k-Cores for game comonads

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Who am I, what do I do and why am I here?

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Finite model theory and **descriptive complexity**, using **game comonads**.

We study finite relational structures and the logical sentences they satisfy using two-player games.
Comonads let us study the non-classical notions of homomorphism arising from this perspective.

What am I going to say?

1. Background and motivation
2. Game comonads and approximations to homomorphism
3. Cores for the pebbling comonad?
4. Regularity, obstacles to core uniqueness and open questions

Background and motivation

Some basic graph notions

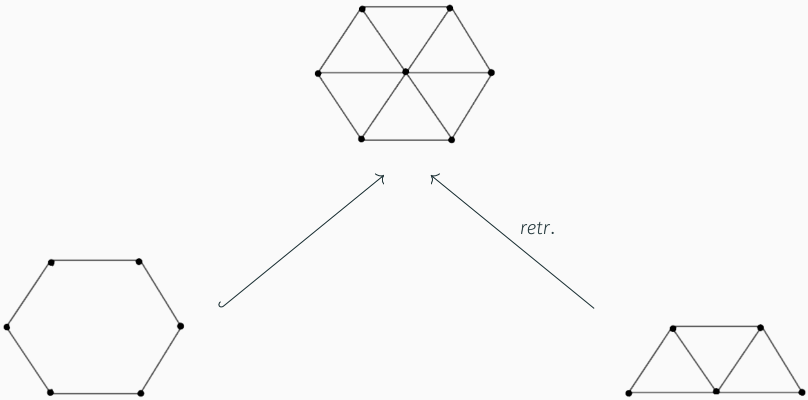
A *graph* G is a set, $V(G)$ of vertices with a single binary relation $E(G) \subset V(G)^2$

A *graph homomorphism* $f : G \to H$ is a function $f : V(G) \to V(H)$ s.t. $vv' \in E(G) \implies f(v)f(v') \in E(H)$

A *subgraph* of G is a graph G' with an injective homomorphism $\iota_{G'} : G' \hookrightarrow G$

A *retract* of G is a graph is a subgraph G' with a surjective homomorphism $\rho_{G'} : G \twoheadrightarrow G'$ s.t. $\rho_{G'} \circ \iota_{G'} = 1_{G'}$

Examples: Subgraphs and Retracts



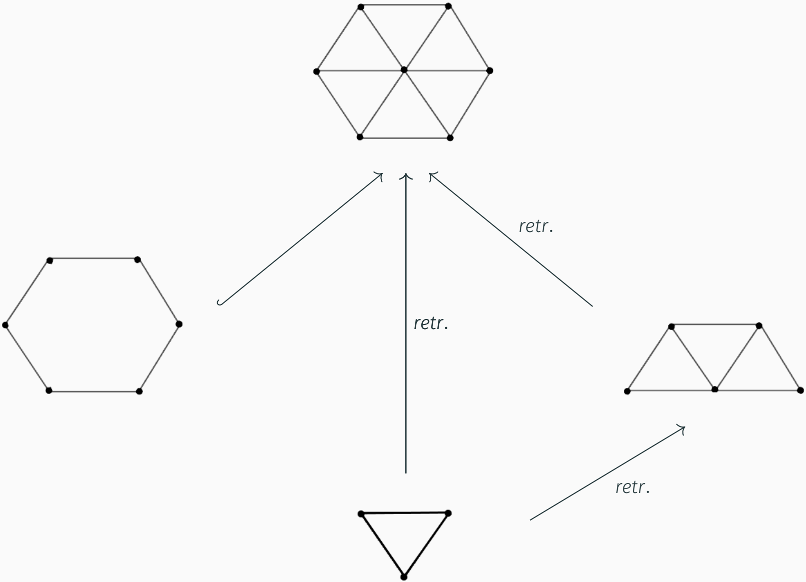
Defining graph cores

Definition

The core H of a graph G is the smallest subgraph which is also a homomorphic image.
Say a graph is a core if it is its own core, i.e. it has no homomorphisms to any of its proper subgraphs.

Note: For finite structures this corresponds to the $\hookrightarrow$-minimal retract of a graph.

Examples: Cores



Why study cores?

Proposition (Properties of cores for finite graphs[1])

i *If H is a core of G then H is a core (of itself)*

ii *Every graph has a core.*

iii *The core of a graph is unique up to isomorphism.*

iv *If graphs G and H have cores G' and H' then $G \to H$ iff $G' \to H'$*

[1] *Hell and Nešetřil, The core of a graph, 1990*

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Advantages:

- Cores are simpler graphs.
- Can be used to classify all graphs.
- $\mathrm{CSP}(G) = \mathrm{CSP}(\mathrm{core}(G))$

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Properties of cores:

Properties of category of graphs:

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4. $G \rightleftarrows \mathrm{core}(G)$

Game comonads and approximations to homomorphism

Introducing games and their comonads

Introduced for testing expressiveness of logics over finite structures.

Played by spoiler and duplicator. Former tries to distinguish two structures, latter tries to show their the same. Duplicator has a winning strategy iff the structures agree on some logic $\mathcal{L}$. Rules of the game vary to capture different $\mathcal{L}$.

Example

- n round Ehrenfeucht-Fraïssé game: Duplicator wins iff $A \equiv_{\mathcal{L}_n} B$
- k pebble game: Duplicator wins iff $A \equiv_{\mathcal{L}^k} B$
- Bijective k-pebble game: Duplicator wins iff $A \equiv_{\mathcal{C}^k} B$
- One-way versions: Duplicator wins iff $A \Rrightarrow_{\mathcal{L}} B$

Compositional view on this

Idea

Have a category $\mathcal{R}(\sigma)$ of relational structures over some signature σ with maps being homomorphism. (e.g graphs)

Also have non-classical homomorphisms arising from games.

Want to relate these together categorically.

The pebbling comonad

Definition

The k pebbling comonad consists of

- an endofunctor $\mathbb{T}_k$ sending A to $(A \times [k])^+$,
  where $(t_1, \dots t_n) \in R^{\mathbb{T}_k A}$ iff $t_i \sqsubset t_j$[2] and $(\epsilon_A(t_1), \dots \epsilon_A(t_n)) \in R^A$
- counit $\epsilon : \mathbb{T}_k \to 1$
- comultiplication $\delta : \mathbb{T}_k \to \mathbb{T}_k \mathbb{T}_k$

[2] t_i should be a prefix of $t_j$or vice versa and the last pebble used in the prefix should not appear in the suffix

coKleisli Category of $\mathbb{T}_k$

$\mathcal{K}(\mathbb{T}_k)$ has the same objects as $\mathcal{R}(\sigma)$ but morphisms $A \to_k B$ are morphisms $\mathbb{T}_k A \to B$.

The identity is given by the counit:

$$A \xrightarrow{1}_k A = \mathbb{T}_k A \xrightarrow{\epsilon_A} A$$

Compositions uses comultiplication:

$$(\mathbb{T}_k B \xrightarrow{g} C) \circ (\mathbb{T}_k A \xrightarrow{f} B) = \mathbb{T}_k A \xrightarrow{\delta_A} \mathbb{T}_k \mathbb{T}_k A \xrightarrow{\mathbb{T}_k f} \mathbb{T}_k B \xrightarrow{g} C$$

Important fact:
$A \to_k B$ iff duplicator has a winning strategy for the one-way k pebble game from A to B.

Locality and contextuality

This collection of comonads allows us to give a categorical perspective on k-local properties of and relations between structures.

For a pair of structures A and B Abramsky et al. define pebble number:

$$\pi_B(A) = \min\{k \mid A \to \mathbb{T}_k B\}$$

and strong consistency number:

$$\mathrm{sc}_B(A) = \max\{k \mid \ \mathbb{T}_k A \to B\}$$

Note that $\mathrm{sc}_B(A) = |A| \iff A \to B$.

So truly *contextual* relations $A \to_k B$ in $\mathcal{K}(\mathbb{T}_k)$ will have $\mathrm{sc}_B(A) < |A|$.

Example: contextuality in $\mathcal{K}(\mathbb{T}_k)$

A	B	C
D	E	F
G	H	I

$$A \bigoplus B \bigoplus C = 0 \quad A \bigoplus D \bigoplus G = 0$$
$$D \bigoplus E \bigoplus F = 0 \quad B \bigoplus E \bigoplus H = 0$$
$$G \bigoplus H \bigoplus I = 0 \quad C \bigoplus F \bigoplus I = 1$$

Table 1: Nine-element magic square seen as a structure M over a signature with two ternary relations R_0 and R_1 with $x \bigoplus y \bigoplus z = i$ meaning $(x, y, z) \in R_i$

Consider $Z2$ as a structure over this signature too, not hard to see:

$$M \not\to Z2$$

but

$$M \to_5 Z2$$

Why do this?

- Compositional language for k-local methods in CSP
  $A \to_k B$ captures the notion of k-locally satisfying constraints of A in domain B (example shows this doesn't capture all tractable cases)
- New connections in descriptive complexity
  $\mathbb{T}_k$-algebras are tree decompositions, $\pi_A(A) = \text{treewidth}(A) + 1$, isomorphism in $\mathcal{K}(\mathbb{T}_k)$ is $\equiv_{\mathcal{C}^k}$,
- More systematic finite model theory
  Rossman and Otto's preservation theorems are more naturally phrased in this setting, perhaps this will lead to simpler proofs.

Cores for the pebbling comonad?

Revisiting cores in this setting

Properties of category of graphs that allowed unique cores:

1. $\hookrightarrow$ (substructure) is closed under composition.
2. $\{H \mid G \xrightarrow{retr.} H\}$ has a $\hookrightarrow$-minimal element.
3. a If H and H' are cores of G, then $H \twoheadrightarrow H'$.
   b *Schröder-Bernstein property* :
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4. $G \rightleftarrows \mathrm{core}(G)$

First challenge: defining $\hookrightarrow$ and $\twoheadrightarrow$

Definition

Define a branch-injective (resp. branch-surjective) winning strategy for the k pebble game as a map $f : \mathbb{T}_k A \to B$ where for every $t \in \mathbb{T}_k A,\ j \in [k]$ the map

$$\psi_{t,j}(x) = f(t : (j, x))$$

is injective (resp. surjective), write $A \to_k^i B$ (resp.$A \to_k^s B$)

First challenge: defining $\hookrightarrow$ and $\twoheadrightarrow$

Lemma

- $\to^{i}_{k}$ *corresponds to monomorphism in* $\mathcal{K}(\mathbb{T}_k)$
- $\to^{s}_{k}$ *corresponds to epimorphism in* $\mathcal{K}(\mathbb{T}_k)$

Lemma

- $A \to^{i}_{k} B \iff A \Rrightarrow_{\exists^{+}\mathcal{C}^k} B$
- $A \to^{s}_{k} B \iff A \Rrightarrow_{+\mathcal{L}^k} B$

Defining cores in $\mathcal{K}(\mathbb{T}_k)$

A *k-pebble retract* of $B \in \mathcal{K}(\mathbb{T}_k)$ is a structure $A \in \mathcal{K}(\mathbb{T}_k)$ with maps $\iota_A : A \to_k^i B$ and $\rho_A : B \to_k^s A$ s.t. $\rho_A \circ \iota_A = 1_A$, write $A \xrightarrow{retr.}_k B$

A *k-pebble core* of B is a $\to_k^i$-minimal k-pebble retract of A.[3]

[3] As structures are finite this is the same as being the smallest structure

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Regularity, obstacles to core uniqueness and open questions

Re-cap

There are "cores" in $\mathcal{K}(\mathbb{T}_k)$ but are they unique?

Equivalently, is it the case that for any two k-pebble cores H, H' of G, there is a surjection $H \rightarrow_k^s H'$

Regularity is a sufficient condition

Definition (Informal)

A category is regular if it admits some form of epi-mono image factorisation, i.e. any morphism factors into an epimorphism followed by a monomorphism as shown below:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, two heads] & & B \\ & im(f) \arrow[ur, hook] & \end{tikzcd}$$

Why is this sufficient?

Suppose H, H' are $\rightarrow^i_k$-minimal k retracts of G then $f = \rho_{H'} \circ \iota_H$ is a map from H to H'. By regularity, this factorizes as an epi followed by a mono. But im(f) is a homomorphic image of G so by $\rightarrow^i_k$-minimality of H' the mono part of the factorisation must also be an epi. Thus f is an epi.

$\mathcal{K}(\mathbb{T}_k)$ is not regular!

Example

There are strategies (i.e. maps in $\mathcal{K}(\mathbb{T}_k)$ for which the "branch" maps $\psi_{t,j}$ have differently sized domains.
This is a problem as any mono (resp. epi) in $\mathcal{K}(\mathbb{T}_k)$ has injective (resp. surjective) branch maps and any epi-mono factorisation must by a simultaneous surj-inj factorisation of all the branch maps.
Consider two directed triangles and the two pebble game ...

Possible fixes?

- Restrict to "balanced"/*special* strategies and show any $A \to_k B$ implies the existence of a "balanced"/*special* strategy. Show that this subcategory is regular.
- Find another way to show property 3 (a)

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There are lots!

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- Is such a notion well defined in other non-classical settings? Which ones have unique cores? (e.g. other game comonads $\mathbb{E}_n$, quantum monads Q_d)
- What's the relationship between regularity and cores?