Continuous-variable non-locality and contextuality

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- CV quantum systems promising candidates for implementing quantum informational tasks.

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- *"Now, it may happen that the two wave functions, ψ_k and ϕ_r, are eigen-functions of two non-commuting operators corresponding to some physical quantities P and Q, respectively. [...] Let us suppose that the two systems are two particles, and that*

$$\psi(x_1, x_2) = \int_{-\infty}^{\infty} e^{\frac{2\pi i}{h}(x_1 - x_2 + x_0)p} \mathrm{d}p$$

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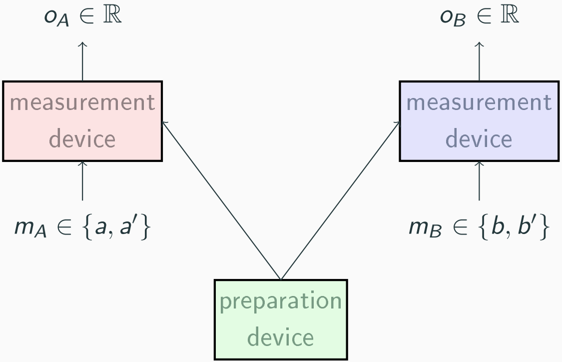
[...] Since we have here the case of a continuous spectrum [...]". [Einstein35]

Framework

Framework

A typical bipartite experiment

Operational depiction



Framework

Measurement scenario

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A measurement scenario is a triple $\langle X, \mathcal{M}, \mathbf{O} \rangle$ where:

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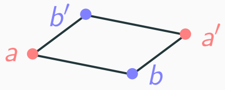

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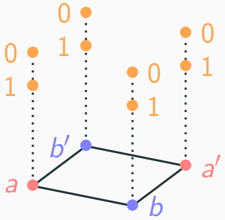

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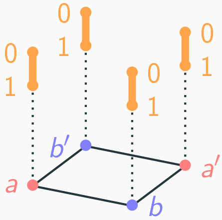
$$O = \mathbb{R} \text{ or } O = [0, 1]$$



Framework

Empirical models

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Definition - empirical model

An *empirical model* e on a measurement scenario $\langle X, \mathcal{M}, \mathbf{O} \rangle$ is a family $e = \{e_C\}_{C \in \mathcal{M}}$ where e_C is a *probability measure* on $\mathbf{O}_C$ which satisfies the compatibility condition:

$$e_C|_{C \cap C'} = e_{C'}|_{C \cap C'}$$

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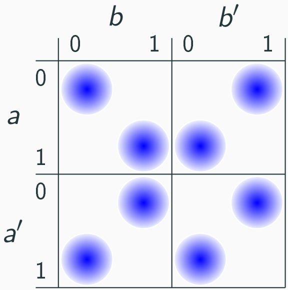
		b = 0	b = 1	b' = 0	b' = 1
a	0	$1/2$	0	0	$1/2$
a	1	0	$1/2$	$1/2$	0
a'	0	0	$1/2$	0	$1/2$
a'	1	$1/2$	0	$1/2$	0



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$$e_C|_{C \cap C'} = e_{C'}|_{C \cap C'}$$

Definition - extendability

An empirical model e is said to be **extendable** (or **noncontextual**) if there exists a probability measure μ on $\mathbf{O}_X$ such that $\forall C \in \mathcal{M}.\ \mu|_C = e_C$.

Framework

CV hidden variable models

Hidden variable models

Definition - hidden variable model

A *hidden variable model* on a measurement scenario $\langle X, \mathcal{M}, \mathbf{O} \rangle$ consists of:

- A measurable space $\boldsymbol{\Lambda} = \langle \Lambda, \mathcal{F}_\Lambda \rangle$ of *hidden variables*.

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- A measurable space $\mathbf{\Lambda} = \langle \Lambda, \mathcal{F}_\Lambda \rangle$ of *hidden variables*.
- A probability measure p on $\mathbf{\Lambda}$.
- For each maximal context $C \in \mathcal{M}$, a probability kernel $k_C : \mathbf{\Lambda} \longrightarrow \mathbf{O}_C$, satisfying the following compatibility condition:

$$\forall \lambda \in \Lambda. \quad k_C(\lambda, -)|_{C \cap C'} = k_{C'}(\lambda, -)|_{C \cap C'}$$

Hidden variable models

Let $\langle \mathbf{\Lambda}, p, k \rangle$ a hidden variable on $\langle X, \mathcal{M}, \mathsf{O} \rangle$. Then empirical model:

$$e_C(B) = \int_{\Lambda} k_C(-, B) \mathrm{d}p = \int_{\lambda \in \Lambda} k_C(\lambda, B) \mathrm{d}p(\lambda)$$

Hidden variable models

Definition - determinism

A hidden variable model $\langle \mathbf{\Lambda}, p, k \rangle$ is said to be **deterministic** if $k_C(\lambda, -) \colon \mathcal{F}_C \longrightarrow [0, 1]$ is a Dirac measure for every $\lambda \in \Lambda$ and for every maximal context $C \in \mathcal{M}$; in other words, there is an assignment $\mathbf{o} \in O_C$ such that $k_C(\lambda, -) = \delta_{\mathbf{o}}$.

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Definition - factorisability

A hidden-variable model $\langle \mathbf{\Lambda}, p, k \rangle$ is said to be **factorisable** if $k_C(\lambda, -) \colon \mathcal{F}_C \longrightarrow [0,1]$ factorises as a product measure for every $\lambda \in \Lambda$ and for every maximal context $C \in \mathcal{M}$. That is, for any family of measurable sets $(B_x \in \mathcal{F}_x)_{x \in C}$,

$$k_C(\lambda, \prod_{x \in C} B_x) = \prod_{x \in C} k_C|_{\{x\}}(\lambda, B_x)$$

where $k_C|_{\{x\}}(\lambda, -)$ is the marginal of the probability measure $k_C(\lambda, -)$ on $\mathbf{O}_C = \prod_{x \in C} \mathbf{O}_x$ to the space $\mathbf{O}_{\{x\}} = \mathbf{O}_x$.

A Fine-Abramsky-Brandenburger theorem in CV

A FAB theorem in CV [Abramsky11]

Theorem

Let e be an empirical model on a measurement scenario $\langle X, \mathcal{M}, \mathbf{O} \rangle$. The following are equivalent:

(1) *e is extendable;*
$\exists \mu$ on $\mathbf{O}_X$ s.t. $\forall C \in \mathcal{M}.\ \mu|_C = e_C$

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Let e be an empirical model on a measurement scenario $\langle X, \mathcal{M}, \mathbf{O} \rangle$*. The following are equivalent:*

(1) *e is extendable;*

(2) *e admits a realisation by a deterministic hidden-variable model;*
$\exists \mathbf{o} \in O_C$ *an assignment s.t.* $k_C(\lambda, -) = \delta_{\mathbf{o}}$

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Theorem

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(1) *e is extendable;*

(2) *e admits a realisation by a deterministic hidden-variable model;*

(3) *e admits a realisation by a factorisable hidden-variable model.* $k_C(\lambda, \prod_{x \in C} B_x) = \prod_{x \in C} k_C|_{\{x\}}(\lambda, B_x)$ *for a family of measurable sets* $(B_x \in \mathcal{F}_x)_{x \in C}$.

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A. Nonlocality special case of contextuality.
Captured by notion of extendability.

Quantifying contextuality

Noncontextual fraction

What fraction of the empirical model e admits a deterministic hidden-variable model? [Abramsky11] [Abramsky17]

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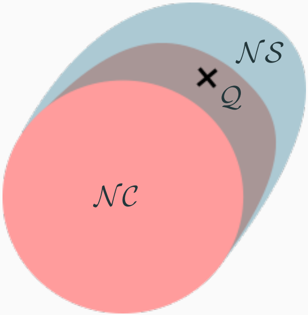

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$$e = \lambda e_{\mathsf{NC}} + (1-\lambda)e'$$

Definition - noncontextual fraction

$$\mathsf{NCF}(e) = \sup\{\mu(O_X) \mid \mu \in \mathbb{M}(\mathbf{O}_X), \forall C \in \mathcal{M}.\ \mu|_C \leq e_C\} \in [0,1]$$

Quantifying contextuality

Linear programming problems

Linear programming

Primal

$$
(\mathrm{P})\begin{cases}
\text{Find} & \mu \in \mathbb{M}_{\pm}(\mathbf{O}_X) \\
\text{maximising} & \mu(O_X) \\
\text{subject to} & \forall C \in \mathcal{M}.\ \mu|_C \leq e_C \\
\text{and} & \mu \geq 0.
\end{cases}
$$

Dual

$$
(\mathrm{D})\begin{cases}
\text{Find} & (f_C)_{C\in\mathcal{M}} \in \prod_{C\in\mathcal{M}} C_0(O_C,\mathbb{R}) \\
\text{minimising} & \sum_{C\in\mathcal{M}} \int_{O_C} f_C \ \mathrm{d}\, e_C \\
\text{subject to} & \sum_{C\in\mathcal{M}} f_C \circ \rho^X_C \geq \mathbf{1} \ \text{ on } O_X \\
\text{and} & \forall C \in \mathcal{M}.\ f_C \geq 0 \ \text{ on } O_C.
\end{cases}
$$

Quantifying contextuality

Bell inequalities

New dual program

$$
(B)\begin{cases}
\text{Find} & (\beta_C)_{C\in\mathcal{M}} \in \prod_{C\in\mathcal{M}} C_0(O_C,\mathbb{R}) \\
\text{maximising} & \sum_{C\in\mathcal{M}} \int_{O_C} \beta_C \;\mathrm{d}\, e_C \\
\text{subject to} & \sum_{C\in\mathcal{M}} \beta_C \circ \rho^X_C \leq 0 \;\text{ on } O_X \\
\text{and} & \forall C \in \mathcal{M}.\; \beta_C \leq |\mathcal{M}|^{-1}\mathbf{1} \;\text{ on } O_C.
\end{cases}
$$

Generalised Bell inequalities

New dual program

$$
(B) \begin{cases}
\text{Find} & (\beta_C)_{C\in\mathcal{M}} \in \prod_{C\in\mathcal{M}} C_0(O_C,\mathbb{R}) \\
\text{maximising} & \sum_{C\in\mathcal{M}} \int_{O_C} \beta_C \ \mathrm{d}\, e_C \\
\text{subject to} & \sum_{C\in\mathcal{M}} \beta_C \circ \rho^X_C \leq 0 \ \text{ on } O_X \\
\text{and} & \forall C \in \mathcal{M}.\ \beta_C \leq |\mathcal{M}|^{-1}\mathbf{1} \ \text{ on } O_C.
\end{cases}
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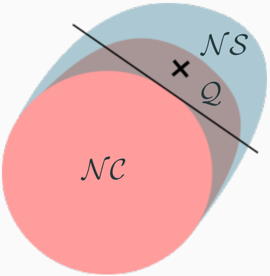

Quantifying contextuality

A hierarchy of Semi-Definite Programming problems

A hierarchy of SDPs [Lasserre09] [Henrion14]

Idea $\rightarrow$ relaxation of the problem:

- Measure $\rightarrow$ moments of the measure and truncated sequence.

A hierarchy of SDPs [Lasserre09] [Henrion14]

Idea $\rightarrow$ relaxation of the problem:

- Measure $\rightarrow$ moments of the measure and truncated sequence.
- Continuous functions $\rightarrow$ SOS polynomials and fixed degree.

A hierarchy of SDPs

Primal

$$
(\mathrm{P}) \begin{cases} \sup_{\mu \in \mathbb{M}_{\pm}(\mathbf{O}_X)} \mu(O_X) & \longrightarrow y_0 \\ \text{s.t. } \forall C \in \mathcal{M}.\ \mu|_C \leq e_C & \longrightarrow M_k(\mathbf{y}^{e,C} - \mathbf{y}_{|C}) \succeq 0 \\ \mu \succeq 0 & \longrightarrow M_k(\mathbf{y}) \succeq 0 \end{cases}
$$

Dual

$$
(\mathrm{D}) \begin{cases} \inf_{(f_c) \in \prod C_0(O_C, \mathbb{R})} \sum_{C \in \mathcal{M}} \int_{O_C} f_C \, \mathrm{d}\, e_C \longrightarrow \inf_{\substack{(f_c) \subset \Sigma^2 \mathbb{R}[x]_k \\ (\sigma_j) \subset \Sigma^2 \mathbb{R}[x]_{k - r_j}}} \sum_{C \in \mathcal{M}} \int_{O_C} f_c \, \mathrm{d}\, e_C \\ \text{s.t. } \sum_{C \in \mathcal{M}} f_C \circ \rho^X_C \geq \mathbf{1} \text{ on } O_X \longrightarrow \sum_{C \in \mathcal{M}} f_C - \mathbf{1} = \sigma_0 + \sum_{j=1}^{m} \sigma_j P_j \\ \forall C \in \mathcal{M}.\ f_C \geq 0 \text{ on } O_C \end{cases}
$$

A hierarchy of SDPs

Primal

$$(\mathrm{SP}_k)\begin{cases} \sup\limits_{\mathbf{y}\in\mathbb{R}^{s(k)}} \; y_0 \; (=\mu(O_X)) \\ \text{s.t. } \forall C \in \mathcal{M}.\; M_k(\mathbf{y}^{e,C} - \mathbf{y}_{|C}) \succeq 0 \\ \quad M_k(\mathbf{y}) \succeq 0 \\ \quad \forall j \in \{1,\dots,m\}.\; M_{k-r_j}(P_j\mathbf{y}) \succeq 0 \end{cases}$$

Dual

$$(\mathrm{SD}_k)\begin{cases} \inf\limits_{\substack{(f_c)\subset\Sigma^2\mathbb{R}[x]_k \\ (\sigma_j)\subset\Sigma^2\mathbb{R}[x]_{k-r_j}}} \; \sum\limits_{C\in\mathcal{M}} \int_{O_C} f_c \,\mathrm{d}\, e_C \\ \text{s.t. } \sum\limits_{C\in\mathcal{M}} f_C - \mathbf{1} = \sigma_0 + \sum\limits_{j=1}^{m} \sigma_j P_j \end{cases}$$

A hierarchy of SDPs

Theorem

The optimal values of the hierarchy of semidefinite programs (SD_k) *provide monotonically decreasing upper bounds on the optimal solution of the linear program* (D) *that converge to its value* $NCF(e)$*. That is,*

$$\inf(\mathrm{SD}_k) \downarrow \inf(\mathrm{D}) = \mathrm{NCF}(e) \quad \text{as } k \to \infty$$

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Also holds for the primal (SP_k):

$$\mathrm{NCF}(e) = \sup(\mathrm{P}) \underset{\substack{\text{strong}\\ \text{duality}}}{=} \inf(\mathrm{D}) \leq \inf(\mathrm{SD}_k)$$

$$\sup(\mathrm{P}) \leq \sup(\mathrm{SP}_k) \leq \inf(\mathrm{SD}_k)$$

Outlook:

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- Relate CV to advantages for quantum computation.

Thank you!

References i

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Elements of measure theory

Some elements of measure theory

- *Measurable space*: pair $\mathbf{X} = \langle X, \mathcal{F}_X \rangle$ e.g. $\langle X, \mathcal{P}(X) \rangle$, $\langle \mathbb{R}, \mathcal{B}(\mathbb{R}) \rangle$

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- Measurable space: pair $\mathbf{X} = \langle X, \mathcal{F}_X \rangle$ e.g. $\langle X, \mathcal{P}(X) \rangle$, $\langle \mathbb{R}, \mathcal{B}(\mathbb{R}) \rangle$
- A *measurable function* f between measurable spaces $\mathbf{X} = \langle X, \mathcal{F}_X \rangle$ and $\mathbf{Y} = \langle Y, \mathcal{F}_Y \rangle$ is a function $f : X \to Y$ s.t. for any $E \in \mathcal{F}_Y$, $f^{-1}(E) \in \mathcal{F}_X$.

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- A *measure* on a measurable space $\mathbf{X} = \langle X, \mathcal{F}_X \rangle$ is a function $\mu : \mathcal{F}_X \to \overline{\mathbb{R}}$. Set of measures: $\mathbb{M}(\mathbf{X})$ (signed $\mathbb{M}_{\pm}(\mathbf{X})$) - probability measures: $\mathbb{P}(\mathbf{X})$. Allow to integrate well-behaved measurable functions: $\int_{\mathbf{X}} f \mathrm{d}\mu$.

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- *Push-forward:* a measurable function $f : \mathbf{X} \to \mathbf{Y}$ carries any measure μ on $\mathbf{X}$ to a measure $f_* \mu$ on $\mathbf{Y}$ s.t. $f_* \mu(E) = \mu(f^{-1}(E))$ for E measurable in $\mathbf{Y}$. Important use: *marginal measure*.

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- Measurable space: pair $\mathbf{X} = \langle X, \mathcal{F}_X \rangle$ e.g. $\langle X, \mathcal{P}(X) \rangle$, $\langle \mathbb{R}, \mathcal{B}(\mathbb{R}) \rangle$
- A measurable function f between measurable spaces $\mathbf{X} = \langle X, \mathcal{F}_X \rangle$ and $\mathbf{Y} = \langle Y, \mathcal{F}_Y \rangle$ is a function $f : X \to Y$ s.t. for any $E \in \mathcal{F}_Y$, $f^{-1}(E) \in \mathcal{F}_X$.
- A measure on a measurable space $\mathbf{X} = \langle X, \mathcal{F}_X \rangle$ is a function $\mu : \mathcal{F}_X \to \overline{\mathbb{R}}$. Set of measures: $\mathbb{M}(\mathbf{X})$ (signed $\mathbb{M}_{\pm}(\mathbf{X})$) - probability measures: $\mathbb{P}(\mathbf{X})$. Allow to integrate well-behaved measurable functions: $\int_{\mathbf{X}} f \mathrm{d}\mu$.
- *Push-forward:* a measurable function $f : \mathbf{X} \to \mathbf{Y}$ carries any measure μ on $\mathbf{X}$ to a measure $f_* \mu$ on $\mathbf{Y}$ s.t. $f_* \mu(E) = \mu(f^{-1}(E))$ for E measurable in $\mathbf{Y}$. Important use: *marginal measure*. $\pi_i : \mathbf{X}_1 \times \mathbf{X}_2 \to \mathbf{X}_i$ then $\mu|_{\mathbf{X}_i} = \pi_{i*} \mu$ and for E measurable in $\mathbf{X}_1$, $\mu|_{\mathbf{X}_1}(E) = \mu(\pi_1^{-1}(E)) = \mu(E \times X_2)$.

Derivation of the LP duality

LP duality

Primal

$$(\mathrm{P}) \begin{cases} \text{Find} & \mu \in \mathbb{M}_{\pm}(\mathbf{O}_X) \\ \text{maximising} & \mu(O_X) \\ \text{subject to} & \forall C \in \mathcal{M}.\ \mu|_C \leq e_C \\ \text{and} & \mu \geq 0. \end{cases}$$

$$\mathcal{L}(\mu, (f_C)) := \underbrace{\mu(O_X)}_{\text{objective}} + \underbrace{\sum_{C \in \mathcal{M}} \int_{O_C} f_C \, \mathrm{d}(e_C - \mu|_C)}_{\text{constraints}}$$

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$$\sup_{\mu} \inf_{(f_C)} \mathcal{L}(\mu, (f_C))$$

LP duality

$$\mathcal{L}\left(\mu, (f_C)\right) := \underbrace{\mu(O_X)}_{\text{objective}} + \underbrace{\sum_{C \in \mathcal{M}} \int_{O_C} f_C \, \mathrm{d}\left(e_C - \mu|_C\right)}_{\text{constraints}}$$

$$\mathcal{L}(\mu, (f_C)) \quad = \quad \mu(O_X) + \sum_{C \in \mathcal{M}} \int_{O_C} f_C \, \mathrm{d}\left(e_C - \mu|_C\right)$$

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$$\begin{aligned}\mathcal{L}(\mu,(f_C)) &= \mu(O_X) + \sum_{C\in\mathcal{M}} \int_{O_C} f_C \;\mathrm{d}\left(e_C - \mu|_C\right) \\ &= \int_{O_X} \mathbf{1} \;\mathrm{d}\,\mu + \sum_{C\in\mathcal{M}} \int_{O_C} f_C \;\mathrm{d}\,e_C - \sum_{C\in\mathcal{M}} \int_{O_C} f_C \;\mathrm{d}\,\mu|_C\end{aligned}$$

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\mathcal{L}(\mu, (f_C)) &= \mu(O_X) + \sum_{C \in \mathcal{M}} \int_{O_C} f_C \ \mathrm{d}(e_C - \mu|_C) \\
&= \int_{O_X} \mathbf{1} \ \mathrm{d}\mu + \sum_{C \in \mathcal{M}} \int_{O_C} f_C \ \mathrm{d}e_C - \sum_{C \in \mathcal{M}} \int_{O_C} f_C \ \mathrm{d}\mu|_C \\
&= \int_{O_X} \mathbf{1} \ \mathrm{d}\mu + \sum_{C \in \mathcal{M}} \int_{O_C} f_C \ \mathrm{d}e_C - \sum_{C \in \mathcal{M}} \int_{O_X} f_C \circ \rho_C^X \ \mathrm{d}\mu
\end{aligned}$$

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&= \int_{O_X} \mathbf{1} \; \mathrm{d}\,\mu + \sum_{C \in \mathcal{M}} \int_{O_C} f_C \; \mathrm{d}\,e_C - \sum_{C \in \mathcal{M}} \int_{O_X} f_C \circ \rho^X_C \; \mathrm{d}\,\mu \\
&= \int_{O_X} \mathbf{1} \; \mathrm{d}\,\mu + \sum_{C \in \mathcal{M}} \int_{O_C} f_C \; \mathrm{d}\,e_C - \int_{O_X} \left(\sum_{C \in \mathcal{M}} f_C \circ \rho^X_C \right) \; \mathrm{d}\,\mu
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&= \int_{O_X} \mathbf{1} \, \mathrm{d}\mu + \sum_{C\in\mathcal{M}} \int_{O_C} f_C \, \mathrm{d} e_C - \sum_{C\in\mathcal{M}} \int_{O_C} f_C \, \mathrm{d}\mu|_C \\
&= \int_{O_X} \mathbf{1} \, \mathrm{d}\mu + \sum_{C\in\mathcal{M}} \int_{O_C} f_C \, \mathrm{d} e_C - \sum_{C\in\mathcal{M}} \int_{O_X} f_C \circ \rho_C^X \, \mathrm{d}\mu \\
&= \int_{O_X} \mathbf{1} \, \mathrm{d}\mu + \sum_{C\in\mathcal{M}} \int_{O_C} f_C \, \mathrm{d} e_C - \int_{O_X} \left(\sum_{C\in\mathcal{M}} f_C \circ \rho_C^X \right) \mathrm{d}\mu \\
&= \sum_{C\in\mathcal{M}} \int_{O_C} f_C \, \mathrm{d} e_C + \int_{O_X} \left(\mathbf{1} - \sum_{C\in\mathcal{M}} f_C \circ \rho_C^X \right) \mathrm{d}\mu
\end{aligned}$$

LP duality

$$\mathcal{L}(\mu, (f_C)) = \sum_{C \in \mathcal{M}} \int_{O_C} f_C \, \mathrm{d}\, e_C + \int_{O_X} \left(\mathbf{1} - \sum_{C \in \mathcal{M}} f_C \circ \rho^X_C \right) \, \mathrm{d}\, \mu$$

$$\inf_{(f_C)} \sup_{\mu} \mathcal{L}(\mu, (f_C))$$

Dual

$$(\mathrm{D}) \begin{cases} \text{Find} & (f_C)_{C \in \mathcal{M}} \in \prod_{C \in \mathcal{M}} C_0(O_C, \mathbb{R}) \\ \text{minimising} & \sum_{C \in \mathcal{M}} \int_{O_C} f_C \, \mathrm{d}\, e_C \\ \text{subject to} & \sum_{C \in \mathcal{M}} f_C \circ \rho^X_C \geq \mathbf{1} \;\text{ on } O_X \\ \text{and} & \forall C \in \mathcal{M}.\; f_C \geq 0 \;\text{ on } O_C. \end{cases}$$

Generalised Bell inequality

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A *generalised Bell inequality* (β, R) on a measurement scenario $\langle X, \mathcal{M}, \mathbf{O} \rangle$ is a family $\beta = (\beta_C)_{C \in \mathcal{M}}$ with $\beta_C \in C_0(O_C, \mathbb{R})$ for all $C \in \mathcal{M}$, together with a bound $R \in \mathbb{R}$, such that for all noncontextual empirical models e on $\langle X, \mathcal{M}, \mathbf{O} \rangle$ it holds that $\langle \beta, e \rangle_2 := \sum_{C \in \mathcal{M}} \int_{O_C} \beta_C \, \mathrm{d}\, e_C \leq R$.

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Normalised violation

The *normalised violation* of a Bell inequality (β, R) by an empirical model e is:

$$\frac{\max\{0, \langle \beta, e \rangle_2\}}{\|\beta\| - R}$$

where: $\|\beta\| = \sum_{C \in \mathcal{M}} \sup\{\beta(\mathbf{o}) | \mathbf{o} \in O_C\}$

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Theorem

Let e be an empirical model.

(i) *The normalised violation by e of any Bell inequality is at most* $CF(e)$*;*

(ii) *if* $CF(e) > 0$ *then for every* $\epsilon > 0$ *there exists a Bell inequality whose normalised violation by e is at least* $CF(e) - \epsilon$*.*