Classifying space for quantum contextuality

Cihan Okay[1]

The University of British Columbia

July 2019

[1] joint with Daniel Sheinbaum **arXiv:1905.07723**

Outline

- Strong contextuality
  1. Topology of contexts
  2. Principal bundles
  3. Homotopical approach[2]
- Contextuality
  1. Empirical models
  2. Twisted representations
  3. Wigner function
  4. Twisted K-theory

[2] Okay and Raussendorf, "arXiv:1905.03822".

Strong contextuality - Part I

- A context is a set of pairwise commuting observables.
- Given a collection of contexts we can construct a chain complex[3] such that

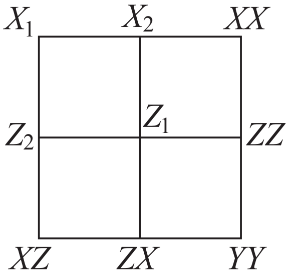
$$[\beta] \neq 0 \in H^2(\mathcal{C}) \Rightarrow \text{strongly contextual}$$

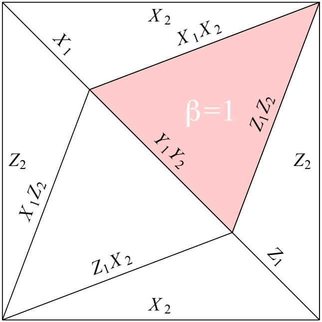
i.e. there is no consistent way of assigning pre-determined measurement outcomes.

[3] "Topological proofs of contextuality in quantum mechanics".

Contexts

Mermin square is interpreted as a torus





Universal construction?

- Given a set of contexts is there a space realization?
- We will give a construction for Pauli observables.

  But the constructions can be done for more general observables.

Pauli observables

- $\mathcal{H}$: n-qudit Hilbert space (local dimension p)
- P_n denotes the n-qudit Pauli group

$$T_v = \begin{cases} i^{v_X \cdot v_Z} Z(v_Z)X(v_X) & p = 2 \\ \omega^{(v_X \cdot v_Z)/2} Z(v_Z)X(v_X) & p > 2 \end{cases}$$

where $\omega = e^{2\pi i/p}$ and

$$v \in \overbrace{\underbrace{(\mathbb{Z}/p)^n}_{Z \text{ part}} \times \underbrace{(\mathbb{Z}/p)^n}_{X \text{ part}}}^{V\text{: phase space}}$$

Isotropic subspaces

- Symplectic vector space $(V, \mathfrak{b})$

$$\mathfrak{b}(v, v') = v_X \cdot v'_Z - v'_X \cdot v_Z$$

- A subspace $I \subset V$ is isotropic if

$$\mathfrak{b}|_I = 0$$

i.e. $\mathfrak{b}(v, v') = 0$ for all $v, v' \in I$.

Contexts in the Pauli group

- For us contexts are specified by a collection of isotropic subspaces

$$\mathcal{I} = \{I_1, I_2, I_3, \cdots\}$$

$\mathcal{I}(V)$ denotes the set of all isotropic subspaces in V.

Ex. Mermin square

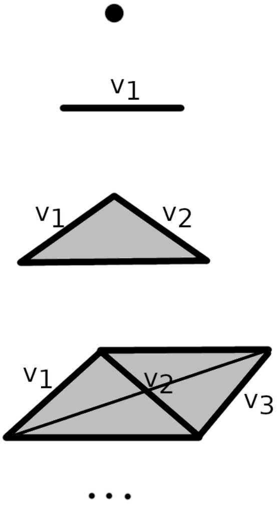
$$\{X_1, X_2, XX\} \Rightarrow \{(0,0;1,0),(0,0;0,1),(0,0;1,1)\}$$

$$\{Z_2, Z_1, ZZ\} \Rightarrow \{(0,1;0,0),(1,0;0,0),(1,1;0,0)\}$$

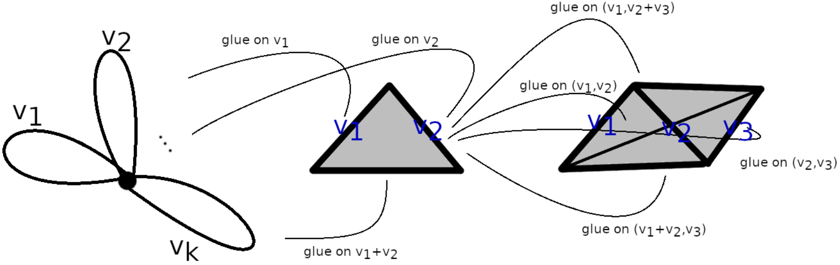
$$\{XZ, ZX, YY\} \Rightarrow \{(0,1;1,0),(1,0;0,1),(1,1;1,1)\}$$

$$\cdots\cdots$$

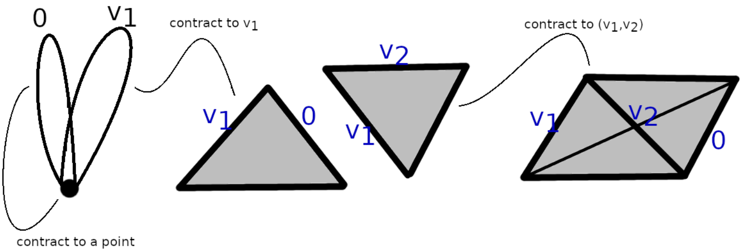
Contexts as simplices



Face maps


v1
v2
vk
glue on v1
glue on v2
glue on v1+v2
v1
v2
glue on (v1,v2+v3)
glue on (v1,v2)
v1
v2
v3
glue on (v2,v3)
glue on (v1+v2,v3)

Degeneracies



Simplicial sets

- This construction is formalized using the language of simplicial sets.
- They can be thought of as "simplicial complexes with degeneracies".
- They are like "non-linear version of chain complexes".

Classifying space for contextuality

- $B_{\text{cx}}(\mathcal{I})$ is the space constructed from n-simplices of the form

$$(v_1, v_2, \cdots, v_n)$$

  where there exists a context $I \in \mathcal{I}$ such that

$$\{v_1, v_2, \cdots, v_n\} \subset I.$$

- We write $B_{\text{cx}}V$ when $\mathcal{I} = \mathcal{I}(V)$.

Classifying space for contextuality

- $B_{\text{cx}}(\mathcal{I})$ is the space constructed from n-simplices of the form

$$(v_1, v_2, \cdots, v_n)$$

where there exists a context $I \in \mathcal{I}$ such that

$$\{v_1, v_2, \cdots, v_n\} \subset I.$$

Remark If we use all $(v_1, \cdots, v_n)$ we obtain the ordinary classifying space BV.

Chain complex

- The chain complex $C(B_{\mathsf{cx}}\mathcal{I})$, which we denote by $C(\mathcal{I})$, is given by

$$\cdots \to C(\mathcal{I})_n \to \cdots \to \underbrace{C(\mathcal{I})_3 \to C(\mathcal{I})_2 \to C(\mathcal{I})_1 \to C(\mathcal{I})_0}_{\text{Studied previously}}$$

An element in degree n has the form

$$\sum_{(v_1,\cdots,v_n)} \alpha_{v_1,\cdots,v_n}[v_1,\cdots,v_n]$$

where $(v_1,\cdots,v_n)$ runs over the n-simplices of $B_{\mathsf{cx}}\mathcal{I}$ and the coefficients $\alpha_{v_1,\cdots,v_n} \in \mathbb{Z}/p$.

[4] "Topological proofs of contextuality in quantum mechanics".

Factorization

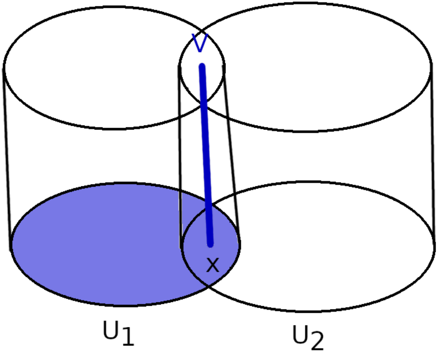
$$
\begin{array}{ccc}
\mathcal{I} & \longrightarrow & C(\mathcal{I}) \\
& \searrow \quad \nearrow & \\
& B_{\mathsf{cx}}(\mathcal{I}) &
\end{array}
$$

Any benefits?

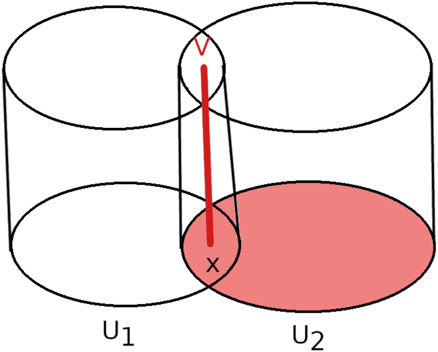
- $B_{\mathsf{cx}}\mathcal{I}$ classifies principal bundles whose transition functions are specified by the contexts[5].
- We will see that probabilities can be included into the picture.

[5] Adem and Gómez, "A classifying space for commutativity in Lie groups".

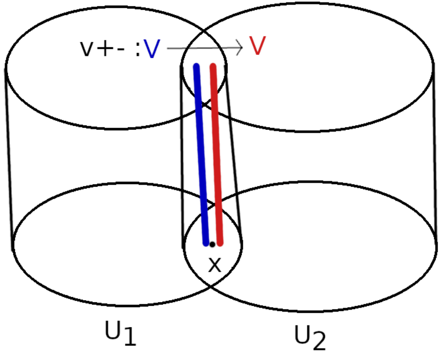
Contexts as transition functions



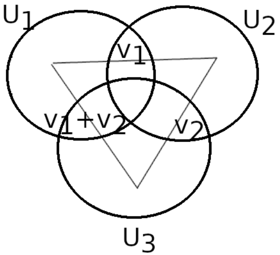
Contexts as transition functions



Contexts as transition functions



Contexts as transition functions



$$\mathrm{Prin}_{\mathrm{cx}}^{V}(X) = [X, B_{\mathrm{cx}} V]$$

Mermin's square

- Let T denote the torus realizing Mermin's square

$$f : T \to B_{\mathsf{cx}} V$$

all vertices are collapsed to a single vertex

$$f^* : H^2(B_{\mathsf{cx}} V) \to H^2(T)$$
$$[\beta] \mapsto [\beta_{\mathsf{Mer}}] \neq 0$$

- The corresponding principal bundle over T is non-trivial!

Formula for β

Theorem

The class $[\beta] \in H^2(B_{cx}V)$ *satisfies*

$$[\beta] = \begin{cases} 0 & p > 2 \\ \mathfrak{q}^2 + \sum_{i=1}^{n} \delta_{x_i} \cup \delta_{z_i} & p = 2 \end{cases}$$

where the 1*-cochains are defined by*

$$\mathfrak{q}(v) = v_X \cdot v_Z$$

$$\delta_{x_i}(v) = (v_X)_i$$

$$\delta_{z_i}(v) = (v_Z)_i$$

Vanishing of β ($p = 2$)

- Let $f : X \to B_{\mathrm{cx}} V$ be a map.
- If $H^1(X, \mathbb{Z}/2) = 0$ then $f^*[\beta] = 0$.

  For example:

  1. X is simply connected, $\pi_1(X) = 1$
  2. More generally, $|\pi_1 X|$ is odd.

Homotopical approach[7]

Theorem

Let X be a space realizing a collection of contexts[6] such that

$$(|\pi_1 X|, d) = 1$$

then we don't have strong contextuality.

[6] unitary matrices $(T_a)^d = I$

[7] Okay and Raussendorf, "Homotopical approach to quantum contextuality".

Contextuality - Part II

- We use the sheaf-theoretic framework[8].

- In this framework contextuality is defined using empirical models:

$$(\rho : \text{quantum state}) \mapsto (e_\rho : \text{probability distribution})$$

[8] Abramsky and Brandenburger, "The sheaf-theoretic structure of non-locality and contextuality".

Formulation

- Sheaf of events

$$\mathcal{E} : \mathcal{I}^{\mathsf{op}} \to \mathbf{Set}$$

where $\mathcal{E}(I)$ is the set of functions $I \to \mathbb{Z}/p$ (outcomes).

- Distributions over outcomes

$$D\mathcal{E} : \mathcal{I}^{\mathsf{op}} \to \mathbf{Set}$$

where $D = D_{\mathbb{R}_{\geq 0}}$ is the distribution monad.

Formulation

- Let us denote the set of compatible families[9] by

$$\varprojlim D\mathcal{E} \subset \prod_{I \in \mathcal{I}} D\mathcal{E}(I)$$

i.e. family of distributions $\{e|_I\}$ satisfying no-signaling

$$(e|_I)|_{I'} = e|_{I'} \quad I' \subset I.$$

[9] also known as an inverse limit

Empirical model

We will think of an empirical model of a state as a function

$$e : \mathrm{Den}(\mathcal{H}) \to \lim_{\overleftarrow{\mathcal{I}}} D\mathcal{E}$$

$$\rho \mapsto e_\rho$$

defined by the formula

$$e_\rho|_I(s) = \mathrm{Tr}(\rho P_s)$$

where P_s is the projection to the common eigenspace of the outcome $s : I \to \mathbb{Z}/p$.

Contextuality

Let $\Sigma(\mathcal{I})$ denote the union of the contexts in $\mathcal{I}$.

There is a function

$$\theta : D\mathcal{E}(\Sigma) \to \lim_{\overleftarrow{\mathcal{I}}} D\mathcal{E}$$

sending d to the collection $\{d|_I\}$.

A state ρ is **contextual** if

$$e_\rho \notin \mathrm{im}(\theta)$$

Sheaf of value assignments

- Observe that

$$e_\rho|_I(s) = \mathrm{Tr}(\rho P_s) = 0$$

if s does not satisfy

$ds(v, v') = s(v) - s(v + v') + s(v') = \beta(v, v').$

- Define

$$\mathcal{E}_\beta(I) = \{s : I \to \mathbb{Z}/p|\ ds = \beta\}$$

which can be regarded as a functor

$$\mathcal{E}_\beta : \mathcal{I}^{\mathrm{op}} \to \mathbf{Set}$$

Empirical model - revised

We will think of an empirical model of a state as a function

$$e : \mathsf{Den}(\mathcal{H}) \to \lim_{\overleftarrow{\mathcal{I}}} D\mathcal{E}_\beta$$

$$\rho \mapsto e_\rho$$

Twisted representations

- Let $s \in \mathcal{E}_\beta(I)$ then we can define a twisted representation $\chi_s : I \to U(1)$ by the formula

$$\chi_s(v) = \omega^{s(v)} = e^{2\pi i s(v)/p}.$$

- We will write $R_\beta(I)$ for the $\mathbb{Z}$-linear combinations of twisted representations[10]

$$\sum_{s \in \mathcal{E}_\beta(I)} \alpha_s \, [\chi_s]$$

[10] Grothendieck group of twisted representations

Twisted representation functor

- We obtain a functor

$$R_\beta : \mathcal{I}^{\mathrm{op}} \to \mathbf{Set}$$

where given $I' \subset I$ we use the restriction of representations

$$\mathrm{res}_{I,I'} : R_\beta(I) \to R_\beta(I')$$
$$[\chi_s] \mapsto [\chi_s|_{I'}]$$

Extending coefficients

- The set of distributions $D\mathcal{E}_\beta(I)$ can be seen as sitting inside the $\mathbb{R}$-vector space $\mathbb{R} \otimes R_\beta(I)$:

$$e \mapsto \sum_s e(s)\,[\chi_s]$$

- Moreover this gives a natural transformation

$$D\mathcal{E}_\beta \to \mathbb{R} \otimes R_\beta$$

Empirical model - revised

We will think of an empirical model of a state as a function

$$e : \mathrm{Den}(\mathcal{H}) \to \lim_{\overleftarrow{\mathcal{I}}} \mathbb{R} \otimes R_\beta$$

the element $e_\rho|_I$ is thought of as

$$\underbrace{\sum_s e_\rho|_I(s)\,[\chi_s]}_{\text{probabilistic combination of twisted representations}}$$

What is the benefit?

- We can use the character map

$$\mathrm{ch} : R_\alpha(G) \to \mathrm{Cl}_\alpha(G)$$

where $\mathrm{Cl}_\alpha(G)$ is the $\mathbb{C}$-vector space of α-class functions i.e. functions $f : G \to \mathbb{C}$ satisfying

$$f(hgh^{-1}) = \frac{\alpha(h, h^{-1})}{\alpha(h, gh^{-1})\alpha(g, h^{-1})} f(g)$$

for all $g, h \in G$.

Space of compatible families

Theorem

There is an isomorphism of $\mathbb{R}$-vector spaces

$$\phi_p : \mathbb{R} \otimes R(V) \xrightarrow{\cong} \varprojlim_{\mathcal{I}(V)} \mathbb{R} \otimes R_\beta$$

$p = 2$ vs $p > 2$

- Definition of ϕ_p depends on whether $p = 2$ or $p > 2$.
  1. When $p > 2$ since $[\beta] = 0$ we can forget about the twisting

     $$\phi_p : \mathbb{R} \otimes R(V) \longrightarrow \lim_{\leftarrow} \mathbb{R} \otimes R$$

     is induced by restriction along $I \subset V$.
  2. For $p = 2$ we have to pass through the character map

     $$\phi_2 : \mathbb{R} \otimes R(V) \cong \mathbb{R}^V \longrightarrow \lim_{\leftarrow} \mathbb{R}^- \cong \lim_{\leftarrow} \mathbb{R} \otimes R_\beta$$

     where $\mathbb{R}^- : I \mapsto \mathbb{R}^I$.

Lifting empirical models

$$
\begin{array}{ccc}
\mathsf{Den}(\mathcal{H}) & \overset{?}{\dashrightarrow} & \mathbb{R} \otimes R(V) \\
& \searrow^{e} & \downarrow \cong \\
& & \varprojlim \mathbb{R} \otimes R_\beta
\end{array}
$$

Lifting empirical models

$$\begin{array}{ccc} \mathrm{Den}(\mathcal{H}) & \overset{?}{\dashrightarrow} & \mathbb{R} \otimes R(V) \\ & \searrow^{e} & \downarrow \cong \\ & & \varprojlim \mathbb{R} \otimes R_\beta \end{array}$$

$R(V)$ consists of $\mathbb{Z}$-linear combinations of $[b_v]$ where

$$b_v : V \to U(1), \quad b_v(u) = \omega^{\mathfrak{b}(u,v)}$$

Lifting empirical models

$$\mathsf{Den}(\mathcal{H}) \overset{?}{\dashrightarrow} \mathbb{R} \otimes R(V)$$

$$\rho \longrightarrow \sum_v \, (?_v) \, [b_v]$$

Wigner function

Theorem

Let $W_\rho : V \to \mathbb{R}$ denote the Wigner function of ρ.

Then the diagram commutes

$$\begin{array}{ccc} Den(\mathcal{H}) & \xrightarrow{W} & \mathbb{R} \otimes R(V) \\ & \searrow^{e} & \downarrow \phi_p \\ & & \underleftarrow{lim}\, \mathbb{R} \otimes R_\beta \end{array}$$

where W is defined by

$$[W_\rho] = \sum_{v \in V} W_\rho(v)\,[b_v]$$

Application - $p > 2$ case

- $W_\rho \geq 0$ if and only if ρ is non-contextual[11].

$$\begin{array}{ccc} \overbrace{D\mathcal{E}_\beta(V)}^{\text{prob. comb.}} & \xrightarrow{\theta} & \varprojlim \mathbb{R}\otimes R_\beta \\ \big\downarrow & \nearrow_{\cong} & \\ \mathbb{R}\otimes R(V) & & \end{array}$$

1. If $W_\rho \geq 0$ then $\theta(W_\rho) = e_\rho$.
2. If $\theta(d) = e_\rho$ then $d \mapsto W_\rho$.

[11] Delfosse et al., "Equivalence between contextuality and negativity of the Wigner function for qudits"; Howard et al., "Contextuality supplies the magic for quantum computation".

Question

- Is there a topological interpretation for the target?

$$e : \mathrm{Den}(\mathcal{H}) \to \varprojlim_{\mathcal{I}} \mathbb{R} \otimes R_\beta$$

Twisted K-group

- $K^\beta(X)$ is the Grothendieck group of twisted vector bundles over X.

  When $[\beta] = 0$ it is the ordinary K-group $K(X)$.

- Atiyah-Segal completion

$$R_\beta(I) \to K^\beta(BI)$$

  twisted K-group can be obtained from $R_\beta(I)$ algebraically.

Twisted K-group

Theorem

There is a commutative diagram of $\mathbb{R}$-vector spaces

$$\begin{array}{ccc} \mathbb{R} \otimes K(BV) & \xrightarrow{\cong} & \mathbb{R} \otimes K^{\beta}(B_{cx}V) \\ \uparrow & & \uparrow \\ \mathbb{R} \otimes R(V) & \xrightarrow{\cong} & \varprojlim \mathbb{R} \otimes R_{\beta} \end{array}$$

Empirical model - final revision

We can think of an empirical model of a state as a function

$$e : \mathrm{Den}(\mathcal{H}) \to \mathbb{R} \otimes K^{\beta}(B_{\mathsf{cx}} V)$$

the element e_ρ corresponds to a class $[e_\rho]$ in the twisted K-group.

Questions

- Homotopy type of $B_{\mathsf{cx}}V$ is well-understood[12].

  Further applications to contextuality?

- Can we physically interpret principal bundles whose transition functions are given by contexts?

- Twisted K-theory is graded, can we interpret elements of $K^{\beta+1}(B_{\mathsf{cx}}V)$?

[12] O., "Spherical posets from commuting elements".