A}{\Gamma, \Delta \vdash PQ : B}$$

Weakening

$$\frac{\Gamma \vdash P : B}{\Gamma, x : A \vdash P : B}$$

Contraction

$$\frac{\Gamma, x : A, y : A \vdash P : B}{\Gamma, z : A \vdash P[x, y \leftarrow z] : B}$$

Permutation

$$\frac{\Gamma, x : A, y : B, \Delta \vdash P : C}{\Gamma, y : B, x : A, \Delta \vdash P : C}$$

Proof invariants

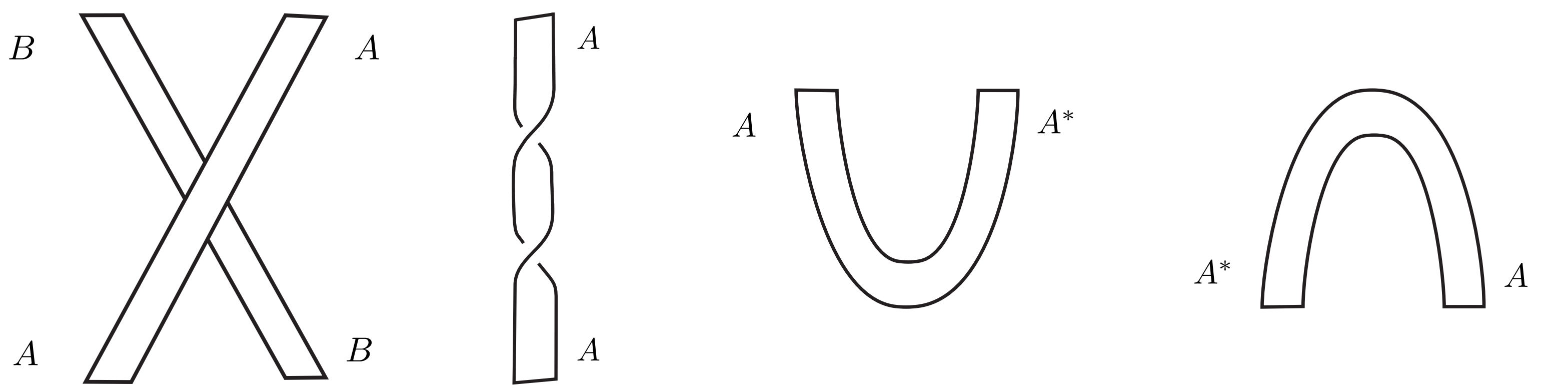
Every ccc $\mathcal{D}$ induces a proof invariant $[-]$ modulo execution.

$$\begin{tikzcd}
\textbf{free-ccc}(\mathcal{C}) \arrow[rr, "{[-]}"] & & \mathcal{D} \\
& \mathcal{C} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Hence the prejudice that proof theory is intrinsically syntactical...

However, a striking similarity with knot invariants

A tortile category is a **monoidal** category with



braiding | twists | duality unit | duality counit

The free tortile category is a category of framed tangles

Knot invariants

Every tortile category $\mathcal{D}$ induces a knot invariant

$$\begin{tikzcd} \textbf{free-tortile}(\mathcal{C}) \arrow[rr, "{[-]}"] & & \mathcal{D} \\ & \mathcal{C} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

A deep connection between algebra and topology
first noticed by Joyal and Street

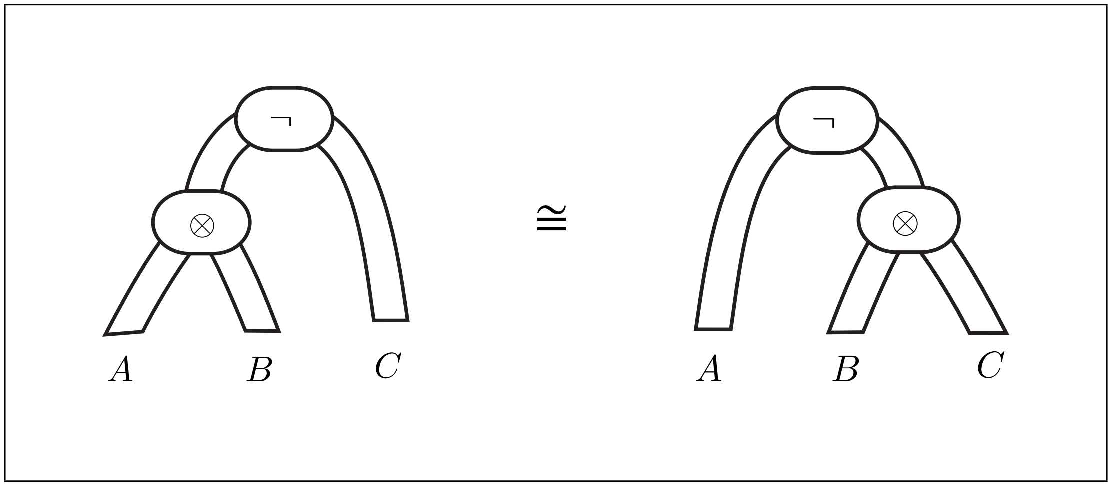
Dialogue categories

A **symmetric monoidal** category $\mathcal{C}$ equipped with a functor

$$\neg \quad : \quad \mathcal{C}^{op} \quad \longrightarrow \quad \mathcal{C}$$

and a natural bijection

$$\varphi_{A,B,C} \quad : \quad \mathcal{C}(A \otimes B\,,\,\neg\,C) \;\cong\; \mathcal{C}(A\,,\,\neg\,(\,B \otimes C\,)\,)$$



The free dialogue category

The objects of the category **free-dialogue**$(\mathcal{C})$ are **dialogue games** constructed by the grammar

$$A, B \quad ::= \quad X \quad | \quad A \otimes B \quad | \quad \neg A \quad | \quad 1$$

where X is an object of the category $\mathcal{C}$.

The morphisms are **total** and **innocent strategies** on dialogue games.

As we will see: proofs are 3-dimensional variants of knots...

A presentation of logic by generators and relations

Negation defines a pair of **adjoint functors**

$$\begin{tikzcd} \mathcal{C} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathcal{C}^{op} \arrow[ll, bend left, "R"] \end{tikzcd}$$

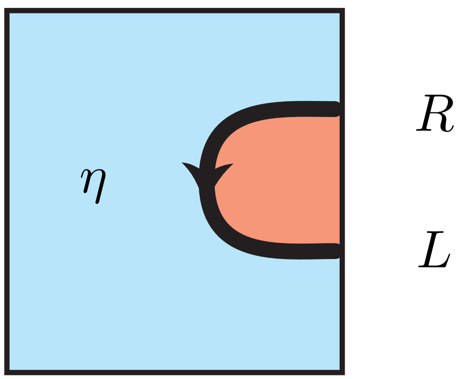
witnessed by the series of bijection:

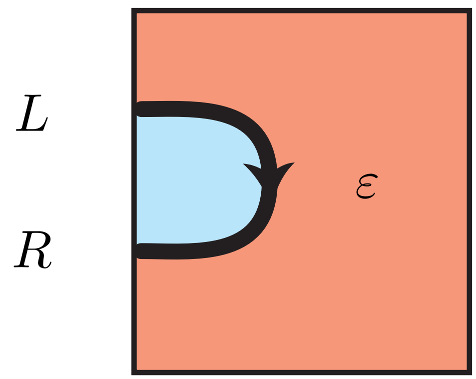
$$\mathcal{C}(A, \neg B) \quad \cong \quad \mathcal{C}(B, \neg A) \quad \cong \quad \mathcal{C}^{op}(\neg A, B)$$

The 2-dimensional topology of adjunctions

The **unit** and **counit** of the adjunction $L \dashv R$ are depicted as

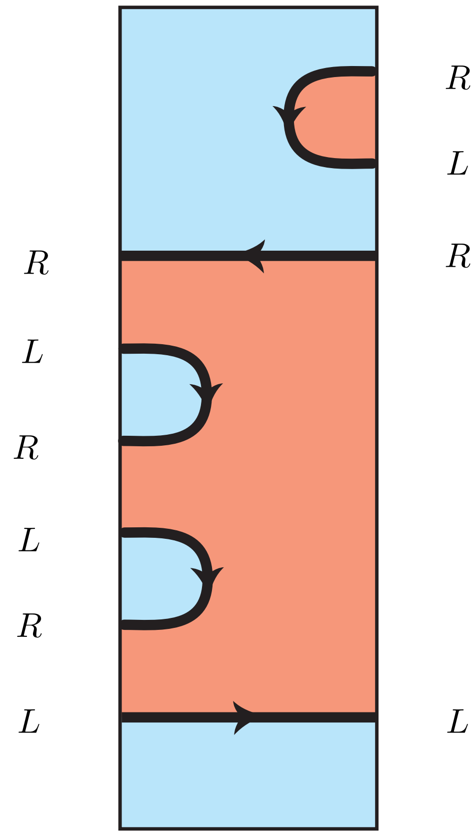
$$\eta : Id \longrightarrow R \circ L \qquad\qquad \varepsilon : L \circ R \longrightarrow Id$$





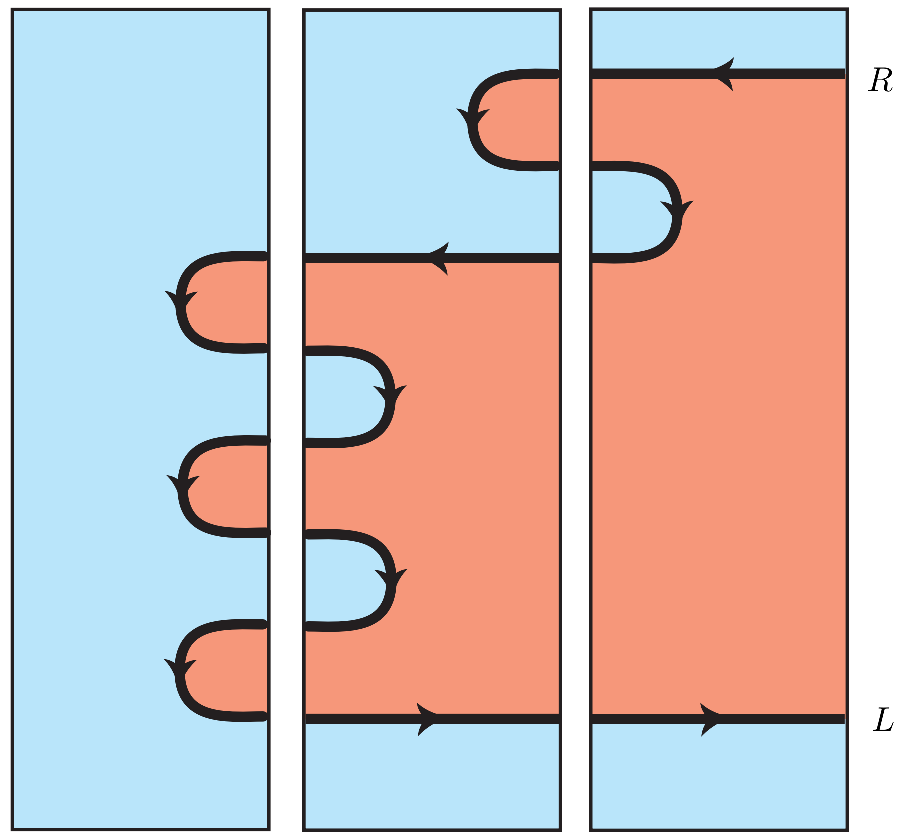
Opponent move = functor R Proponent move = functor L

A typical proof

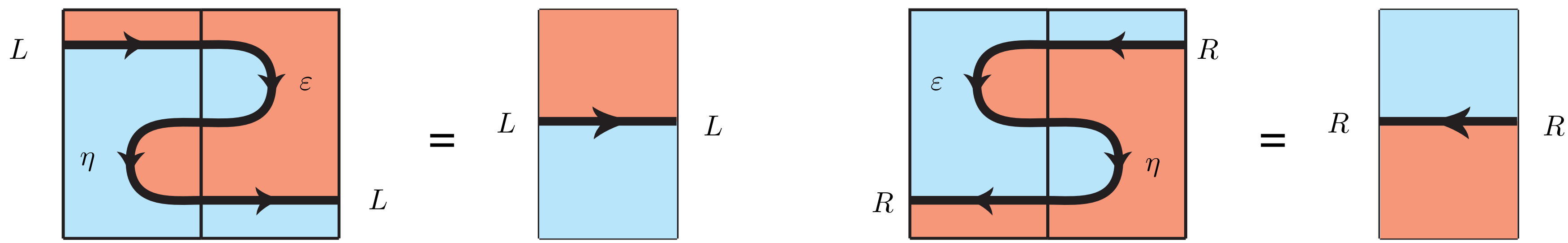


Reveals the algebraic nature of game semantics

A purely diagrammatic cut elimination



The 2-dimensional dynamic of adjunction



Recovers the usual way to compose strategies in game semantics

Interlude: a combinatorial observation

Fact: there are just as many canonical proofs

$$\overbrace{\neg \cdots \neg}^{2p} \; A \quad \vdash \quad \overbrace{\neg \cdots \neg}^{2q} \; A$$

as there are increasing functions

$$[p] \longrightarrow [q]$$

between the ordinals $[p] = \{0 < 1 < \cdots < p - 1\}$ and $[q]$.

This fragment of logic has the same combinatorics as simplices.

The two generators of a monad

Every increasing function is composite of **faces** and **degeneracies**:

$$\eta \quad : \quad [0] \vdash [1]$$

$$\mu \quad : \quad [2] \vdash [1]$$

Similarly, every proof is composite of the two generators:

$$\eta \quad : \quad A \vdash \neg\neg A$$

$$\mu \quad : \quad \neg\neg\neg\neg A \vdash \neg\neg A$$

The unit and multiplication of the double negation monad

The two generators in sequent calculus

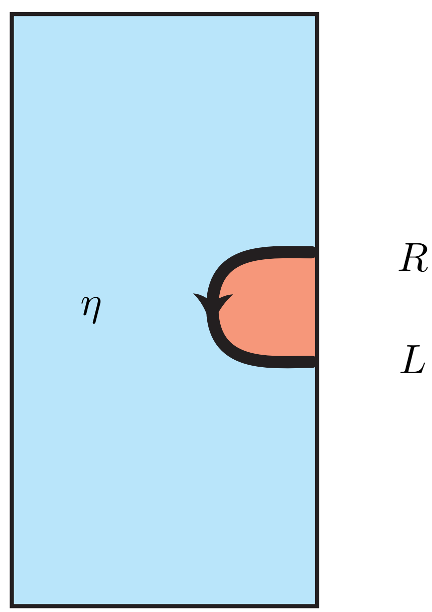
$$
\dfrac{\dfrac{A \vdash A}{A\,,\, \neg A \vdash}\;2}{A \vdash \neg\neg A}\;1
$$

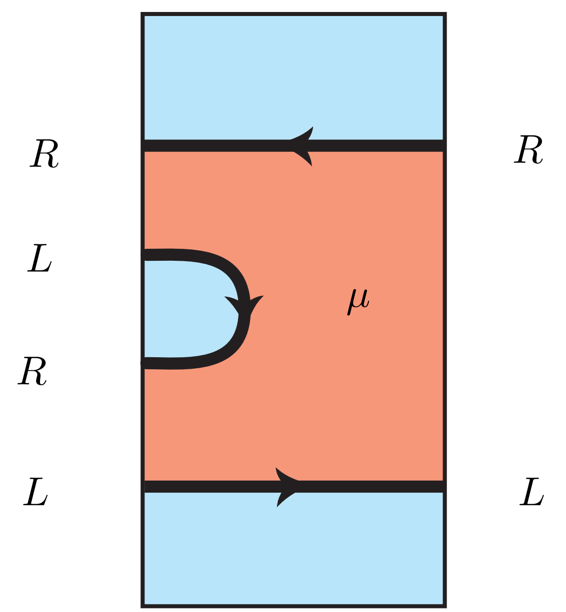
$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A\,,\, \neg A \vdash}\;6}{\neg A \vdash \neg A}\;5}{\neg A\,,\, \neg\neg A \vdash}\;4}{\neg A \vdash \neg\neg\neg A}\;3}{\neg\neg\neg\neg A\,,\, \neg A \vdash}\;2}{\neg\neg\neg\neg A \vdash \neg\neg A}\;1
$$

The two generators in string diagrams

The **unit** and **multiplication** of the monad $R \circ L$ are depicted as

$$\eta : Id \longrightarrow R \circ L \qquad\qquad \mu : R \circ L \circ R \circ L \longrightarrow R \circ L$$





Part 2

Tensor and negation

An atomist approach to proof theory

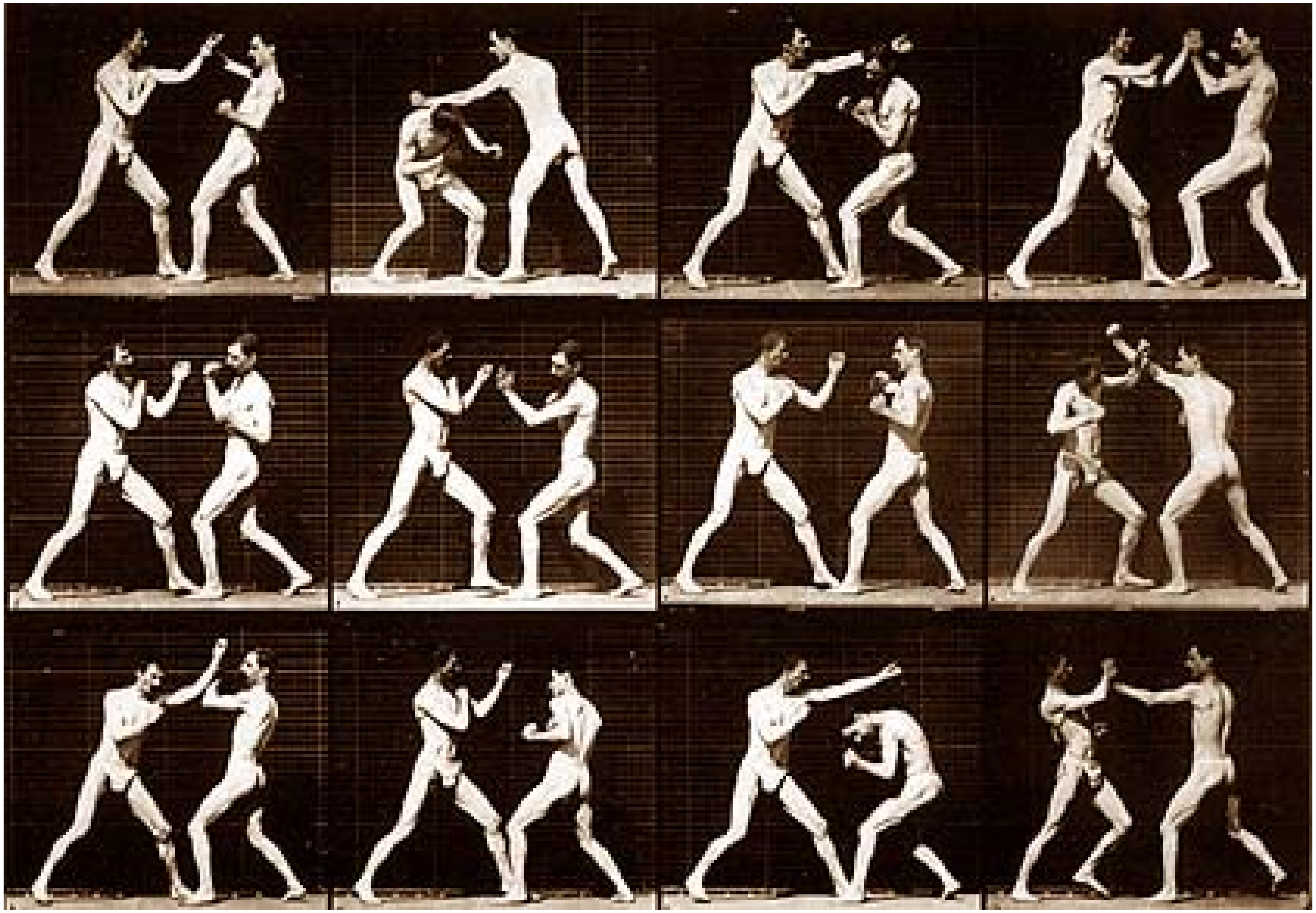
Guiding idea

A proof $\pi : A \vdash B$ is a linguistic choreography where

Proponent tries to convince Opponent

Opponent tries to refute Proponent

which we would like to **decompose** in elementary particles of logic



The linear decomposition of the intuitionistic arrow

$$A \Rightarrow B \quad = \quad (!A) \multimap B$$

[1] a proof of $A \multimap B$ uses its hypothesis A exactly once,

[2] a proof of $!A$ is a bag containing an infinite number of proofs of A.

Andreas Blass discovered this decomposition as early as 1972...

Four primitive components of logic

[1]	the negation	$\neg$
[2]	the linear conjunction	$\otimes$
[3]	the repetition modality	$!$
[4]	the existential quantification	$\exists$

Logic = Data Structure + Duality

Tensor vs. negation

A well-known fact: the continuation monad is strong

$$\boxed{(\neg\neg\, A) \,\otimes\, B \;\longrightarrow\; \neg\neg\,(A \otimes B)}$$

The starting point of the algebraic theory of side effects

Tensor vs. negation

Proofs are generated by a **parametric strength**

$$\kappa_X \quad : \quad \neg\,(X \otimes \neg\,A) \otimes B \longrightarrow \neg\,(X \otimes \neg\,(A \otimes B))$$

which generalizes the usual notion of **strong monad** :

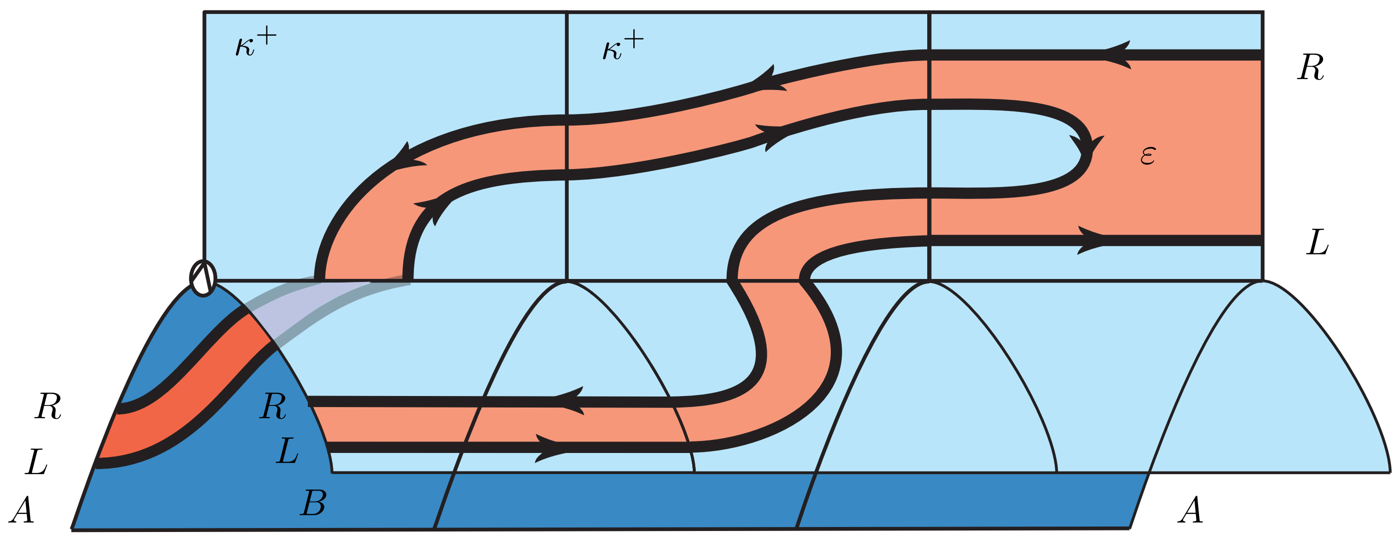
$$\kappa \quad : \quad \neg\neg\,A \otimes B \longrightarrow \neg\neg\,(A \otimes B)$$

Proofs as 3-dimensional string diagrams

The left-to-right proof of the sequent

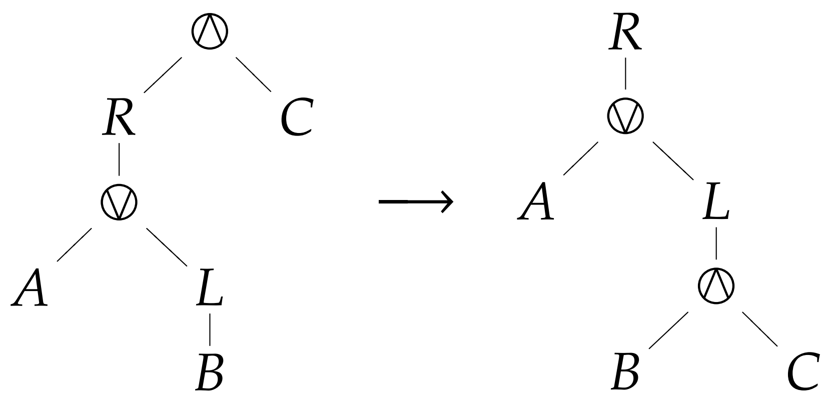
$$\neg\neg A \otimes \neg\neg B \quad \vdash \quad \neg\neg(A \otimes B)$$

is depicted as



Tensor vs. negation – conjunctive strength

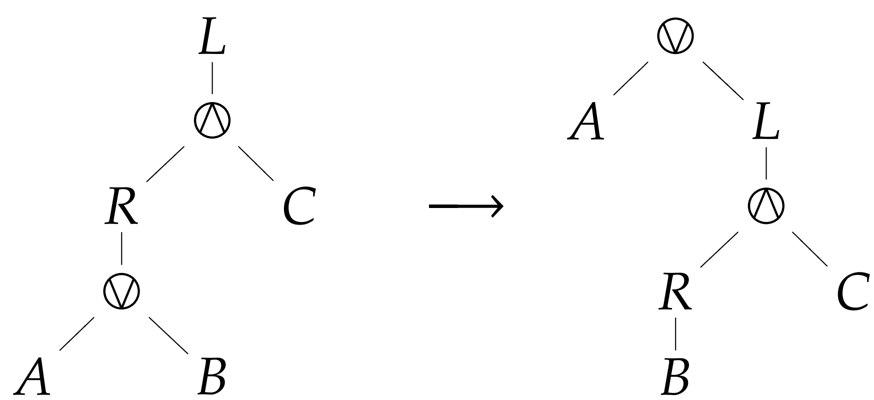
$$\kappa^{+} \quad : \quad R(A \;\ovee LB) \owedge C \;\longrightarrow\; R(A \;\ovee L(B \owedge C))$$



Linear distributivity in a continuation framework

Tensor vs. negation – disjunctive strength

$$\kappa^{-} \quad : \quad L(R(A \ovee B) \owedge C) \longrightarrow A \ovee L(R(B) \owedge C)$$



Linear distributivity in a continuation framework

A factorization theorem

The four proofs η, ϵ, κ^+ and κ^- generate every proof of the logic. Moreover, every such proof

$$X \xrightarrow{\epsilon} \xrightarrow{\kappa^+} \xrightarrow{\epsilon} \xrightarrow{\epsilon} \xrightarrow{\eta} \xrightarrow{\eta} \xrightarrow{\kappa^-} \xrightarrow{\epsilon} \xrightarrow{\eta} \xrightarrow{\epsilon} \xrightarrow{\kappa^-} \xrightarrow{\eta} \xrightarrow{\eta} Z$$

factors uniquely as

$$X \overset{\kappa^+}{\twoheadrightarrow} \overset{\epsilon}{\twoheadrightarrow} \overset{\eta}{\twoheadrightarrow} \overset{\kappa^-}{\twoheadrightarrow} Z$$

Corollary: two proofs are equal iff they are equal as strategies.

Part 3

Revisiting the negative translation

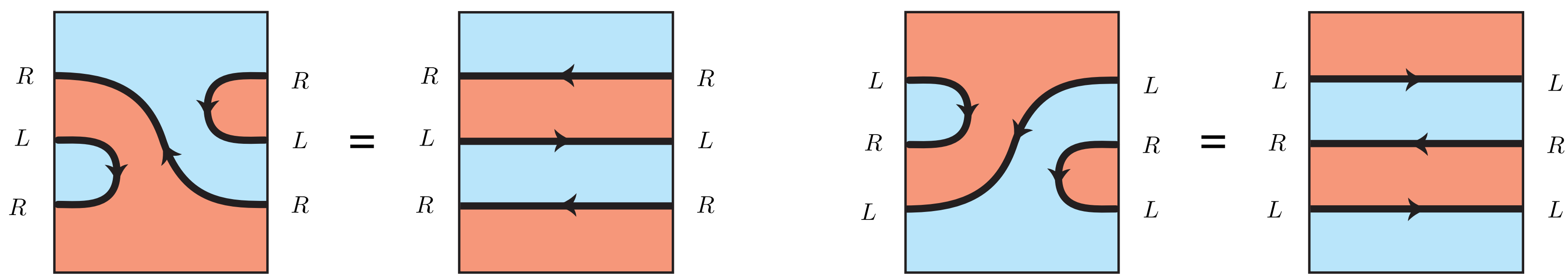
A rational reconstruction of linear logic

The algebraic point of view (in the style of Boole)

$\boxed{\text{The negated elements of a Heyting algebra form a Boolean algebra.}}$

The algebraic point of view (in the style of Frege)

A double negation monad is **commutative** iff it is **involutive**. This amounts to the following diagrammatic equations:



$\boxed{\text{In that case, the negated elements form a } *\text{-autonomous category.}}$

The continuation monad is strong

$$(\neg\neg A) \otimes B \xrightarrow{lst} \neg\neg(A \otimes B)$$

$$A \otimes \neg\neg B \xrightarrow{rst} \neg\neg(A \otimes B)$$

The continuation monad is not commutative

There are two canonical morphisms

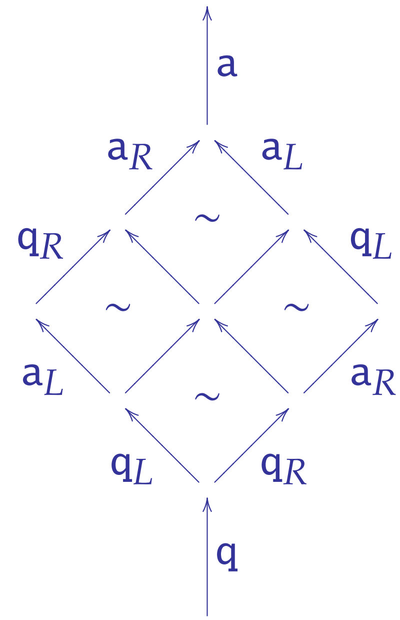
$$\neg\neg A \otimes \neg\neg B \rightrightarrows \neg\neg(A \otimes B)$$

$$\boxed{\begin{array}{ccccc}
\neg\neg A & \otimes & \neg\neg B & \longrightarrow & \neg\neg(A \otimes B) \\
 & & & & \mathtt{q} \\
\mathtt{q} & & & & \\
\mathtt{a} & & & & \\
 & & \mathtt{q} & & \\
 & & \mathtt{a} & & \\
 & & & & \mathtt{a}
\end{array}}
\qquad
\boxed{\begin{array}{ccccc}
\neg\neg A & \otimes & \neg\neg B & \longrightarrow & \neg\neg(A \otimes B) \\
 & & & & \mathtt{q} \\
 & & \mathtt{q} & & \\
 & & \mathtt{a} & & \\
\mathtt{q} & & & & \\
\mathtt{a} & & & & \\
 & & & & \mathtt{a}
\end{array}}$$

Left strict and

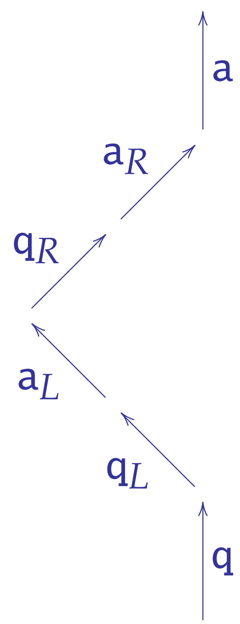
Right strict and

Asynchronous games



Arena game models extended to propositional linear logic by identifying the two strategies — hence mystifying the innocent audience.

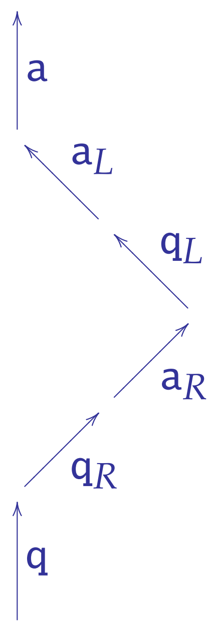
Asynchronous games



Left and

Arena game models extended to propositional linear logic by identifying the two strategies — hence mystifying the innocent audience.

Asynchronous games



Right and

Arena game models extended to propositional linear logic by identifying the two strategies — hence mystifying the innocent audience.

Hence, the schism between games and linear logic

The isomorphism

$$A \quad \cong \quad \neg\neg A$$

means that linear logic is **static** and **a posteriori.**

Imagine that a discussion is defined by the set of its final states

Linear logic originates from a model of PCF

Tensorial logic

$$
\begin{aligned}
\text{tensorial logic} \quad &= \quad \text{a logic of tensor and negation} \\
&= \quad \text{linear logic without } A \cong \neg\neg A \\
&= \quad \text{the very essence of polarized logic}
\end{aligned}
$$

Offers a synthesis of linear logic, games and continuations

Research program: recast every aspect of linear logic in this setting

A lax monoidal structure

Typically, the family of n-ary connectives

$$
\begin{aligned}
(A_1 \parr \cdots \parr A_n) \quad &:= \quad \neg\,(\,\neg A_1 \otimes \cdots \otimes \neg A_k\,) \\
&:= \quad R\,(\,LA_1 \varovee \cdots \varovee LA_k\,)
\end{aligned}
$$

is still associative, but all together, and in the lax sense.

A general phenomenon of adjunctions

Given a 2-monad T and an adjunction

$$\begin{tikzcd} \mathcal{A} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathcal{B} \arrow[ll, bend left, "R"] \end{tikzcd}$$

every lax T-algebraic structure

$$T\mathcal{B} \xrightarrow{b} \mathcal{B}$$

induces a lax T-algebraic structure

$$T\mathcal{A} \xrightarrow{TL} T\mathcal{B} \xrightarrow{b} \mathcal{B} \xrightarrow{R} \mathcal{A}$$

A general phenomenon of adjunctions

Consequently, every adjunction with a monoidal category $\mathcal{B}$

$$\begin{tikzcd} \mathcal{A} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathcal{B} \arrow[ll, bend left, "R"] \end{tikzcd}$$

induces a lax action of $\mathcal{B}$ on the category $\mathcal{A}$

$$\mathcal{B} \times \mathcal{A} \xrightarrow{\mathcal{B}\times L} \mathcal{B} \times \mathcal{B} \xrightarrow{\otimes_B} \mathcal{B} \xrightarrow{R} \mathcal{A}$$

Enscopes the double negation monad and the arrow.

Distributivity laws seen as lax bimodules

The distributivity law

$$R(A \ \varovee LB) \varowedge C \xrightarrow{\kappa} R(A \ \varovee L(B \varowedge C))$$

may be seen as a lax notion of bimodule:

$$(A \triangleright B) \triangleleft C \longrightarrow A \triangleright (B \triangleleft C)$$

A useful extension of the notion of strong monad (case $A = *$)

Part 4

A relaxed notion of Frobenius algebra

After Brian Day and Ross Street

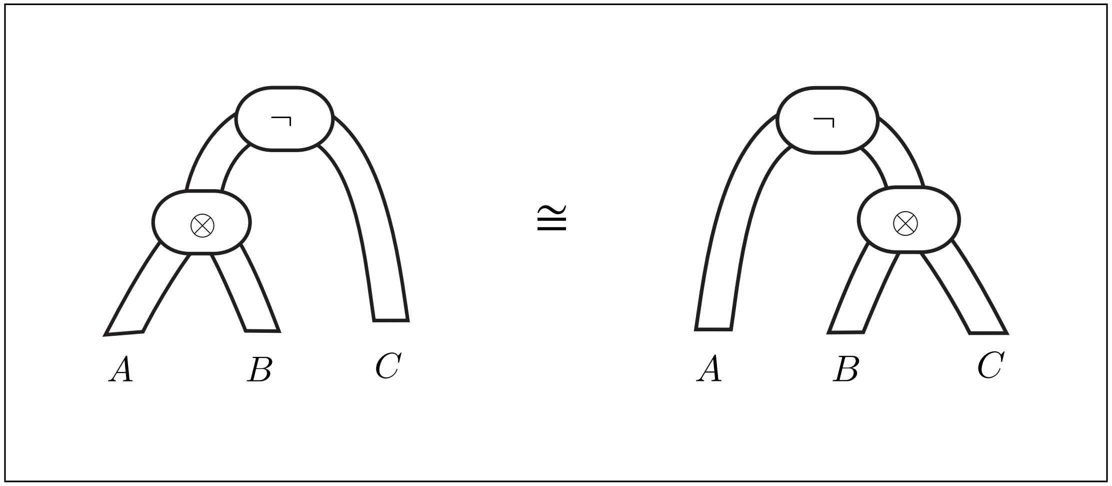
Dialogue categories

A **symmetric monoidal** category $\mathcal{C}$ equipped with a functor

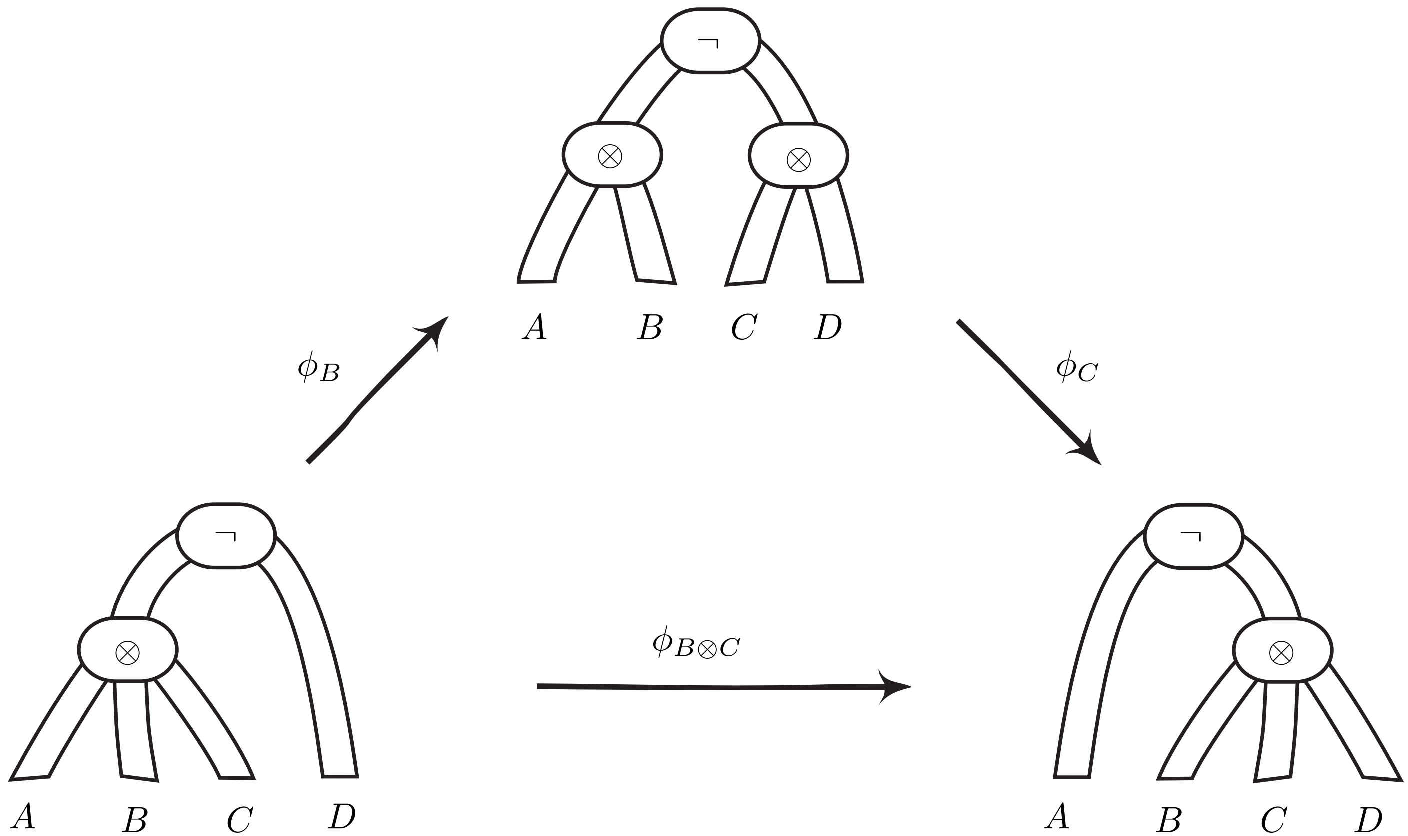
$$\neg \quad : \quad \mathcal{C}^{op} \quad \longrightarrow \quad \mathcal{C}$$

and a natural bijection

$$\varphi_{A,B,C} \quad : \quad \mathcal{C}(A \otimes B\,,\, \neg\, C) \;\cong\; \mathcal{C}(A\,,\, \neg\,(\,B \otimes C\,)\,)$$

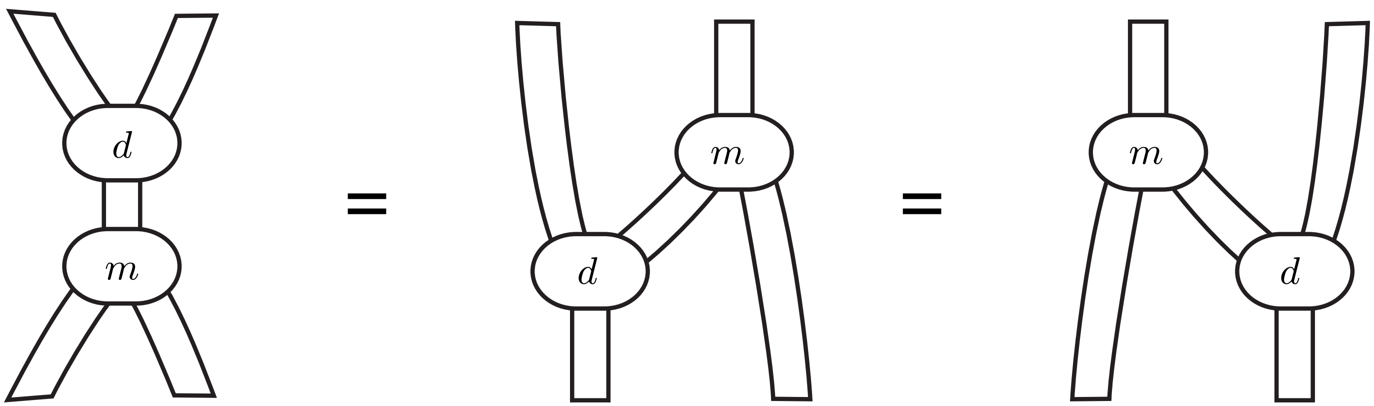


A coherence law



Frobenius objects

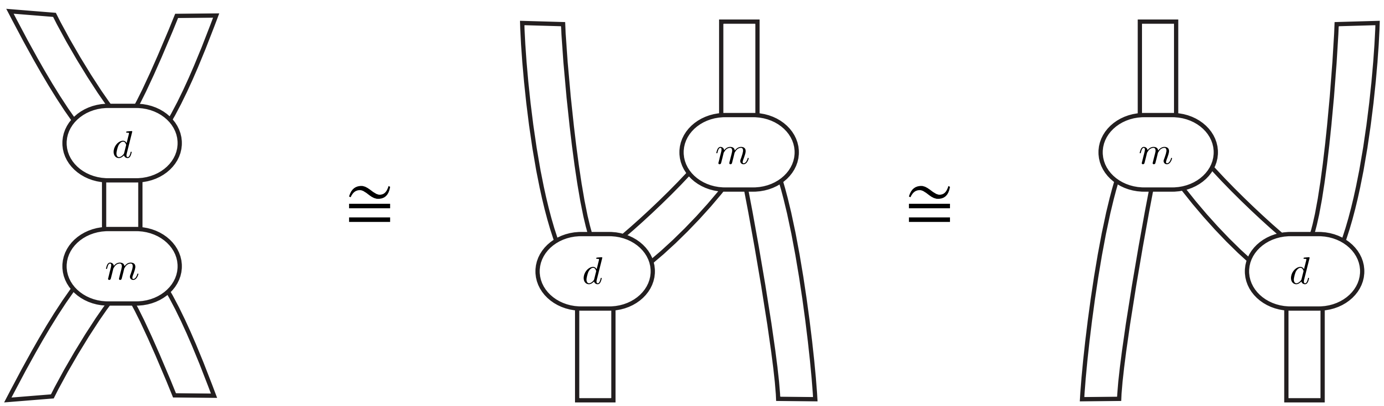
A Frobenius object F is a monoid and a comonoid satisfying



an alternative formulation of cobordism

$*$-autonomous categories

A Frobenius object in the bicategory of modules



An observation by Brian Day and Ross Street (2003)

Lax Frobenius algebras

Relax the self-duality equivalence

$$\mathcal{C} \quad \cong \quad \mathcal{C}^{op}$$

into an adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{C}^{op} \arrow[l, bend left, "R"] \end{tikzcd}$$

this connects game semantics and quantum algebra

Quantum groupoids

A lax monoidal adjunction in the bicategory Comod (k-Vect)

$$
\begin{tikzcd}
E \arrow[rr, bend left, "f_*"] \arrow[rr, phantom, "\perp"] & & V^{op} \otimes V \arrow[ll, bend left, "f^*"]
\end{tikzcd}
$$

where the pseudomonoids E and $V^{op} \otimes V$ and map f are $*$-autonomous.

$$V^{op} \quad \dashv \quad V \quad \dashv \quad V^{op}$$

A definition by Brian Day and Ross Street (2003)

Conclusion

Logic = Data Structure + Duality

This point of view is accessible thanks to 2-dimensional algebra