CONSTRAINT SATISFACTION PROBLEMS
INCLUDING QUANTUM RELAXATIONS

A TUTORIAL

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ALGORITHMIC HIERARCHIES

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Constraint Satisfaction Problems (CSPs)

Domain of values and constraint language:

$D = \{b_1, \ldots, b_q\}$ where $q \geq 2$,
$\Gamma = \{R_1, R_2, \ldots\}$ where $R_i \subseteq D^{r_i}$.

An instance over Γ:

$$\exists X_1 \cdots \exists X_n (C_1(t_1) \wedge \ldots \wedge C_m(t_m))$$

where

each X_j is a D-valued variable; $V = \{X_1, \ldots, X_n\}$

each $C_j \in \Gamma \cup \{=\}$ is a constraint relation

each $t_j \in (V \cup D)^{r_j}$ is a constraint scope

The solution space:

$$f : V \to D \quad \text{with} \quad f(t_j) \in C_j \quad \text{for } j = 1, \ldots, m.$$

Example 1:

System of linear equations over $\mathbb{Z}_2$:

$$\begin{aligned} X_1 + X_2 + X_3 &\equiv 0 \pmod 2 \\ X_2 + X_4 + X_5 &\equiv 1 \pmod 2 \\ X_3 + X_4 + X_2 &\equiv 1 \pmod 2 \end{aligned}$$

Here

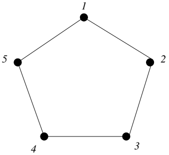
$$D = \{0, 1\} \text{ and } \Gamma = \{R_0, R_1\}$$

where

$$\begin{aligned} R_0 &= \{(a, b, c) \in D^3 : a + b + c \equiv 0 \pmod 2\} \\ R_1 &= \{(a, b, c) \in D^3 : a + b + c \equiv 1 \pmod 2\} \end{aligned}$$

Example 2

Graph 3-colorability:



$$\begin{array}{l} X_1 \neq X_2 \\ X_2 \neq X_3 \\ X_3 \neq X_4 \\ X_4 \neq X_5 \\ X_5 \neq X_1 \end{array} \quad \text{with } X_i \in \{\color{red}\bullet, \color{green}\bullet, \color{blue}\bullet\}$$

Here

$$D = \{\color{red}\bullet, \color{green}\bullet, \color{blue}\bullet\} \text{ and } \Gamma = \{\neq\}.$$

CSP and Contextuality

Empirical models (example: the PR Box)

		00	01	10	11
a	b	$1/2$	0	0	$1/2$
a	b'	$1/2$	0	0	$1/2$
a'	b	$1/2$	0	0	$1/2$
a'	b'	0	$1/2$	$1/2$	0

It is quantum realizable: $T(uv|XY) = \langle\psi|XY|\psi\rangle$
It is not classically realizable: $T(uv|XY) \neq \mu(XY = uv)$

Following [Abramsky 2011] we use CSPs to:

1) express/witness necessary conditions for classical realizability
2) express/witness sufficient conditions for classical unrealizability
3) hence, provide proofs of non-locality and contextuality.

Possibilistic Empirical Models and CSPs

PR-Box (possibilistic):

		00	01	10	11
a	b	1	0	0	1
a	b'	1	0	0	1
a'	b	1	0	0	1
a'	b'	0	1	1	0

$$\exists a \exists b \exists a' \exists b' (S_1(ab) \wedge S_1(ab') \wedge S_1(a'b) \wedge S_2(a'b'))$$

$D = \{0, 1\}$,
$\Gamma = \{S_1, S_2\}$
$S_1(XY) = \neg(X \oplus Y)$
$S_2(XY) = X \oplus Y$

Possibilistic Empirical Models and CSPs

Hardy (possibilistic):

		00	01	10	11
a	b	1	1	1	1
a	b'	0	1	1	1
a'	b	0	1	1	1
a'	b'	1	1	1	0

$$\exists a \exists b \exists a' \exists b' (R_1(ab) \wedge R_2(ab') \wedge R_2(a'b) \wedge R_3(a'b'))$$

$D = \{0, 1\}$,
$\Gamma = \{R_1, R_2, R_3\}$
$R_1(XY) = 1$
$R_2(XY) = X \vee Y$
$R_3(XY) = \neg X \vee \neg Y$.

Possibilistic Empirical Models and CSPs

GHZ (possibilistic):

			000	001	010	011	100	101	110	111
a	b	c	1	0	0	1	0	1	1	0
a	b'	c'	0	1	1	0	1	0	0	1
a'	b	c'	0	1	1	0	1	0	0	1
a'	b'	c	0	1	1	0	1	0	0	1

$$\exists abca'b'c'(T_1(abc) \wedge T_2(ab'c') \wedge T_2(a'bc') \wedge T_2(a'bc') \wedge T_2(a'b'c))$$

$D = \{0, 1\}$,
$\Gamma = \{T_1, T_2\}$
$T_1(XYZ) = \neg(X \oplus Y \oplus Z)$
$T_2(XYZ) = X \oplus Y \oplus Z$

Possibilistic Empirical Models and CSPs

18 Vector Kochen-Specker:

				1000	0100	0010	0001	0000	$\cdots$
a	b	c	d	1	1	1	1	0	$\cdots$
a	e	f	g	1	1	1	1	0	$\cdots$
h	i	c	j	1	1	1	1	0	$\cdots$
h	k	g	l	1	1	1	1	0	$\cdots$
b	e	m	n	1	1	1	1	0	$\cdots$
i	k	n	o	1	1	1	1	0	$\cdots$
p	q	d	j	1	1	1	1	0	$\cdots$
p	r	f	l	1	1	1	1	0	$\cdots$
q	r	m	o	1	1	1	1	0	$\cdots$

$$\exists ab \cdots r (U(abcd) \wedge U(aefg) \wedge \cdots \wedge U(qrmo))$$

$D = \{0, 1\}$,
$\Gamma = \{U\}$
$U(XYZW) =$ ONE-IN-FOUR$(XYZW)$

STRUCTURE OF THE TALK

Part I	Feder-Vardi Dichotomy Conjecture (now Theorem!)
Part II	pp-definitions and closure operations
Part III	Post's Lattice and the 2-valued case
Part IV	pp-interpretations and the q-valued case
Part V	bounded width
Part VI	quantum relaxations

Part I: Feder-Vardi Dichotomy Conjecture

CSP(Γ): Given an instance Φ over Γ, does Φ have a solution satisfying all constraints?

Feder-Vardi Dichotomy Conjecture (1993):

For every constraint language Γ,
CSP(Γ) is either in P or NP-complete.

A very short history:

– True for $\Gamma \in$ Two-valued	by Schaefer 1978
– True for $\Gamma \in$ Three-valued	by Bulatov 2006
– True for $\Gamma \in$ Graphs	by Hell-Nesetril 1990
– True for $\Gamma \in$ Smooth-digraphs	by Barto-Kozik-Niven 2009
– ...	
– **True**.	by Zhuk 2017, Bulatov 2017.

Part II: pp-definitions and closure operations

Primitive positive definitions (pp-definitions):

$$(X_1, \ldots, X_r) \in R \iff \exists Y_1 \cdots \exists Y_s (C_1(t_1) \wedge \cdots \wedge C_m(t_m))$$

where

each X_i and Y_i ranges over D; $V = \{X_i\}_{i=1}^r$ and $W = \{Y_i\}_{i=1}^s$
each $t_i \in (V \cup W \cup D)^{r_i}$
each $C_i \in \Gamma \cup \{=\}$

We say:

R is pp-definable in Γ.

Example 1 of pp-definition

Example 1:

- $D = \{0, 1\}$,
- $\Gamma =$ 3-LIN := both 3-ary parity equations $= \{R_0, R_1\}$
- pp-definition of 4-ary parity equations in Γ:

$$X_1 \oplus X_2 \oplus X_3 \oplus X_4 = a$$
$$\Longleftrightarrow$$
$$\exists Y (X_1 \oplus X_2 \oplus Y = 0 \wedge Y \oplus X_3 \oplus X_4 = a)$$

Example 2 of pp-definition

Example 2:

- $D = \{0, 1\}$,
- $\Gamma =$ ONE-IN-THREE $:= \{R_{1/3}\}$ where $R_{1/3} := \{001, 010, 100\}$,
- pp-definition of OR_2 in Γ:

$$X_1 \vee X_2$$
$$\Longleftrightarrow$$
$$\exists Y_1 \exists Y_2 \exists Y_3 (R_{1/3}(X_1, Y_1, 0) \wedge R_{1/3}(X_2, Y_2, 0) \wedge R_{1/3}(Y_1, Y_2, Y_3)).$$

Gadget pp-based reductions

Fact:

If R is pp-definable in Γ,
then $\mathrm{CSP}(\Gamma \cup \{R\}) \leq_m^{\mathrm{L}} \mathrm{CSP}(\Gamma)$.

Proof:

Locally replace each R-constraint by its pp-definition. QED

Closure operation (a.k.a. polymorphism)

$R \subseteq D^r$ is invariant under $f : D^k \to D$ if

whenever

$$\begin{matrix} (a_{1,1}, & \cdots, & a_{1,r}) \in R \\ \vdots & \ddots & \vdots \\ (a_{k,1}, & \cdots, & a_{k,r}) \in R \end{matrix}$$

also

$$(f(a_{1,1}, \ldots, a_{k,1}), \cdots, f(a_{1,r}, \ldots, a_{k,r})) \in R$$

Example 0 of closure operation

- arbitrary D,
- arbitrary $R \subseteq D^r$,
- R is invariant under all projections

$$\begin{gathered}\mathrm{proj}_{k,i} : \{0,1\}^k \to \{0,1\} \\ (a_1, \ldots, a_k) \mapsto a_i\end{gathered}$$

Whenever

$$\begin{gathered}(a_{1,1}, \ldots, a_{1,r}) \in R \\ (a_{2,1}, \ldots, a_{2,r}) \in R \\ \vdots \\ (a_{k,1}, \ldots, a_{k,r}) \in R\end{gathered}$$

(obviously) also

$$(a_{i,1}, \ldots, a_{i,r}) \in R$$

Example 1 of closure operation

- $D = \{0, 1\}$,
- $R = \mathrm{OR}_2 = \{01, 10, 11\}$,
- R is invariant under any odd-arity majority

$$\mathrm{maj}_{2k+1} : \{0,1\}^{2k+1} \to \{0,1\}$$
$$(a_1, \ldots, a_{2k+1}) \mapsto \text{majority}(a_1, \ldots, a_{2k+1})$$

Whenever

$$(a_1, b_1) \in \mathrm{OR}_2$$
$$(a_2, b_2) \in \mathrm{OR}_2$$
$$\vdots$$
$$(a_{2k+1}, b_{2k+1}) \in \mathrm{OR}_2$$

also

$$(a, b) \in \mathrm{OR}_2$$

Example 2 of closure operation

- $D = \{0, 1\}$,
- $R_0 =$ "$X_1 \oplus X_2 \oplus X_3 = 0$" and $R_1 =$ "$X_1 \oplus X_2 \oplus X_3 = 1$",
- both R_0 and R_1 are invariant under any odd-arity parity

$$\mathrm{xor}_{2k+1} : \{0,1\}^{2k+1} \to \{0,1\}$$
$$(a_1, \ldots, a_{2k+1}) \mapsto a_1 \oplus \cdots \oplus a_{2k+1}$$

Whenever

$$\begin{gathered} a_1 \oplus b_1 \oplus c_1 = d \\ a_2 \oplus b_2 \oplus c_2 = d \\ \vdots \\ a_{2k+1} \oplus b_{2k+1} \oplus c_{2k+1} = d \end{gathered}$$

also

$$a \oplus b \oplus c = d$$

Closure operations and pp-definability

$\mathrm{Pol}(R)$: set of all idempotent closure operations of R
$\mathrm{Inv}(f)$: set of all relations that are invariant under f

$\mathrm{Pol}(R_1, R_2, \ldots) := \bigcap_i \mathrm{Pol}(R_i)$
$\mathrm{Inv}(f_1, f_2, \ldots) := \bigcap_i \mathrm{Inv}(f_i)$

Theorem [Geiger 1968, Bodnarchuk et al. 1969]:

R is pp-definable in Γ
if and only if
R is in $\mathrm{Inv}(\mathrm{Pol}(\Gamma))$.

Example 1 application of Geiger's Theorem

Corollary:

OR_2 is not pp-definable in 3-LIN.

Proof:

OR_2 is not invariant under xor_3:

$$\frac{\begin{array}{c}(0,1) \in \mathrm{OR}_2 \\ (1,0) \in \mathrm{OR}_2 \\ (1,1) \in \mathrm{OR}_2\end{array}}{(0,0) \notin \mathrm{OR}_2.}$$

Example 2 application of Geiger's Theorem

Corollary:

If $\mathrm{Pol}(\Gamma) =$ "all the projections",
then $\mathrm{Inv}(\mathrm{Pol}(\Gamma)) =$ "all the relations".

Reductions

Theorem [Jeavons]:

If $\mathrm{Pol}(\Gamma) \subseteq \mathrm{Pol}(\Gamma')$,
then $\mathrm{CSP}(\Gamma') \leq_m^{\mathrm{L}} \mathrm{CSP}(\Gamma)$.

Corollary:

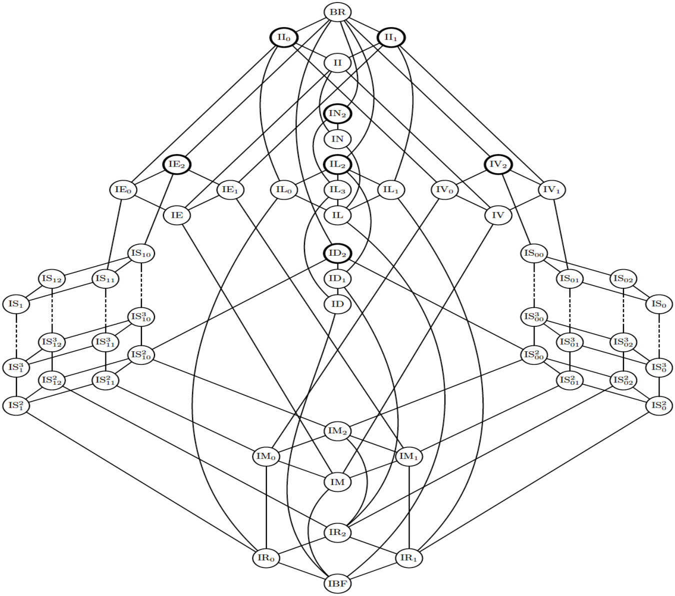
If $\mathrm{Pol}(\Gamma) =$ "all the projections",
then $\mathrm{CSP}(\Gamma)$ is NP-complete.

Part III: Post's Lattice and the 2-valued case

Theorem [Post 1941]

There are countably many sets $\mathrm{Inv}(\mathrm{Pol}(\Gamma))$ with $D = \{0, 1\}$.
Moreover, they make a lattice under set inclusion.
And we know who they are.

Post's lattice of closed sets of Boolean relations



Example 1 of application of Post's Lattice

Theorem [Schaefer 1978]:
Let Γ have $|D| = 2$.
Then $\mathrm{CSP}(\Gamma)$ is in P if

every $R \in \Gamma$ is 0-valid	(closed under 0),	or
every $R \in \Gamma$ is 1-valid	(closed under 1),	or
every $R \in \Gamma$ is bijunctive	(closed under maj$_3$),	or
every $R \in \Gamma$ is Horn	(closed under and$_2$),	or
every $R \in \Gamma$ is dual-Horn	(closed under or$_2$),	or
every $R \in \Gamma$ is affine	(closed under xor$_3$).	

Else $\mathrm{CSP}(\Gamma)$ is NP-complete.

Corollary:

ONE-IN-THREE-SAT is NP-complete

Part IV: pp-interpretations and the q-valued case

A pp-interpretation of (D', Γ') in (D, Γ) is:

– a partial surjective map $h : D^k \to D'$ s.t.
– $\mathrm{Dom}(h)$ is pp-definable in Γ,
– $h^{-1}(R)$ is pp-definable in Γ,
– $h^{-1}(=)$ is pp-definable in Γ,
– $h^{-1}(\{b\})$ is pp-definable in Γ for every $b \in D$.

Theorem [Bulatov-Jeavons-Krokhin 2005]:

If (D', Γ') is pp-interpretable in (D, Γ),
then $\mathrm{CSP}(\Gamma') \leq_m^{\mathrm{L}} \mathrm{CSP}(\Gamma)$.

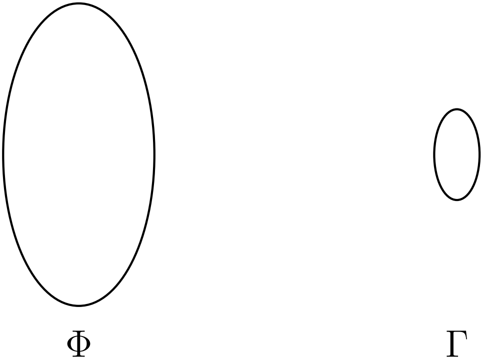
Part V : Bounded width

Game-theoretic formulation:

Let Γ be a constraint language with $|D|$ values.

Let Φ be an instance over Γ with $|V|$ variables.

Fix an integer k, $1 \leq k \leq |V|$.



Local Consistency

Definition:

A CSP instance Φ is called k-locally-consistent if Duplicator has a winning strategy in the existential k-pebble game.

Observations:

1) $|V|$-locally-consistent = satisfiable
2) $k+1$-locally-consistent $\Rightarrow$ k-locally-consistent
3) k-local-consistency is decidable in time $O(|D|^k|V|^k)$

Two more observations:

4) k-local-consistency $\Rightarrow$ $\frac{k}{\text{arity}}$-consistency [à la Abramsky et al]
5) but not at all vice-versa: the gap can be arbitrarily large.

Local Consistency

Algorithmic formulation:

1. Start with $H = \{h : V \to D \mid |\mathrm{Dom}(h)| \leq k\}$.
2. Remove each h from H that falsifies some constraint.
3. Remove each h from H such that
   a. $\exists g \subseteq h$ with $g \notin H$, or
   b. $|\mathrm{Dom}(h)| < k$ and $\exists x \in V$ with $h \cup \{x \mapsto b\} \notin H$ for all $b \in D$.
4. Repeat step 3 until H stabilizes.
5. If $H = \emptyset$, assert that Φ is unsatisfiable.
6. If $H \neq \emptyset$, say that Φ is k-locally-consistent.

Local Consistency

Equational formulation:

Variables:

A 0-1 variable X_h for each $h : V \to D$ w/ $\mathrm{Dom}(h) \leq k$.

Equations (over the Boolean algebra $(\{0,1\}, \leq, \wedge, \vee)$):

$$
\begin{array}{ll}
X_h = 0 & \text{if } h \text{ falsifies some constraint} \\
X_h \leq X_g & \text{if } g \subseteq h \\
X_h \leq \bigvee_{b \in D} X_{h \cup \{x \mapsto b\}} & \text{if } |\mathrm{Dom}(h)| < k \text{ and } x \in V
\end{array}
$$

$$X_\emptyset \stackrel{?}{=} 1.$$

Schaeffer's Theorem revisited

Theorem
Let Γ have $|D| = 2$.
Then $\mathrm{CSP}(\Gamma)$ is in P if

every $R \in \Gamma$ is 0-valid	(**bounded width**),	or
every $R \in \Gamma$ is 1-valid	(**bounded width**),	or
every $R \in \Gamma$ is bijunctive	(**bounded width**),	or
every $R \in \Gamma$ is Horn	(**bounded width**),	or
every $R \in \Gamma$ is dual-Horn	(**bounded width**),	or
every $R \in \Gamma$ is affine	(**NO bounded width**).	

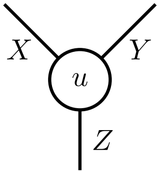
Else $\mathrm{CSP}(\Gamma)$ is NP-complete (**NO bounded width**).

Construction of k-locally-consistent instances

Tseitin construction [T68]:

G is an undirected graph.
$\sigma : V(G) \to \{0,1\}$ is a 0-1 labelling of the nodes of G.

There is a variable at every edge.
There is an equation at every node:



$$\Longrightarrow X \oplus Y \oplus Z = \sigma(u)$$

Observations:

1) each variable appears in exactly two equations
2) if $|\sigma^{-1}(1)|$ is odd, then the system is unsatisfiable.

Construction of k-locally-consistent instances

Theorem [A05]

If treewidth$(G) \geq k$ and Φ is a Tseitin instance based on G, then Φ is $\Omega(k)$-locally-consistent.

Proof: Play the Robber-Cops game. QED

Other tractability criteria

1. Φ has bounded treewidth (**bounded width**)
2. core(Φ) has bounded treewidth ($\equiv$ **bounded width**)
3. two occurrences and $\Gamma =$ ONE-IN-THREE
4. two occurrences and Γ is a Δ-matroid
5. ...