Simulations between contextual resources




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Contextuality as a resource in quantum computation
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Overview of the talk

- Simulations
- Main results:
  - Categorical characterizations of contextuality
  - Equivalence with free operations
  - No-cloning
- Further topics:
  - More expressive simulations?
  - No-catalysis?
  - Further questions

Structure of resources

Two perspectives:

1. **Resource theories** (coming from Physics):
   Algebraic theory of '**free operations**' that do not introduce more of the resource in question.

   Resource B can be obtained from resource A if it can be built from A using free operations.

   '*Contextual fraction as a measure of contextuality*', Abramsky, B, Mansfield, PRL, 2017.
   '*Noncontextual wirings*', Amaral, Cabello, Terra Cunha, Aolita, PRL, 2018.

2. **Simulations or reducibility** (coming from Computer Science):
   Notion of **simulation** between behaviours of systems.

   One resource can be reduced to another if it can be simulated by it.
   Cf. (in)computability, degrees of unsolvability, complexity classes.

- Intuitively, some correlations can be used to simulate others. Here's some examples:
  - Any two-outcome bipartite box can be simulated with PR boxes.

    '*Popescu-Rohrlich correlations as a unit of nonlocality*'
    Barrett, Pironio, PRL 2005.
  - An explicit two-outcome three-partite box that cannot be simulated with PR boxes (same source).
  - No finite set of bipartite boxes can simulate all of them

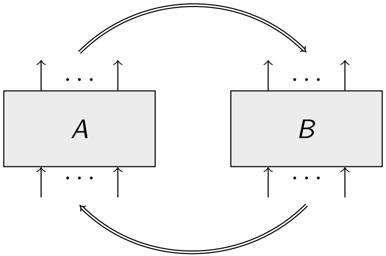
    '*No nonlocal box is universal*'
    Dupuis et al, Journal of Mathematical Physics 2007.

Morphisms as simulations

- Think of a measurement scenario as a concrete experimental setup, where for each measurement there is a grad student responsible for it.
- The grad student responsible for measuring $x \in X$, should have instructions which measurement $\pi(x) \in Y$ to use instead.
- Given a result for those measurements, should be able to determine the outcome to output.
- The outcome statistics should be identical to those of e.

We first define basic simulations. Dependencies on multiple measurements and stochastic processing added as a comonadic effect.

Basic simulations



To simulate B using A:

- map inputs of B (measurements) to inputs of A
- run A
- map outputs of A (measurement outcomes) to outputs of B

Basic simulations formally

A morphism of scenarios $(\pi, h) : \langle X, \Sigma, O\rangle \to \langle Y, \Theta, P\rangle$ is given by:

- A simplicial map $\pi : \Theta \to \Sigma$.
- For each $y \in Y$, a map $h_y : O_{\pi(y)} \to P_y$.

Simpliciality of π means that contexts in Θ are mapped to contexts in Σ.

Basic simulations formally

A morphism of scenarios induces a natural action on empirical models:

Given $e \colon \langle X, \Sigma, O \rangle$, then $(\pi, h)^* e \colon \langle Y, \Theta, P \rangle$ is given by, for $\tau \in \Delta$:

$$((\pi, h)^* e)_\tau = \mathrm{Prob}(\gamma)(e_{\pi(\tau)})$$

the push-forward of the probability measure $e_{\pi(\tau)}$ along the map

$$\gamma : \prod_{x \in \pi(\tau)} O_x \longrightarrow \prod_{y \in \tau} P_y$$

given by $\gamma(s)_y = h_y \left(s_{\pi(y)} \right)$.

This gives a category **Emp**, with:

- objects are empirical models $e \colon \langle X, \Sigma, O \rangle$,
- morphisms $e \to d$ are simulations $(\pi, h) : \langle X, \Sigma, O \rangle \to \langle Y, \Theta, P \rangle$ such that $(\pi, h)^* e = d$.

Basic simulations

This is a simple notion of simulation, but already covers several things:

- It allows for relabelling of measurements and outcomes (**isomorphisms**).
- Since the map on measurements needn't be surjective, it allows for simulating B using only part of A (**restrictions**).
- Since the map on outcomes needn't be injective, it allows for **coarse-graining** of outcomes.

Note that mappings of inputs go backward, of outputs forward:

- Akin to the Hom functor being **contravariant** in its first argument, **covariant** in its second.
- Logically, to reduce one implication to another, one must **weaken** the antecedent and **strengthen** the consequent.

Beyond deterministic maps

Basic simulations are useful, but limited.

To allow **adaptive** use of the resource, we introduce **measurement protocols**.

These protocols proceed iteratively by first performing a measurement over the given scenario, and then conditioning their further measurements on the observed outcome.

Note that different paths can lead into different, **incompatible** contexts.

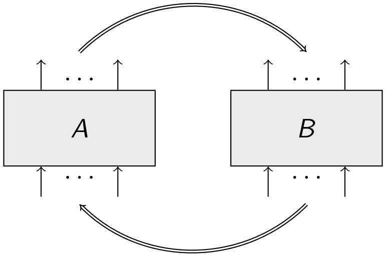
Thus they incorporate adaptive classical processing, of the kind used e.g. in Measurement-Based Quantum Computing.

Previously considered in:

'*A combinatorial approach to nonlocality and contextuality*'
Acin, Fritz, Leverrier, Sainz, Communications in Mathematical Physics, 2015.

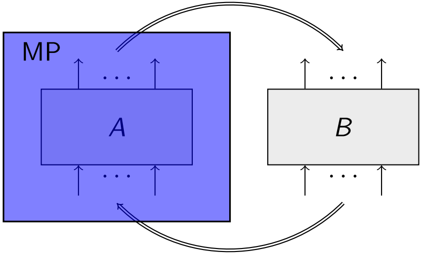
Formally, we construct a **comonad** MP on the category of empirical models, where $\mathsf{MP}(e \colon \langle X, \Sigma, O \rangle)$ is the model obtained by taking all measurement protocols over the given scenario.

Basic simulations

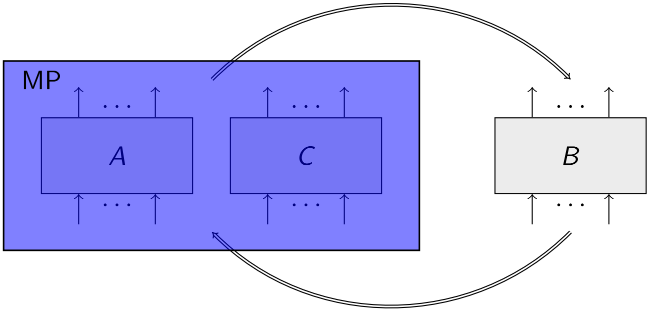


For more expressive simulations, increase the power of A **classically**

CoKleisli simulations



Cokleisli simulations with shared randomness



Requirement: C is noncontextual.

The MP construction

Given a scenario $\mathbf{X} = \langle X, \Sigma, O \rangle$ we build a new scenario $\mathrm{MP}(\mathbf{X})$, where:

- measurements are the measurement protocols on $\mathbf{X}$

$$\mathrm{MP}(\langle X, \Sigma, O \rangle) ::= \emptyset \mid (x, f) \quad \text{where } x \in X \text{ and } f \colon O_x \to \mathrm{MP}(\langle X \setminus \{x\}, \mathrm{lk}_x \Sigma, O \rangle).$$

- outcomes are the joint outcomes observed during a run of the protocol

- measurement protocols are compatible if they can be combined consistently
  - A run is a sequence $\bar{x} = (x_i, o_i)_{i=1}^{l}$ with $x_i \in X$, $o_i \in O_{x_i}$
  - $\sigma_{\bar{x}} = \{x_1, x_2, \ldots, x_l\} \in \Sigma$.
  - Two runs (of different protocols) are consistent if they agree on common measurements
  - Protocols $\{Q_1, \ldots, Q_n\}$ are compatible if for any choice of pairwise consistent runs $\bar{x}_i$ from Q_i, we have $\bigcup_i \sigma_{\bar{x}_i} \in \Sigma$

The MP construction

Empirical models in $\mathbf{X}$ are then naturally lifted to this scenario $\mathsf{MP}(\mathbf{X})$.

Proposition

MP *defines a comonoidal comonad on the category* **Emp** *of empirical models.*

Roughly: comultiplication $\mathsf{MP}(\mathbf{X}) \to \mathsf{MP}^2(\mathbf{X})$ by "flattening", unit $\mathsf{MP}(\mathbf{X}) \to \mathbf{X}$, and $\mathsf{MP}(\mathbf{X} \otimes \mathbf{Y}) \to \mathsf{MP}(\mathbf{X}) \otimes \mathsf{MP}(\mathbf{Y})$

General simulations

Given empirical models e and d, a **simulation** of e by d is a map

$$d \otimes c \to e$$

in $\mathbf{Emp}_{\mathrm{MP}}$, the coKleisli category of MP, i.e. a map

$$\mathrm{MP}(d \otimes c) \to e$$

in **Emp**, for some noncontextual model c.

The use of the noncontextual model c is to allow for classical randomness in the simulation.

We denote the existence of a simulation of e by d as $d \rightsquigarrow e$, read "d simulates e".

Results

In the resulting category/resource theory,

- Contextual fraction is a monotone
- an empirical model is contextual iff it cannot be simulated from the trivial model
- an empirical model is logically noncontextual iff it cannot be simulated possibilistically from the trivial model
- an empirical model e is strongly contextual iff no model e' contained in the support of e can be simulated from the trivial model

Moreover, no-cloning holds: a simulation $e \to e \otimes e$ exists iff e is noncontextual.

The viewpoints agree

Theorem

Let $e : \mathbf{X}$ and $d : \mathbf{Y}$ be empirical models. Then $d \rightsquigarrow e$ if and only if there is a typed term $a : \mathbf{Y} \vdash t : \mathbf{X}$ such that $t[d/a] \simeq e$.

Proof.

(Sketch) If $d \rightsquigarrow e$, then e can be obtained from $\mathrm{MP}(d \otimes x)$ by a combination of a coarse-graining and a measurement translation. There is a term representing x and MP can be built by repeated controlled measurements.

For the other direction, build a simulation $d \to t[d/a]$ inductively on the structure of t. $\square$

Properties of NCF

- $\mathrm{NCF}(e) = 1$ iff e is noncontextual
- $\mathrm{NCF}(\mathrm{MP}(e)) = \mathrm{NCF}(e)$
- $\mathrm{NCF}(e \otimes d) = \mathrm{NCF}(e)\mathrm{NCF}(d)$
- $d \rightsquigarrow e$ implies $\mathrm{NCF}(d) \leq \mathrm{NCF}(e)$

No-cloning

Theorem (No-cloning)

$e \rightsquigarrow e \otimes e$ if and only if e is noncontextual.

Proof.

The direction "$\Leftarrow$" is easy as there is an obvious deterministic simulation $\mathrm{MP}(e \otimes e) \to (e \otimes e)$
For the other direction, we rule out $0 < \mathrm{NCF}(e) < 1$ and $\mathrm{NCF}(e) = 0$. So assume $0 < \mathrm{NCF}(e) < 1$ and $e \rightsquigarrow e \otimes e$. Then

$$\mathrm{NCF}(e) \leq \mathrm{NCF}(e \otimes e) = \mathrm{NCF}(e)^2 < \mathrm{NCF}(e),$$

a contradiction. $\square$

Cont.

Remains to prove $\mathrm{NCF}(e) > 0$. Now $e \rightsquigarrow e \otimes e$ implies $e \rightsquigarrow e^{\otimes n}$ for any n. Moreover, we may assume that the noncontextual resource gets measured first. But then it merely randomizes which deterministic simulation $\mathrm{MP}(\mathbf{X}^{\otimes} n) \to \mathrm{MP}(X)$ to use, and there is only finitely many of these. Thus, for large enough n and some possible value of the random data, too many of the components end up using the same deterministic simulation. As the underlying map $\Sigma^{*n} \to \Sigma$ is simplicial, they must thus use a a compatible (and hence noncontextual) part of X. This gives $\mathrm{NCF}(e) > 0$. $\square$

The talk so far is more or less based on

'*A comonadic view of simulation and quantum resources*'
Abramsky, Barbosa, Karvonen, Mansfield, LiCS 2019.

Let's do something new!

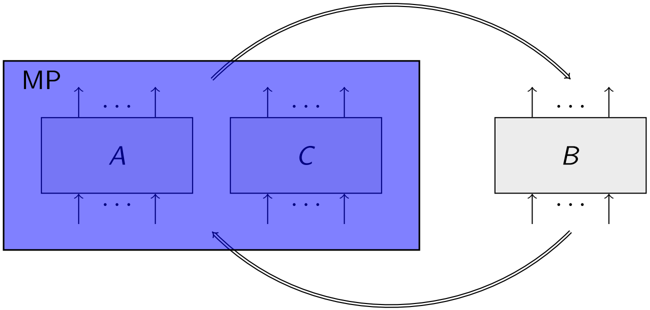
More expressive simulations?

When asking if d simulates e, several further relaxations are possible:

- does (π, h) take d to (a subset of) e on the level of supports?
- does d simulate e approximately?
- does $d^{\otimes n}$ simulate e for some n?
- is there a sequence of approximate simulations $d^{\otimes n} \to e$ converging to e?
- what if we allow the free resource c to come from some larger class of models?

We will discuss the last one.

More expressive simulations?



Requirement: ~~C is noncontextual.~~ C is in some fixed class $\mathcal{F}$.

More expressive simulations?

Let $\mathcal{F}$ be a class of models closed under $\otimes$ that contains noncontextual models. Given empirical models d and e, an $\mathcal{F}$-simulation $d \to e$ consists of a deterministic simulation $\mathrm{MP}(d \otimes f) \to e$ for some $f \in \mathcal{F}$. If d $\mathcal{F}$-simulates e, we write $d \rightsquigarrow_{\mathcal{F}} e$

One could ask if $\rightsquigarrow_F$ allows for catalysis: a situation where e doesn't $\mathcal{F}$-simulate f yet $d \otimes e \rightsquigarrow_F d \otimes f$.

Theorem (No-catalysis)

If $d \otimes e \rightsquigarrow_{\mathcal{F}} d \otimes f$, then $e \rightsquigarrow_F f$

NB: The case when $d = f$ and e is non-contextual is no-cloning.

Contrast with entanglement, where catalysis is possible:

‘*Entanglement-assisted local manipulation of pure quantum states*’
Jonathan, Plenio, PRL 1999.

Further questions

- Study the preorder induced by $d \rightsquigarrow e$.
- Study various relaxed simulations
- Comparison with other approaches to contextuality.
- Multipartite non-locality
- Graded structure on the comonad?
- MBQC?
- Generating all empirical models?
- Bell inequalities?

Summary

- Intraconversions of contextual resources formalized in terms of
  - free operations
  - simulations
- These viewpoints agree and capture known examples
- A no-cloning result
- No-catalysis
- Several avenues for further work