Magic States and Contextuality

Mark Howard (NUI Galway, Ireland)

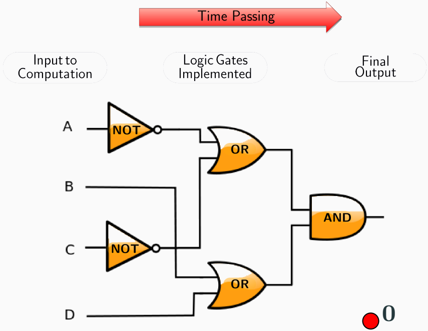
THE ROYAL SOCIETY

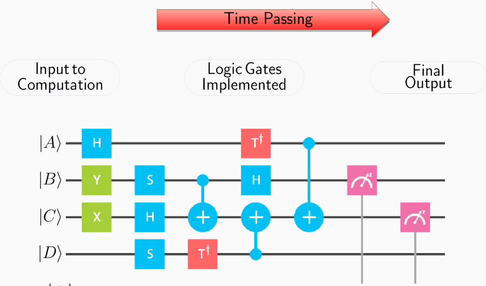
Fondúireacht Eolaíochta Éireann
Dá bhfuil romhainn
Science Foundation Ireland
For what's next

Outline of Talk

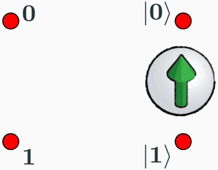
1. Universal Quantum Computing via Magic States
2. Quantum Contextuality
3. How 1 and 2 are related (esp. qudits)

Quantum Computing

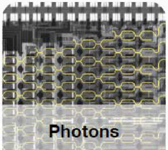


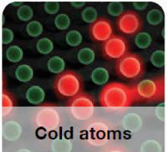


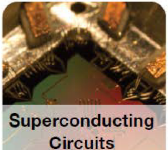
(Classical) Bits
Probabilistic $0/1$



Qubits (Quantum)
$\alpha|0\rangle + \beta|1\rangle$







Qubits vs Qudits (Relevant Later)

Let's define a qudit to be a p-level quantum system ($p =$**odd** prime)

$$\begin{cases} \text{Qubit:} & \alpha_0|0\rangle + \alpha_1|1\rangle \\ \text{Qudit:} & \displaystyle\sum_{k=0}^{p-1} \alpha_k|k\rangle \end{cases} \qquad \sum_{k=0}^{p-1} |\alpha_k| = 1, \quad \alpha_k \in \mathbb{C}$$

Mixed/impure states: $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i| \qquad \left(\sum_i p_i = 1\right)$

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Qudits...

- ✓ Are naturally occurring in many physical systems
- × Are (probably) more difficult to experimentally prepare, control & measure than qubits
- ✓ Have nice symmetries, advantageous for fault-tolerance

Qubits...

- ✓ Are what people actually want to use...

Pauli Matrices:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, Y = iXZ = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

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Weyl-Heisenberg/Pauli Group:

$a = [a_1, a_2, \ldots, a_n], b = [b_1, b_2, \ldots, b_n] \in \mathbb{F}_2^n$ Binary Vectors

$\Updownarrow$

$D(a,b) = X^{a_1} Z^{b_1} \otimes X^{a_2} Z^{b_2} \otimes \cdots \otimes X^{a_n} Z^{b_n}$ Pauli Operators

$WH_{2^n} = \{i^\kappa D(a,b) \mid a, b \in \mathbb{F}_2^n, \kappa \in \mathbb{Z}_4\}$ Pauli/WH Group

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The Clifford Group:

$$\mathsf{Cliff}_{2^n} = \left\{ g \in \mathbb{U}_{2^n} \mid gD(a, b)g^\dagger \in WH_{2^n}, \forall D(a, b) \in WH_{2^n} \right\}$$

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It turns out that

$$gD(a, b)g^\dagger = \pm D([a, b]F_g) \text{ where } F_g \text{ is a } 2n \times 2n \text{ binary matrix}$$

$$\mathsf{Cliff}_{2^n}/WH_{2^n} \approx \mathsf{Sp}(2n, \mathbb{F}_2)$$

Classical Simulation

Rough Intuition: Simulating quantum theory is hard because

- States' description exponentially long: $|\psi\rangle \in \mathbb{C}^{2^n}$
- Evolution governed by $2^n \times 2^n$ Unitary Matrices.

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Heisenberg Representation:
Describe a state by its stabilizer instead…

$$\mathcal{S}(\psi) = \begin{cases} \{s_j \in WH_{2^n} \mid s_j|\psi\rangle = |\psi\rangle,\ 1 \leq j \leq 2^n\} \\ \left\{[a, b]_j \in \mathbb{F}_2^{2n} \mid s_j|\psi\rangle = |\psi\rangle,\ 1 \leq j \leq 2^n\right\} \end{cases}$$

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Upshot:
Large class of states & operations efficiently/$poly(n)$ simulable

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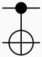
$\mathrm{Cliff}_{2^n} = \langle$  , Symmetry group of  $\rangle \neq \mathrm{UQC}$

Fault-Tolerant Logic

Use Quantum Error-Correcting to protect a 2^k-dim subspace

$$\left\{ |\psi\rangle \in \mathbb{C}^{2^n} \;\middle|\; s_j|\psi\rangle = |\psi\rangle,\ 1 \leq j \leq 2^{n-k} \right\}$$

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Encode 1 logical qubit in 3: $\alpha|0_L\rangle + \beta|1_L\rangle \rightsquigarrow \alpha|000\rangle + \beta|111\rangle$

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In general $U_L \in$ "Normalizer of $\{s_j\}$ in $\mathbb{U}$"

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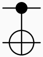
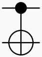
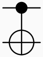
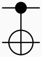
Will show how to supplement Cliff_{2^n} with additional " T " gate

$\Big\langle$  , Symmetry group of  $\Big\rangle \neq \mathrm{UQC} = \Big\langle$  , Symmetry group of  , $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \Big\rangle$

The Magic State model of Quantum Computing

How to supplement Cliff_{2^n} with additional " T " gate

$\Big\langle$  , Symmetry group of 

$\Big\rangle \neq \mathrm{UQC} = \Big\langle$  , Symmetry group of 

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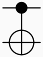
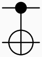
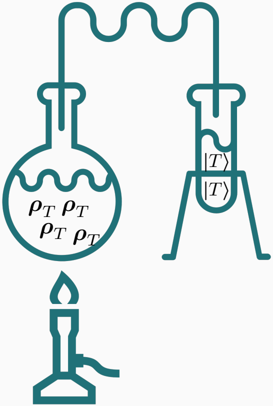
$\Big\langle$  , Symmetry group of  $\Big\rangle \neq \mathrm{UQC} = \Big\langle$  , Symmetry group of  , $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \Big\rangle$

Use "Magic State Distillation"
to complete Univeral gate set



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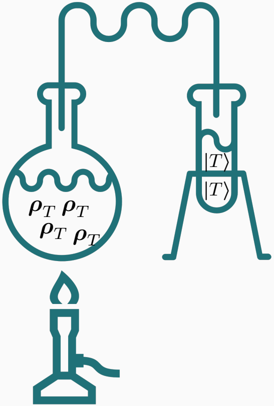

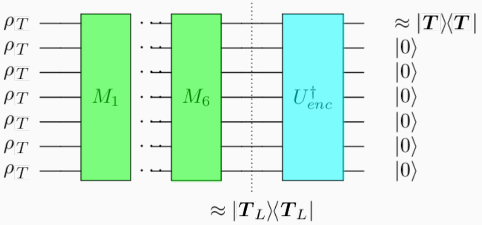

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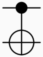
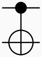
$\langle$  , Symmetry group of  $\rangle \neq \text{UQC} = \langle$  , Symmetry group of  $, T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \rangle$

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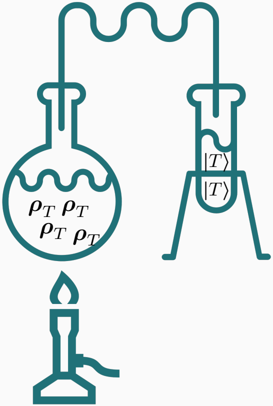

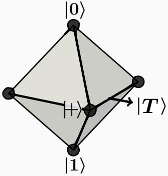

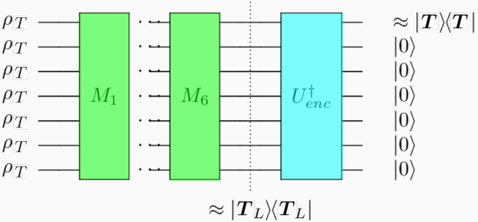

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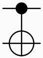
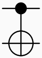
$\Big\langle$  , Symmetry group of  $\Big\rangle \neq \mathrm{UQC} = \Big\langle$  , Symmetry group of  , $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \Big\rangle$

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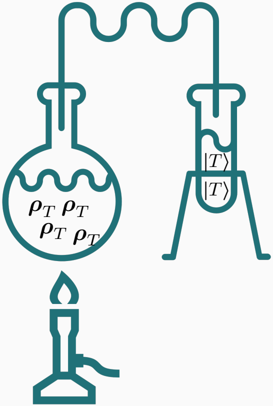
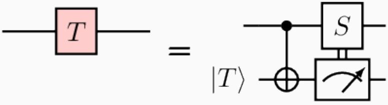
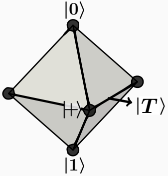
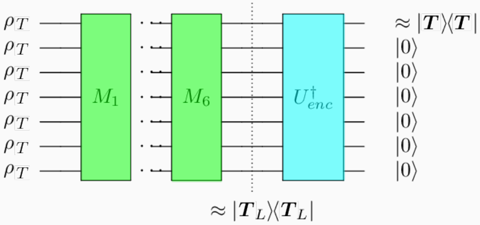




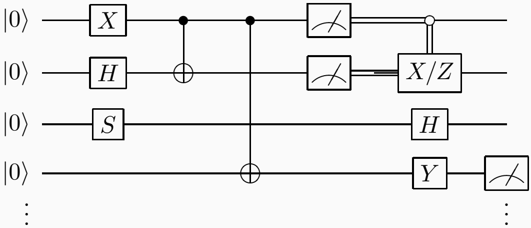




Universality vs. Simulability (Older Results for Qubits)

Stabilizer Circuits comprise

- Preparation of $|0\rangle$
- Applying Clifford (unitary) Gates
- Measuring in computational basis (and feed-forward of results)



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Stabilizer Circuits $\leq$ *Classical Circuits*

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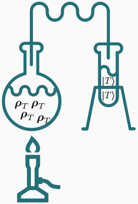

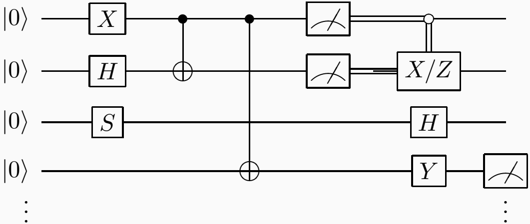

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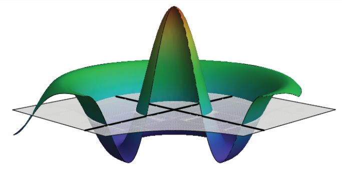
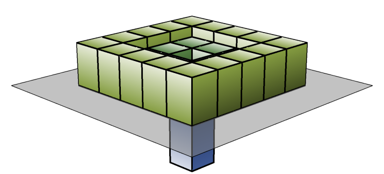
Known:
Allowing multiple uses of suitable states ρ boosts stabilizers to universal QC (via "**Magic State Distillation**")

Suitable ρ?
If ρ is a mixture of stabilizer states $\Rightarrow$ unsuitable

Background Machinery for Qudits: Discrete Wigner Function





Continuous W.F.

- Continuous phase space
- $[\hat{q}, \hat{p}] = i\hbar$
- Gaussian States

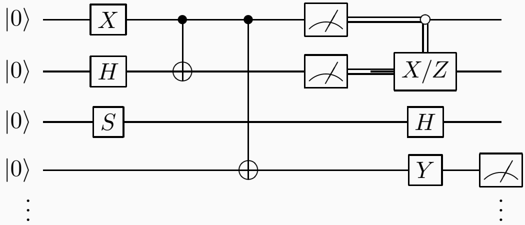
Discrete W.F.

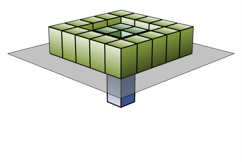
- Discrete phase space
- $|\langle \mathcal{B}_a^1 | \mathcal{B}_b^2 \rangle| = \frac{1}{\sqrt{d}}$
- Stabilizer States

D. Gross,
"Hudson's theorem for finite-dimensional quantum systems"
J. Math. Phys. **47**, number 12, 122107, (2006).

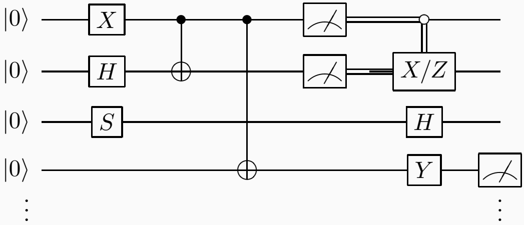
K. S. Gibbons, M. J. Hoffman, and W. K. Wootters,
"Discrete phase space based on finite fields"
Phys. Rev. A. **70**, 062101, (2004).

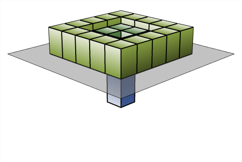
Universality vs. Simulability (More Recent Results for Qudits)





Theorem (Veitch et al., Mari & Eisert)
Positive Ancillas & Stabilizer Circuits $\leq$ *Classical Circuits*





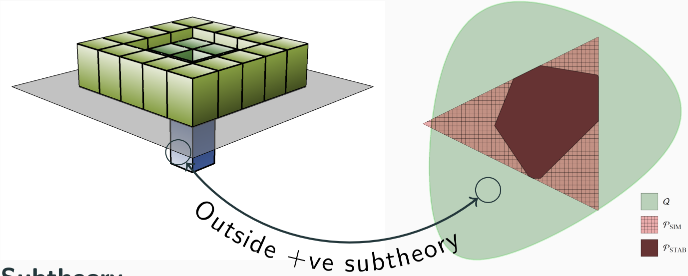
Theorem (Veitch et al., Mari & Eisert)
Positive Ancillas & Stabilizer Circuits $\leq$ *Classical Circuits*

How about adding in suitable ρ for MSD?
If ρ is positively represented in Gross DWF $\Rightarrow$ unsuitable

Wigner Negativity is a necessary resource for UQC
All the “magic ingredient” is in Magic ancillas

Sufficiency is somewhat open (MSD routines)

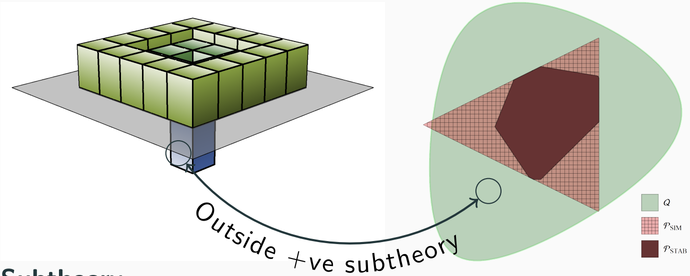
Contextuality enters the scene...



Subtheory
QM + restrictions on allowed states and operations

DWF Subtheory
Positively represented states and Ops inc Pauli Mmts

Contextuality enters the scene. . .



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This closed subtheory of QM contains

- Superposition
- Entanglement
- Post-mmt collapse
- Teleportation

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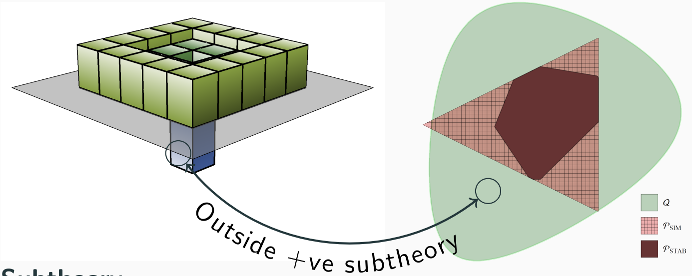

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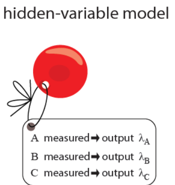
all of the above have classical interpretation in terms of LHV.
Contextuality is the only inherently QM feature missing.

Quantum contextuality (Kochen-Specker proofs)

- Spectral decomposition says:
  observable $A = \sum_a \lambda_a \Pi_a$ where Π_a is a projector onto λ_a eigenspace.
- Consider B, C such that $\begin{cases} [A,B] = 0 & \ldots\text{compatible} \\ [A,C] = 0 & \ldots\text{compatible} \\ [B,C] \neq 0 & \ldots\text{incompatible} \end{cases}$

* Commuting/compatible observables can be jointly/sequentially measured without mutual disturbance: $ABAAB \to \lambda_a\lambda_b\lambda_a\lambda_a\lambda_b$ etc
* Free to measure A then decide whether to measure B or C.

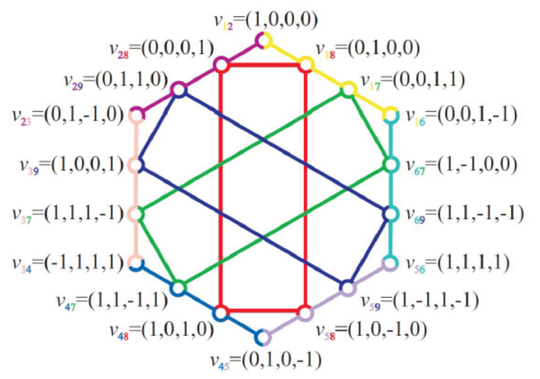
- Born rule says Prob$(a|\psi) = ||\Pi_a|\psi\rangle||^2$
- Natural(?) to have a mental model whereby quantum state $|\psi\rangle$ possesses a value $v(A) \in \{\lambda_a\}$ revealed by measurement of A (irrespective of context)



Quantum contextuality (Kochen-Specker proofs)

If we pursue this idea that $v(A)$ exists ahead of measurement then

- $v(A+B) = v(A) + v(B)$
- $v(AB) = v(A)v(B)$
- $v(\mathbb{I}) = 1$
- **NCHV:** Any measurement $M = \{\Pi_1, \Pi_2, \ldots, \Pi_k\}$ satisfying $\sum_i \Pi_i = \mathbb{I}$ exactly one of $\{\Pi_1, \Pi_2, \ldots, \Pi_k\}$ is true: $v(\Pi_i) = 1$ and $v(\Pi_{j\neq i}) = 0$



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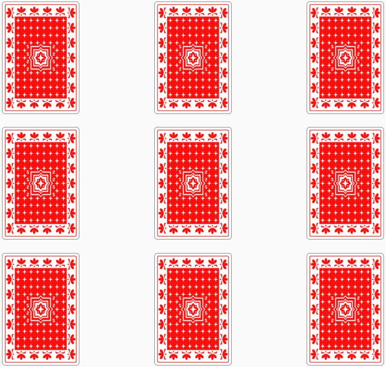

- Find an arrangement of cards such that both (i) and (ii) hold
  (i) the number of black suits in each row is odd
  (ii) the number of black suits in each column is even

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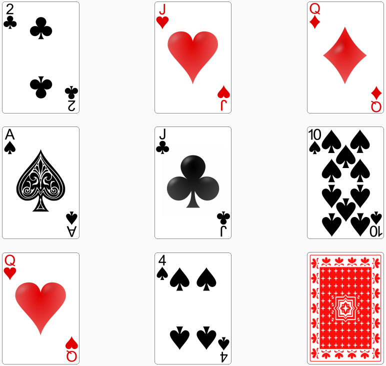

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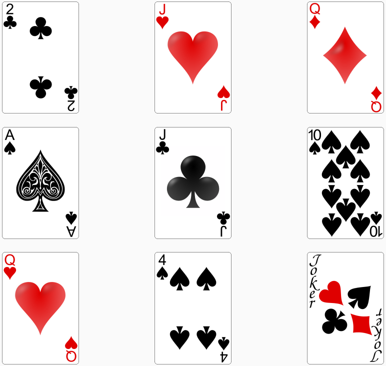

- Find an arrangement of cards such that both (i) and (ii) hold
    - (i) the number of black suits in each row is odd
    - (ii) the number of black suits in each column is even
- This is not satisfiable

Quantum contextuality (Kochen-Specker proofs)

$X \otimes Y$	$Y \otimes X$	$Z \otimes Z$
$Y \otimes Z$	$Z \otimes Y$	$X \otimes X$
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Quantum version of Card Arrangement

- Arrange 9 Pauli/stabilizer observables,
  all of which have outcomes ± 1 (think black/red)
- Ensure that triples in the same row or same column mutually commute
  ($\rightsquigarrow$ Ordering of measurements within triple is irrelevant)

- This arrangement of observables apparently satisfies both (i) and (ii)
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- Contextuality tells us how NOT to think about quantum mechanics
- Measurement is not merely revealing a pre-existing value
  Measurement is not like turning over a card!

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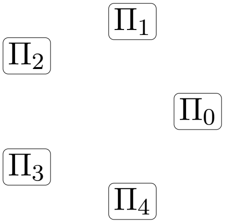
- Contextuality tells us how NOT to think about quantum mechanics
- Measurement is not merely revealing a pre-existing value
  Measurement is not like turning over a card!
- If we insist that measurement results are pre-existing, we must take into account the whole context of the experiment . . . i.e., $v_B(A) \neq v_C(A)$
- Can think of contextuality as a generalization of Bell non-locality (i.e., non-locality is a special case where commuting observables are spatially separated)

Quantum Contextuality (state-dependent)



- $\{\Pi_i\}$ corresponds to a set of yes/no propositions
- In QM represent Π_i by projectors with $\lambda(\Pi_i) \in \{1, 0\}$
- Commuting rank-1 $\Pi_i \leftrightarrow$ mutually exclusive propositions

A. Klyachko, M. Can, S. Binicioğlu, A. Shumovsky
"Simple Test for Hidden Variables in Spin-1 Systems"
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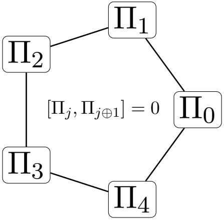

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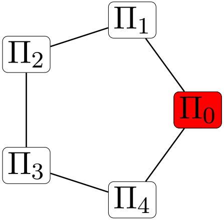

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$$\langle \Sigma_\Gamma \rangle_{\max}^{\mathrm{NCHV}} = \alpha(\Gamma),$$

Pentagon: $\alpha(\Gamma) = 2,$

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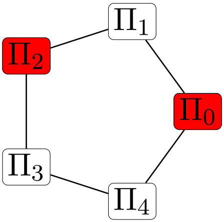

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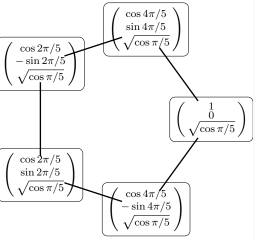

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Pentagon: $\alpha(\Gamma) = 2, \quad \vartheta(\Gamma) = \sqrt{5} \approx 2.24,$

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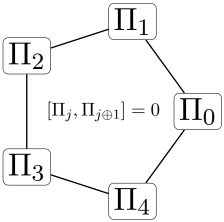

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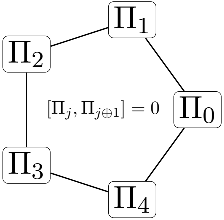

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CHSH graph:	$\alpha(\Gamma) \mapsto 2,$	$\vartheta(\Gamma) \mapsto 2\sqrt{2},$	$\alpha^*(\Gamma) \mapsto 4$

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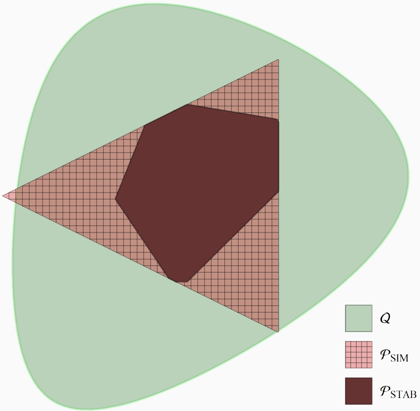
Quantum Contextuality (state-dependent)

Recap:
Given a set of observables, we can construct a non-contextuality inequality that identifies certain states as contextual (with respect to this set of observables)

Inequalities are of the form :

$$\rho \text{ is contextual if it violates } \mathrm{Tr}(\rho \Sigma_{\Gamma}) \leq \alpha(\Gamma)$$

Quantum Contextuality (state-dependent)



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Inequalities are of the form :

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Relevance to magic states:
For stabilizer measurements, previous work has established that $\rho \in \mathcal{P}_{\mathrm{SIM}}$ never exhibits contextuality

Our result:
All $\rho \notin \mathcal{P}_{\mathrm{SIM}}$ exhibit contextuality with respect to stabilizer measurements

Contextuality and the possibility of speed-up coincide exactly

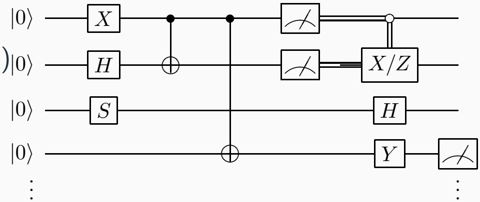
Contextuality for quantum computation

The largest subtheory of QM describable in terms of noncontextual hidden variables is the stabilizer subtheory.
Any combination of the following is allowed:

- Preparation of $|0\rangle$ (Pauli eigenstates)
- Applying Clifford (unitary) Gates
- Measuring in computational basis (and feed-forward of results)

We can simulate such a circuit efficiently using a classical computer!

However: Adding access to a supply of magic states (not Pauli eigenstates) enables the full quantum power



- All of the "Magic Ingredient" is in the magic states
- Characterize UQC-enabling ancillas $\rightsquigarrow$ fundamental insights? i.e., **what quantum phenomena drive UQC?**

Contextuality for quantum computation

- Conceptually satisfying answer (esp. for qu$d$its) is that **quantum contextuality is necessary for speed-up**
- The ancillas that are useless/simulable are exactly those that can never exhibit contextuality (wrt Pauli mmts)
- Contextuality is necessary & possibly sufficient property of ancillas

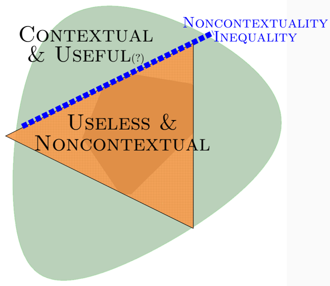
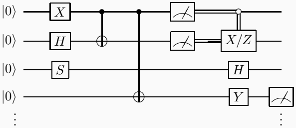


Figure 1: Slice through state space of Magic ancilla (wrt. stab. mmts)



If we add noncontextual ancillas to Clifford circuit we **never** see

1. Violation of NCI
2. Quantum Speed-up (simulable)

Contextuality for quantum computation

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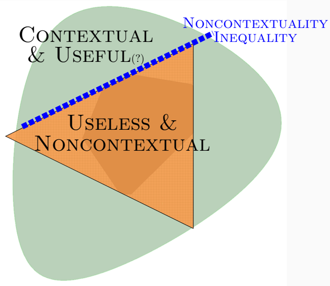

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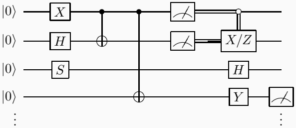


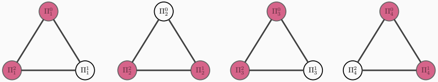
If we add contextual ancillas to Clifford circuit we **do** see

1. Violation of NCI (always)
2. Quantum Speed-up (sometimes,always?)

Proof Sketch (using qutrits)

1. The boundary of $\mathcal{P}_{\mathrm{SIM}}$ is given by p^2 facets $A^{\vec{r}} \in \mathcal{A}_{SIM}$ which decompose

$$-A^{\vec{r}} \propto$$



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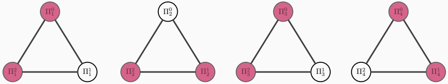

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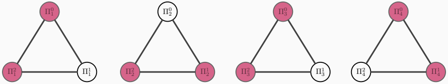

2. Such a single-qudit construction might have worked, but doesn't

$$\underbrace{\mathrm{Tr}\left(\Sigma^{\vec{r}}\rho\right) \leq \alpha(\Gamma^{\vec{r}})}_{\substack{\text{Noncontextuality}\\ \text{inequality}}} \quad \text{vs.} \quad \underbrace{\mathrm{Tr}\left[A^{\vec{r}}\rho\right] \geq 0}_{\substack{\text{Boundary}\\ \text{of } \mathcal{P}_{\mathrm{SIM}}}}$$

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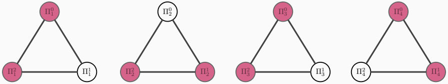

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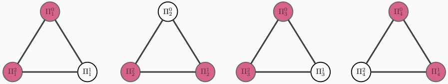

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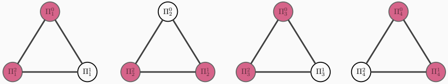
$$\underbrace{\mathrm{Tr}\left(\Sigma^{\vec{r}}\rho\right) \leq \alpha(\Gamma^{\vec{r}})}_{\substack{\text{Noncontextuality}\\ \text{inequality}}} \iff \underbrace{\mathrm{Tr}\left[A^{\vec{r}}\rho\right] \geq -1}_{\substack{\text{Parallel to}\\ \text{Boundary of } \mathcal{P}_{\mathrm{SIM}}}}$$

3. What does work, however, is a two-qudit construction:

Proof Sketch (using qutrits)

1. The boundary of $\mathcal{P}_{\text{SIM}}$ is given by p^2 facets $A^{\vec{r}} \in \mathcal{A}_{SIM}$ which decompose

$\Gamma^{\vec{r}} =$ 

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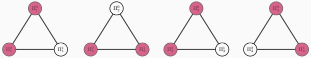
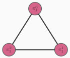
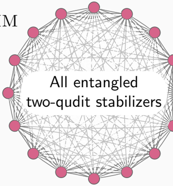
$\{\Pi\}^{\vec{r}} = \Big\{$  $\Big\} \otimes \Big\{$  $\Big\} \cup$ 

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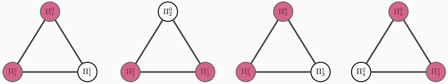

2. Such a single-qudit construction might have worked, but doesn't

$$\underbrace{\mathrm{Tr}\left(\Sigma^{\vec{r}}\rho\right) \leq \alpha(\Gamma^{\vec{r}})}_{\text{Noncontextuality inequality}} \iff \underbrace{\mathrm{Tr}\left[A^{\vec{r}}\rho\right] \geq -1}_{\text{Parallel to Boundary of } \mathcal{P}_{\mathrm{SIM}}}$$

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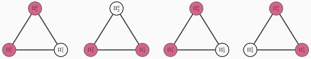
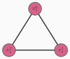


$$\Bigg\} \otimes \Bigg\{$$



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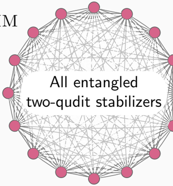

$$\underbrace{\mathrm{Tr}\left[\Sigma^{\vec{r}}\left(\rho \otimes \sigma\right)\right] \leq p^3}_{\text{Noncontextuality inequality}} \iff \underbrace{\mathrm{Tr}\left[A^{\vec{r}}\rho\right] \geq 0}_{\text{Boundary of } \mathcal{P}_{\mathrm{SIM}}}$$

Statement of Results

The independence number of the exclusivity graph is $\alpha(\Gamma^{\vec{r}}) = p^3$ for all $A^{\vec{r}} \in \mathcal{A}_{\text{SIM}}$ and all $p \geq 2$ so that, relative to our construction, exactly the states $\rho \notin \mathcal{P}_{\text{SIM}}$ are those that exhibit contextuality.

For qudits of odd prime dimension there does not exist any construction using stabilizer measurements that characterizes any $\rho \in \mathcal{P}_{\text{SIM}}$ as contextual, so that the conditions for contextuality and the possibility of quantum speed-up via magic state distillation coincide exactly.

Furthermore

$$\langle \Sigma^{\vec{r}} \rangle^{2-\text{QUDIT}}_{\max} = \vartheta(\Gamma^{\vec{r}}) = \alpha^*(\Gamma^{\vec{r}}) = p^3 + 1, \quad (p > 2)$$

which means maximally contextual states saturate the bound on contextuality associated with post-quantum generalized probabilistic theories.

What's the deal with qubits? (State-independence)

$\mathcal{P}_{\mathrm{STAB}} = \mathcal{P}_{\mathrm{SIM}} =$ 

$X \otimes Y$	$Y \otimes X$	$Z \otimes Z$
$Y \otimes Z$	$Z \otimes Y$	$X \otimes X$
$Z \otimes X$	$X \otimes Z$	$Y \otimes Y$

Generalized Pauli: $X|j\rangle = |j+1\rangle, \quad Z|j\rangle = \omega^j |j\rangle \qquad (\omega = e^{\frac{2\pi i}{p}})$

Weyl-Heisenberg Group: $\begin{cases} \{i^\lambda X^x Z^z | x, z \in \mathbb{Z}_2, \lambda \in \mathbb{Z}_4\} & p = 2 \\ \{\omega^\lambda X^x Z^z | x, z, \lambda \in \mathbb{Z}_p\} & p > 2 \end{cases}$

"All primes are odd except 2, which is the oddest of all"

Improvements

Nicer/More-general proof of "$-$"DWF $\iff$ contextuality
Equivalence between contextuality and negativity of the Wigner function for qudits

N. Delfosse, C. Okay, J. Bermejo-Vega, DE. Browne, R. Raussendorf

arXiv preprint arXiv:1610.07093

Generalizes DWF$\leftrightarrow$Simulation connection for qubits and qudits
Phase space simulation method for quantum computation with magic states on qubits

R. Raussendorf, J. Bermejo-Vega, E. Tyhurst, C. Okay, M. Zurel

arXiv preprint arXiv:1905.05374