Categories and nonassociative C^*-algebras in Quantum Field Theory

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Outline

1. Quantum mechanics and C*-algebras.
2. Gel'fand's Theorem and the Dixmier-Douady obstruction.
3. Twisted compact operators.
4. T-duality.
5. Monoidal categories.
6. Nonassociative C*-algebras.

The observables generate an algebra of operators on a Hilbert space $\mathcal{H}$,
closed under addition, multiplication, and adjoints.

One can restrict to bounded operators $\mathcal{B}(\mathcal{H})$:

$$\|A\| = \sup\{\|A\psi\| : \|\psi\| = 1\} < \infty$$

A C^*-algebra is a norm-closed subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

Example: the compact operators ∼ the CCR Algebra

The compact operators $\mathcal{K}(\mathcal{H})$ are the C*-algebra generated by the rank-one operators in $\mathcal{B}(\mathcal{H})$, i.e. those of the form

$$|\xi\rangle\langle\eta| : \psi \mapsto \xi\langle\eta, \psi\rangle.$$

If $\mathcal{H} = L^2(X)$, (normalisable wave functions on X, the compact operators K can be represented as integral operators

$$(K\psi)(x) = \int_X K(x, y)\psi(y)\, dy$$

where the integral kernel $K(x, y)$ is a limit of separable kernels

$$K_N(x, y) = \sum_{j=1}^{N} \alpha_j(x)\beta_j(y).$$

The composition is then

$$(K_1 \circ K_2)(x,z) = \int_X K_1(x,y) K_2(y,z))\, dy$$

The adjoint is

$$K^*(x,y) = \overline{K(y,x)}$$

cf matrices when integral is replaced by a sum.

Example: function algebras

Take E a locally compact Hausdorff topological space with a measure μ,

let $\mathcal{H} = L^2(E, \mu)$,

multiplication by compactly supported, continuous, complex-valued functions $f \in C_K(E)$

$$(f.\psi)(x) = f(x)\psi(x)$$

for $x \in E$, $\psi \in L^2(E, \mu)$ gives a subalgebra of $\mathcal{B}(\mathcal{H})$, with

$$(f_1 \circ f_2)(x) = f_1(x) f_2(x), \qquad f^*(x) = \overline{f(x)}$$

Gel'fand's Theorem

Every commutative C*-algebra is $C_K(E)$ for some locally compact Hausdorff space E, and

the category of commutative C*-algebras is contravariantly equivalent to the category of locally compact Hausdorff spaces, via the functors

$$\begin{array}{rcl} \mathrm{spec}(\mathcal{A}) & \to & \mathcal{A} \\ E & \leftarrow & C_K(E) \end{array}$$

where the spectrum of $\mathcal{A}$

$\mathrm{spec}(\mathcal{A}) =$ equivalence classes of irreducible representations $\sim$ maximal ideals.

Continuous trace algebras

What if $\mathcal{A}$ is not commutative?

There is a broader class, the continuous trace C*-algebras which are given by algebra-valued functions over the spectrum.

Continuous trace C*-algebra $\mathcal{A} \sim$ sections of $\mathcal{K}(\mathcal{H})$-bundle over spectrum $E = \mathrm{spec}\mathcal{A}$ (equivalence classes of irreducible representations).

The bundle structure is trivial if and only if the Dixmier–Douady obstruction $\delta \in H^2(E, \mathbb{T}) \cong H^3(E, \mathbb{Z})$ is trivial, (Brauer 1927, . . ., Dixmier–Douady 1964)

Dixmier–Douady Theorem.

For every such E and $\delta \in H^3(E, \mathbb{Z})$ there is a C*-algebra $\mathcal{A} = CT(E, \delta)$ with spectrum E and Dixmier–Douady obstruction δ, and it is unique up to Morita equivalence.

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Algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are Morita equivalent if there is an additive equivalence between the categories of $\mathcal{A}_1$-modules and $\mathcal{A}_2$-modules.

Raeburn, Kumjian, Muhly, Renault, Williams: groupoid C* algebra proof.

Morita Equivalence

Theorem (Morita–Rieffel) For each additive equivalence from $\mathcal{A}_1$-modules to $\mathcal{A}_2$-modules there exists a left $\mathcal{A}_2$-right $\mathcal{A}_1$-bimodule E such that the equivalence is given by $E \otimes_{\mathcal{A}_1} \cdot$.

The compact operators $\mathcal{K}(\mathcal{H})$ are Morita equivalent to $\mathbb{C}$ via the left $\mathcal{K}(\mathcal{H})$-right $\mathbb{C}$-bimodule $\mathcal{H}$.

(Uniquenes of the Canonical Commutation Relations)

Summary

loc. cpt Hausdorff space	$\longleftrightarrow$	comm. C*-algebra
cpt Hausdorff space	$\longleftrightarrow$	comm. C*-algebra with 1
E	$\longrightarrow$	$C_0(E)$
spectrum spec$(\mathcal{A})$	$\longleftarrow$	algebra $\mathcal{A}$
noncommutative geometry	$\longleftrightarrow$	continuous trace C*-algebra
flux $H \in H^3(E, \mathbb{Z})$	$\longleftrightarrow$	DD class $\delta \in H^2(E, \mathbb{T})$

Twisted compact operators

Take the same integral operators $\mathcal{K}(L^2(X))$, but with a composition

$$(K_1 * K_2)(x,z) = \int_X \frac{K_1(x,y)K_2(y,z))}{\phi(x,y,z)}\,dy$$

for a scalar function $\phi : X \times X \times X \to U(1) = \{z \in \mathbb{C} : |z| = 1\}$.

Problem this is not generally associative:

$$(K_1 * K_2) * K_3 \neq K_1 * (K_2 * K_3)$$

unless $\phi(x,y,z)\phi(x,z,w) = \phi(x,y,w)\phi(y,z,w)$

Einstein's principle of General Covariance:

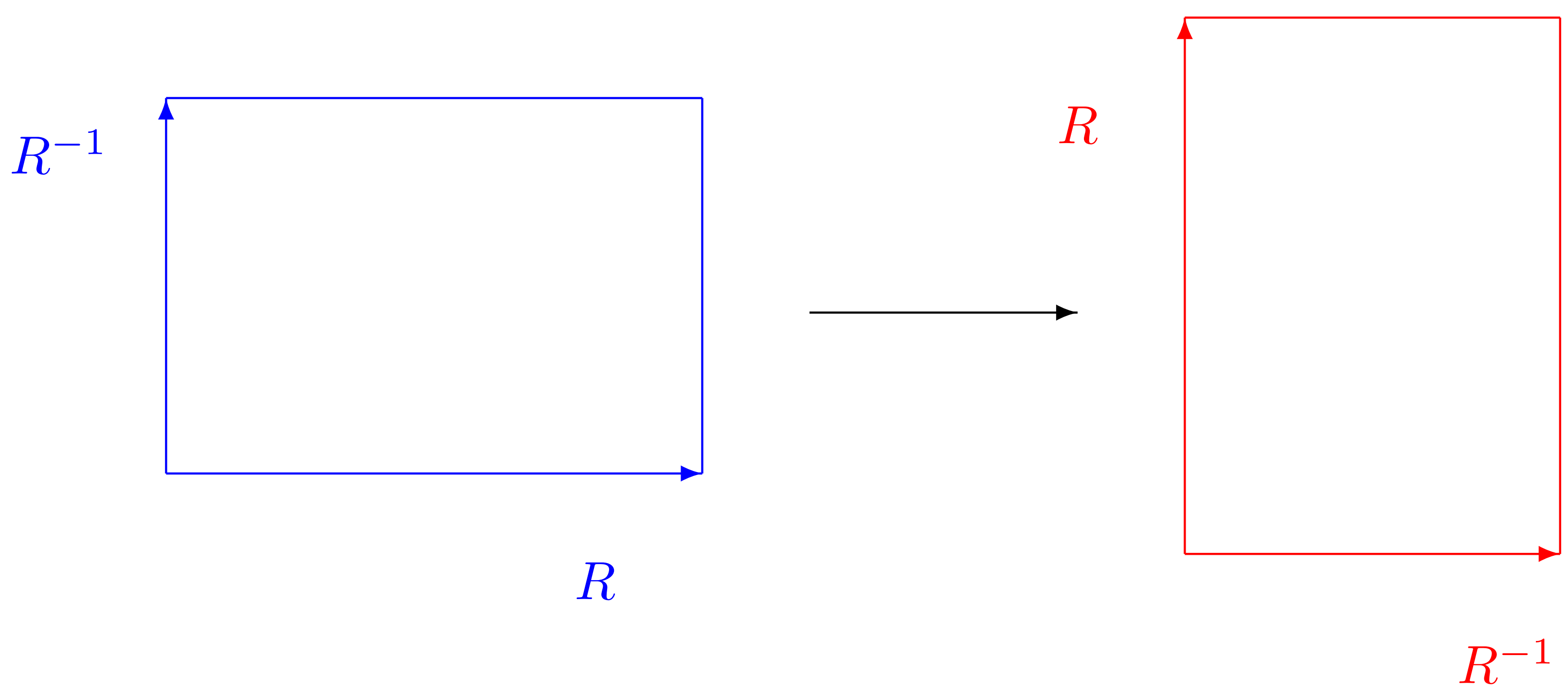
Physical theories should be completely invariant under coordinate transformations.

Quantum Field Theory and String Theory:

Symmetries extend to phase space.

$$\text{configuration space } \mathbb{R}^D \longleftrightarrow \text{momentum space } \widehat{\mathbb{R}}^D$$

T-duality: $R \leftrightarrow R^{-1}$ preserves the physics



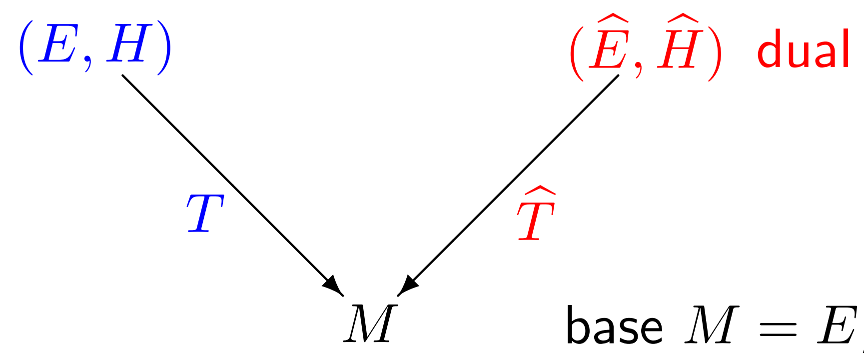
String theory (Hull and Townsend)

T-duality: Momentum and winding number interchange

Added ingredient: flux $H \in H^3(E)$

Rough picture of T-duality

T-duality interchanges two principal torus bundles over the same base, and interchanges the curvature of each with the H-flux ($H \in \Omega^3(E)$ or $\widehat{H} \in \Omega^3(\widehat{E})$) of the other.



base $M = E/T \cong \widehat{E}/\widehat{T}$

$$T = \mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k \cong \widehat{T}$$

Examples

It will suffice to consider $E = \mathbb{T}^3$, $T = \mathbb{T}^k$ ($k \leq 3$), and $M = \mathbb{T}^{3-k}$, with H k times the volume 3-form.

$$\begin{array}{cc} & \mathbb{T}^3 \\ \mathbb{T}^k & \downarrow \\ & \mathbb{T}^{3-k} \end{array}$$

Known geometric dual principal torus bundles

$$H = H_3 + H_2 + H_1 + H_0 \text{ where } H_p \in \Omega^p(M, \wedge^{3-p}\widehat{\mathfrak{t}})$$

with $\widehat{\mathfrak{t}}$ the dual of the Lie algebra of T.

$\dim T$	H		
1	arbitrary	Bouwknegt, Evslin, Mathai	2004
arbitrary	$H_1, H_0 = 0$	Bouwknegt, KCH, Mathai	2004
arbitrary	$H_0 = 0$	Mathai, Rosenberg	2004
arbitrary	arbitrary	Bouwknegt, KCH, Mathai	2005,6

Example

$\mathbb{T}^3$ with volume form: volume is generated by $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ is $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}.\mathbf{b} \times \mathbf{c}$. for $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$.

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- as a $\mathbb{T}^2$-bundle over $\mathbb{T}$: $H = H_1$: noncommutative dual (Mathai–Rosenberg 2004)
- as a $\mathbb{T}^3$ bundle over a point: $H = H_0$: nonassociative dual (Bouwknegt, KCH, Mathai 2005,6).

H-FLUX

Gel'fand's theorem allows one to replace E by a C*-algebra.

The exact sequence of groups

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to 0$$

gives an isomorphism $H^3(E, \mathbb{Z}) \cong H^2(E, \mathbb{T})$:

Identify H with the Dixmier–Douady class δ and replace (E, H) by $CT(E, \delta)$.

Automorphism groups

Now consider a principal $T = G/N$-bundle E over M.

Does G act as automorphisms of $CT(E, \delta)$?

Suppose $\alpha : G \to \mathrm{Aut}(\mathcal{A})$, and the same subgroup N stabilises each irreducible.

Then $E = \mathrm{spec}(\mathcal{A})$ is a $T = G/N$-bundle over $M = \mathrm{spec}(\mathcal{A})/G$.

If $G = \mathbb{R}$ every principal G/N-bundle arises in this way, but for general groups G this is not always true.

Crossed products

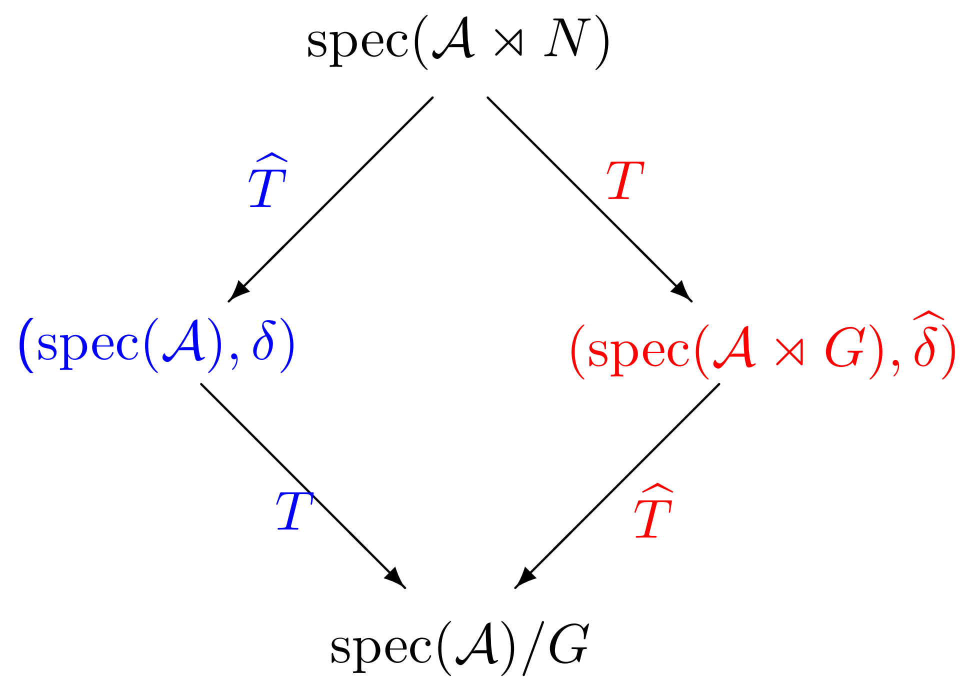
Crossed product $\mathcal{A} \rtimes G = C_0(G, \mathcal{A})$

$$(f * g)(x) = \int_G f(y)\alpha_y[g(y^{-1}x)]\,dy, \qquad f^*(x) = \alpha_x[f(x^{-1})]^*$$

Facts.

1. Under suitable assumptions $\widehat{\mathcal{A}} = \mathcal{A} \rtimes G$ is also a continuous trace algebra with an action of the dual group $\widehat{G}$;
2. (Takai-Takesaki duality) $\widehat{\mathcal{A}} \rtimes \widehat{G} \cong \mathcal{A} \otimes \mathcal{K}(L^2(G)) \sim_M \mathcal{A}$.

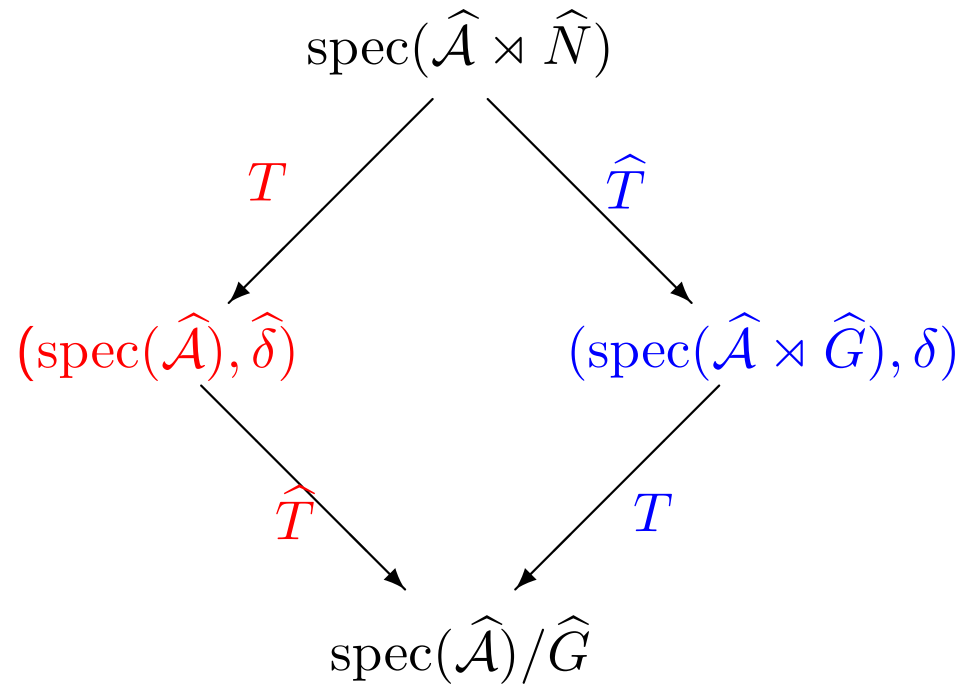
Duality



$\widehat{T}$ is isomorphic to the group-theoretic dual of N.

Connes' Thom isomorphism theorem: $K_*(\mathcal{A} \rtimes \mathbb{R}^D) \cong K_{*+D}(\mathcal{A})$.

Duality



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Remaining case

Puzzle:

$H_0 \neq 0$ never seems to show up in C*-algebra literature.

There are spaces with any H, so problem must lie with group action.

Nonassociative case

Inner automorphisms act trivially on spectrum.

$$G \to \mathrm{Out}(\mathcal{A}) = \mathrm{Aut}(\mathcal{A})/\mathrm{Inn}(\mathcal{A})$$

Lift to $\alpha : G \to \mathrm{Aut}(\mathcal{A})$: $\alpha_x \alpha_y = \mathrm{ad}(u(x,y))\alpha_{xy}$

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$$\mathrm{ad}(u(x,y))\mathrm{ad}(u(xy,z))\alpha_{(xy)z} = \mathrm{ad}(\alpha_x[u(y,z)])\mathrm{ad}(u(x,yz)\alpha_{x(yz)}$$

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$$\phi(x,y,z)u(x,y))u(xy,z) = \alpha_x[u(y,z)]u(x,yz)$$

Properties of ϕ

Central, and satisfies pentagonal cocycle identity

$$\phi(x,y,z)\phi(x,yz,w)\phi(y,z,w) = \phi(xy,z,w)\phi(x,y,zw)$$

ϕ is independent of liftings up to coboundaries

$$\eta(x,y)\eta(xy,z)/\eta(y,z)\eta(x,yz)$$

so only $H^3(G,\mathbb{T})$ class of ϕ matters (but cf. Majid)

$$\phi(\exp(X),\exp(Y),\exp(Z)) = \exp(iH_0(\xi_X,\xi_Y,\xi_Z))$$

where ξ_X is vector field generated by X.

For $\mathbb{T}^3$ with $k \times \mathrm{vol}$: $\quad \phi(\mathbf{a},\mathbf{b},\mathbf{c}) = \exp(2\pi i k[\mathbf{a},\mathbf{b},\mathbf{c}])$

†-Category $\mathcal{C}_G$ of $\widehat{G}$-modules $\sim C_0(G)$-modules with G-morphisms, and

- module tensor product ($f \in C_0(G)$ acting via comultiplication $(\Delta f)(x,y) = f(xy)$), † action multiplies by $f^*(x) = \overline{f(x)}$;

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- identity object: trivial module $\mathbb{C}$ (action by the counit $\epsilon(f) = f(1)$);
- associator $\Phi : A \otimes (B \otimes C) \to (A \otimes B) \otimes C \sim$ the action of

$$\phi \in C(G \times G \times G) = C(G) \otimes C(G) \otimes C(G).$$

The pentagonal identity

The pentagonal cocycle identity for ϕ gives

$$
\begin{array}{ccccc}
 & \mathcal{A}\otimes(\mathcal{B}\otimes(\mathcal{C}\otimes\mathcal{D})) & \longrightarrow & (\mathcal{A}\otimes\mathcal{B})\otimes(\mathcal{C}\otimes\mathcal{D}) & \\
 & \swarrow & & \searrow & \\
\mathcal{A}\otimes((\mathcal{B}\otimes\mathcal{C})\otimes\mathcal{D}) & & & & ((\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C})\otimes\mathcal{D} \\
 & \searrow & & \nearrow & \\
 & & (\mathcal{A}\otimes(\mathcal{B}\otimes\mathcal{C}))\otimes\mathcal{D} & &
\end{array}
$$

Algebras

Def. An *algebra* in $\mathcal{C}_G$ is an object $\mathcal{A}$ with a morphism $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ consistent with Φ:

$$\begin{array}{ccccc} \mathcal{A} \otimes (\mathcal{A} \otimes \mathcal{A}) & \longrightarrow & \mathcal{A} \otimes \mathcal{A} & \longrightarrow & \mathcal{A} \\ \Phi \downarrow & & & & \downarrow \\ (\mathcal{A} \otimes \mathcal{A}) \otimes \mathcal{A} & \longrightarrow & \mathcal{A} \otimes \mathcal{A} & \longrightarrow & \mathcal{A} \end{array}$$

The action of $\widehat{G}$ is automatically by automorphisms.

$\mathcal{C}_G$ is a star/bar/dagger category and so one can also define C*-algebras and Hilbert spaces in $\mathcal{C}_G$.

- Torus bundle $\mathbb{T}^3$ over a point, with $H_0 = k\mathrm{vol}$

  Associated antisymmetric form on $\mathbf{a}, \mathbf{b}, \mathbf{c} \in t = \mathbb{R}^3$ is then given by

$$\phi(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \exp(-2\pi i f(\mathbf{a}, \mathbf{b}, \mathbf{c})) = \exp(-2k\pi i[\mathbf{a}, \mathbf{b}, \mathbf{c}])$$

Examples

- Torus bundle $\mathbb{T}^3$ over a point, with $H_0 = k\text{vol}$
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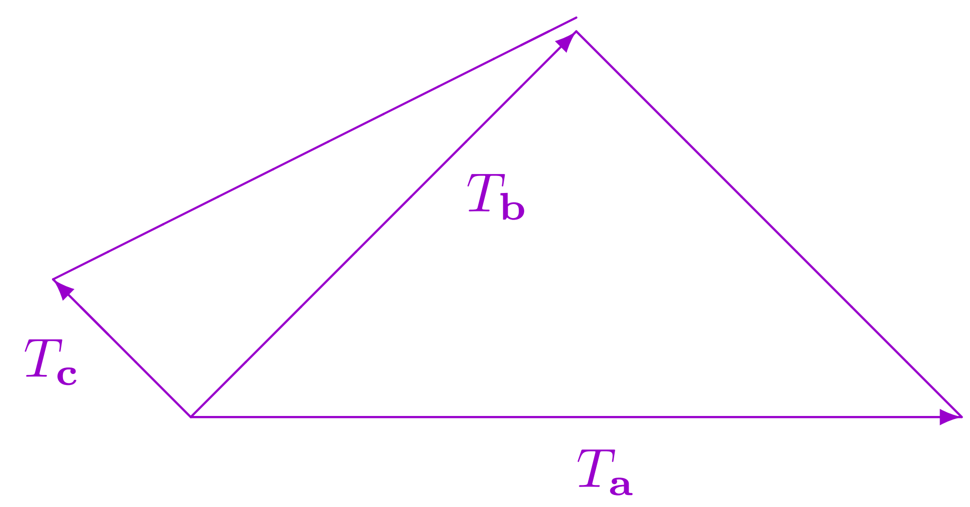
- $\mathcal{A}_0 = \mathbb{C}$, $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \sim \{0,1\}^3 \subseteq \mathbb{R}^3$

$$\phi(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (-1)^{[\mathbf{a}, \mathbf{b}, \mathbf{c}]},$$

  suitable u gives the *octonions* (cf Albuquerque and Majid).

Fadeev/Gauss associative anomaly

Magnetic translations $T_{\mathbf{a}} = \exp(2\pi i \mathbf{a} \cdot \nabla)$



$T_{\mathbf{a}}$ and $T_{\mathbf{b}}$ fail to commute by a factor $\exp(\pi i \Phi)$, where Φ is the flux through face spanned by $\mathbf{a}$ and $\mathbf{b}$.

$T_{\mathbf{a}}$, $T_{\mathbf{b}}$, and $T_{\mathbf{c}}$ fail to associate by a factor $\exp(\pi i \Phi)$, where Φ is the flux out of the tetrahedron spanned by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

Dirac's monopole argument.

But cf. Carey/Mickelsson

Modules

Def. An $\mathcal{A}$-*module* in $\mathcal{C}_G$ is an object M with a morphism $\mathcal{A} \otimes M \to M$ consistent with Φ:

$$\begin{array}{ccccc} \mathcal{A} \otimes (\mathcal{A} \otimes M) & \longrightarrow & \mathcal{A} \otimes M & \longrightarrow & M \\ \Phi \downarrow & & & & \downarrow \\ (\mathcal{A} \otimes \mathcal{A}) \otimes M & \longrightarrow & \mathcal{A} \otimes M & \longrightarrow & M \end{array}$$

The actions of $\mathcal{A}$ and $\widehat{G}$ on M are automatically consistent in that $g[am] = (g[a])(g[m])$, for all $a \in \mathcal{A}$ and $m \in M$, that is one has a covariant representation of $(\widehat{G}, \mathcal{A})$, which is really a representation of $\mathcal{A} \rtimes \widehat{G}$.

The twisted compact operators

Let $\mathcal{H}$ be a Hilbert space, as right $\mathbb{C}$-module (with $\mathbb{C}$ the identity object).

Use the Rieffel construction to obtain rank-one operators

$$|\xi\rangle\langle\eta|\zeta = \Phi(\xi\langle\eta,\zeta\rangle)$$

These rank-one operators generate the twisted compact operators having $\mathcal{H}$ as a module.

When $\mathcal{H} = L^2(X)$ these twisted compact operators can be represented as integral operators with product

$$(K_1 * K_2)(x,z) = \int_X \frac{K_1(x,y)K_2(y,z)}{\phi(x,y,z)}\, dy$$

The twisted bounded operators

In the usual case the bounded operators can be characterised as the *adjointable operators.*

Now an adjointable operator A is one for which there exists $A^* : \xi \to A^*\xi \equiv \xi A$ satisfying

$$\langle A^*\xi, \eta\rangle \equiv \langle \xi A, \eta\rangle = \Phi(\langle \xi, A\eta\rangle$$