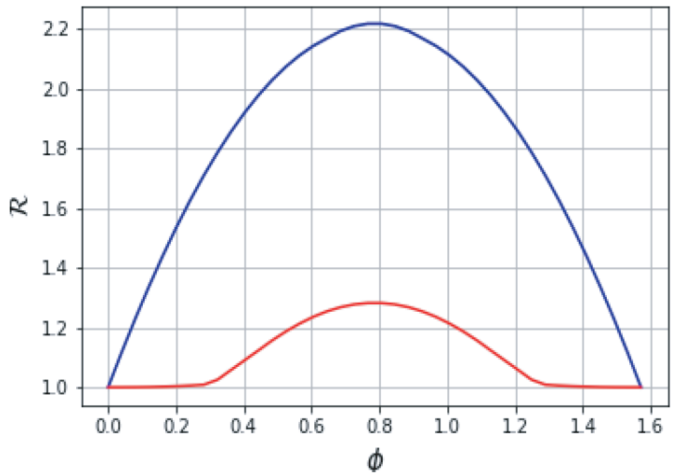
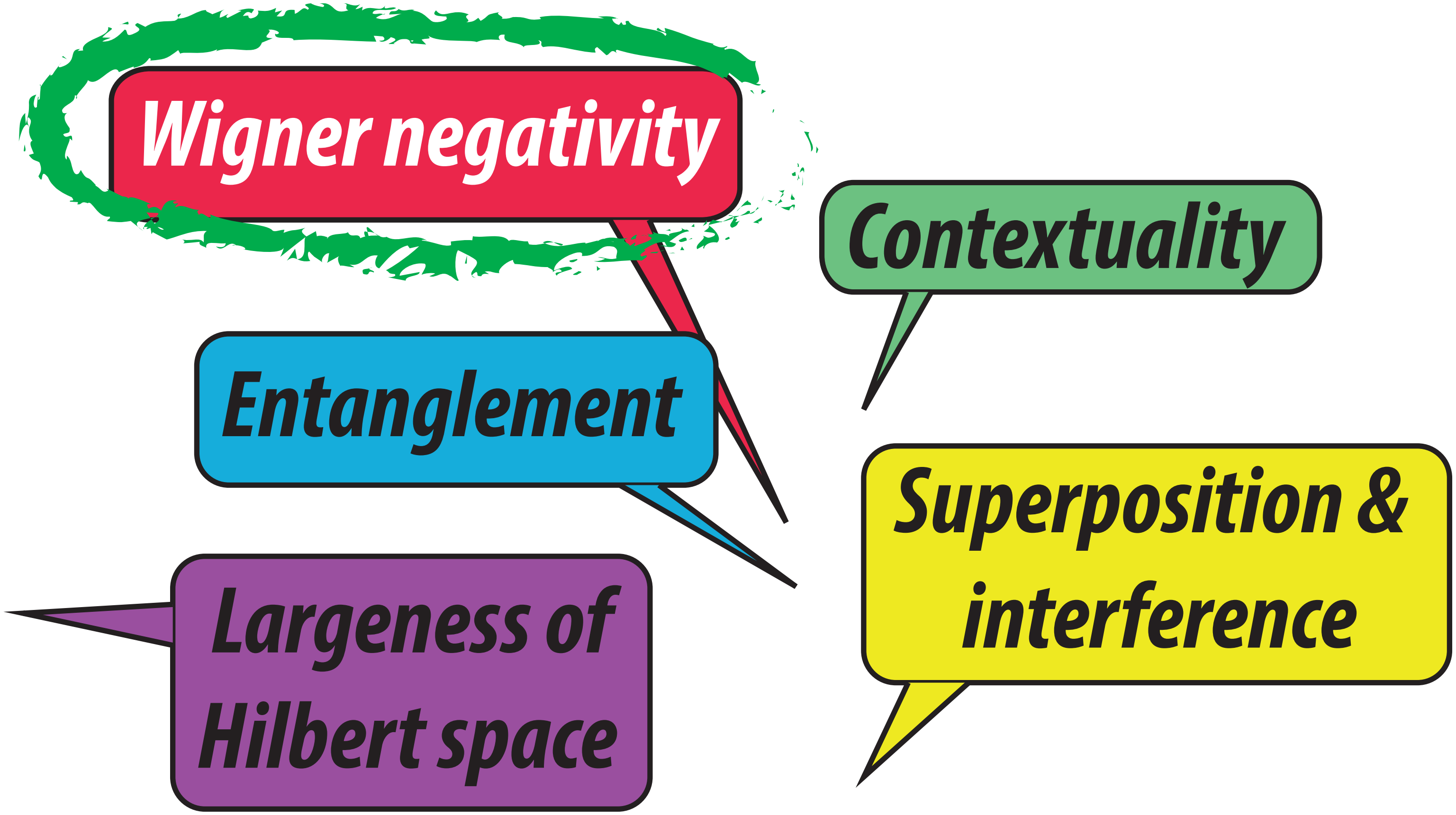
Phase space simulation method for quantum computation with magic states on qubits



Robert Raussendorf, UBC Vancouver

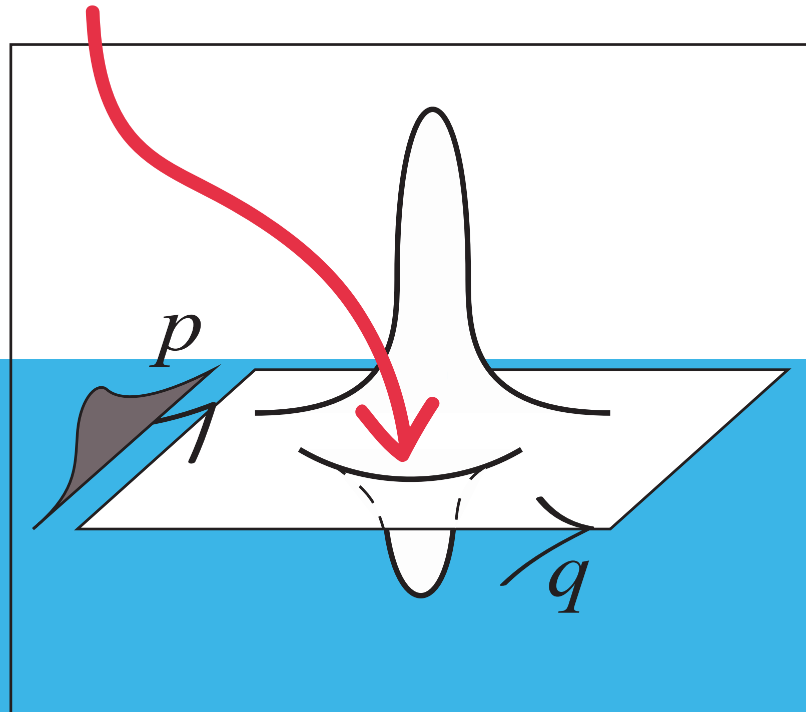
Oxford, July 2019

Joint work with Cihan Okay, Juani Bermejo-Vega, Emily Tyhurst, Michael Zurel



What makes quantum computing work?

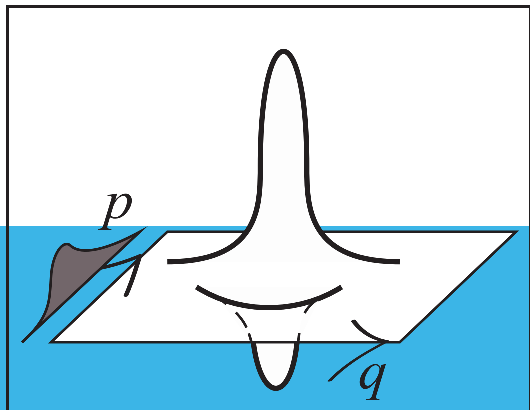
Negativity of the Wigner function
is an indicator of quantumness *

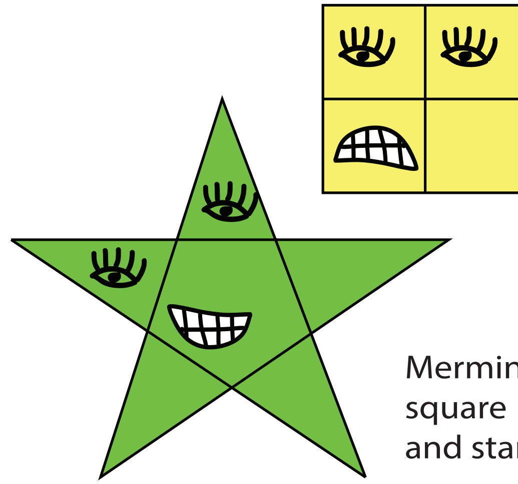


*: This even holds in quantum computation

Role of the Hilbert space dimension

quantum optics:
Hilbert space dimension infinite





Mermin's
square
and star

Odd: all nice & safe

quantum computation:
Hilbert space dimension finite

Even: monsters lurking

Result

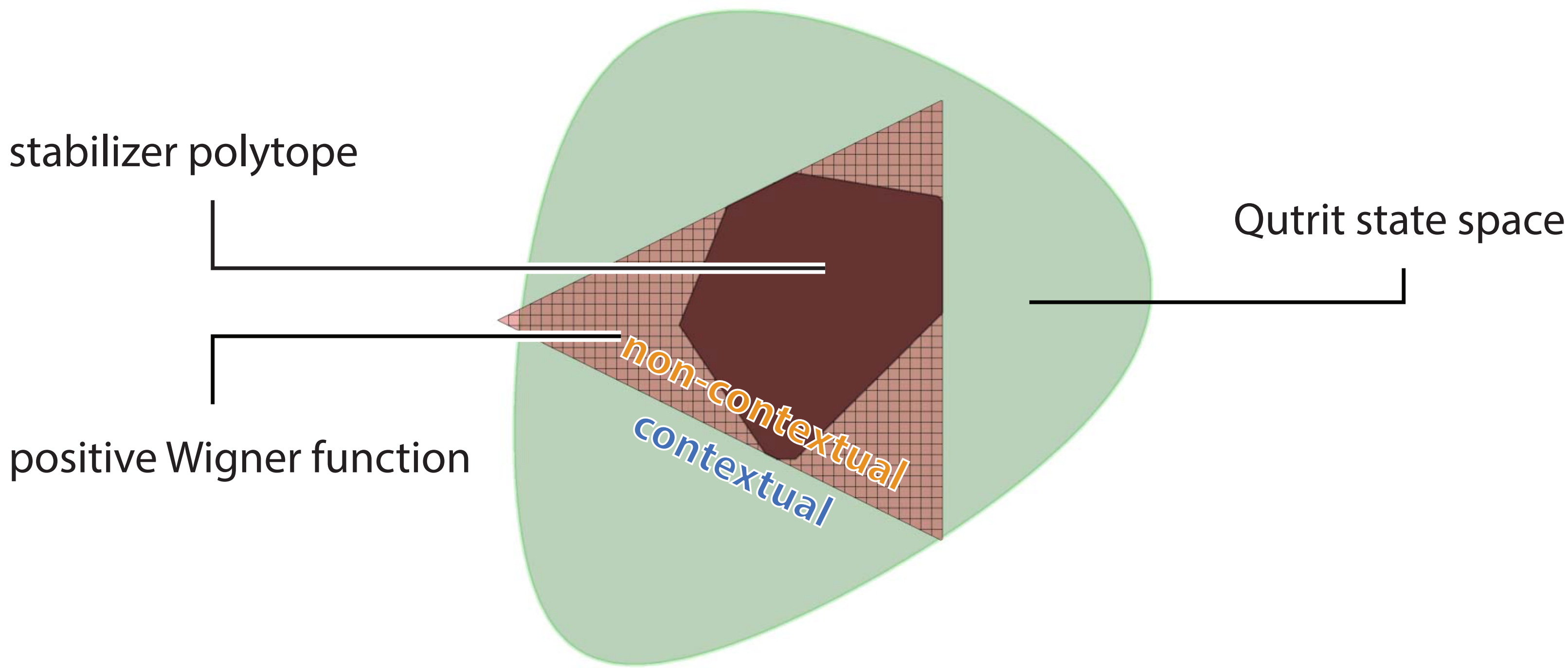
Theorem[1]–[3]: *Quantum computation with magic states can have a quantum speedup only if the Wigner function of the initial magic states is negative.*

Negativity in the Wigner function
is a resource
for quantum computation

[1] Qudits in odd d: V. Veitch *et al.*, New J. Phys. 14, 113011 (2012).
[2] Rebits: N. Delfosse *et al.*, Phys. Rev. X 5, 021003 (2015).
[3] Qubits: R. Raussendorf *et al.*, arXiv:1905.05374.

Contextuality

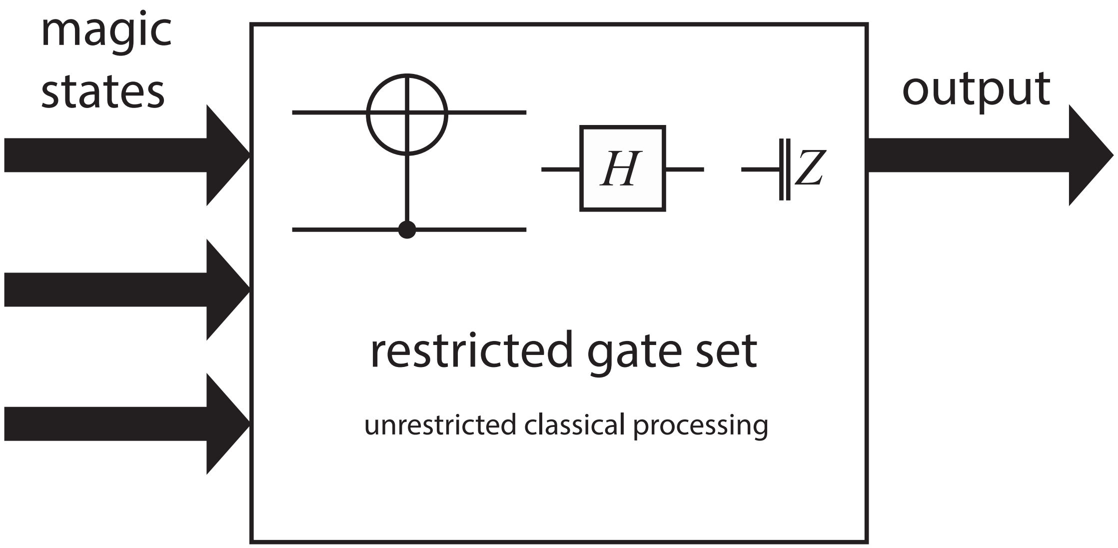
The case of odd prime local Hilbert space dimension



Wigner negativity = contextuality in odd d

M. Howard *et al.*, Nature 510, 351 (2014)

Quantum Computation with magic states



- Non-universal restricted gate set: *e.g. Clifford gates*.
- Universality reached through injection of *magic states*.

+ *As of now, leading scheme for fault-tolerant QC.*

Computational power is shifted from gates to states

Outline

1. Review: the case of odd local dimension

   (a) Wigner functions in finite dimension

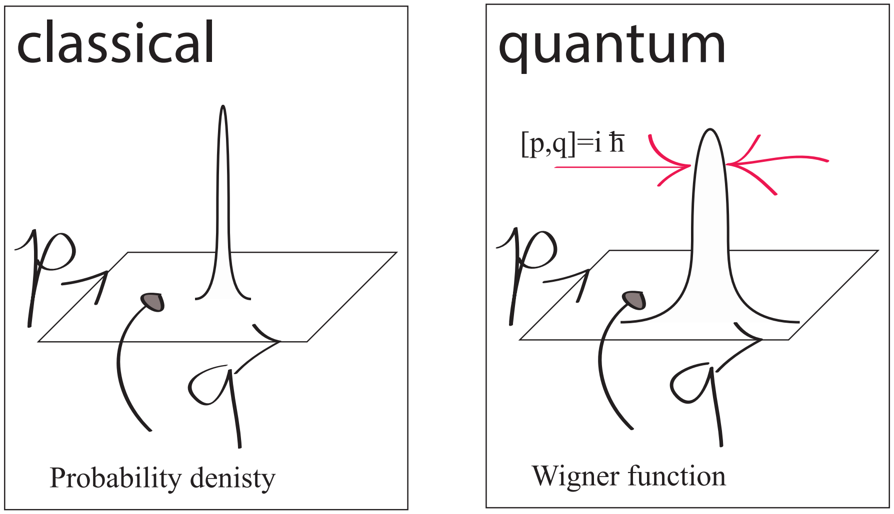
   (b) Wigner function negativity as a resource

2. The trouble with qubits

3. Quantum computation with magic states in $d = 2$

   - Overcoming Mermin's Monsters

[quantum] mechanics in phase space



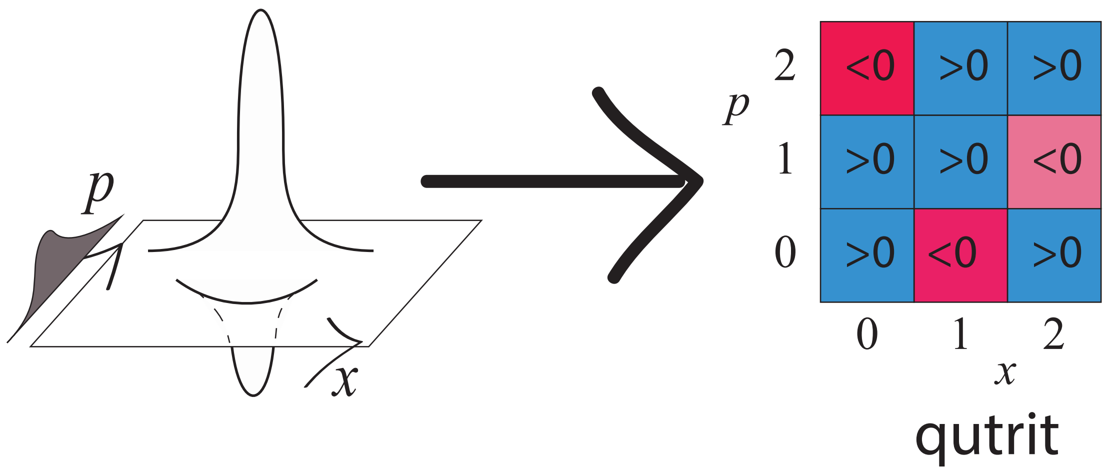
- The Wigner function

$$W_\psi(p,q) = \frac{1}{\pi}\int d\xi\, e^{-2\pi i \xi p}\psi^\dagger(q-\xi/2)\psi(q+\xi/2).$$

is a quasi-probability distribution.

It is the closest quantum counterpart to the classical probability distribution over phase space.

Wigner functions for qudits



Wigner functions can be adapted to finite-dimensional state spaces.

- The Wigner function W is linear in ρ.
- The marginals of W are probability distributions.

Wigner function for qudits

The n-qudit state state space is $V = \mathbb{Z}_d^n \times \mathbb{Z}_d^n$.

For every $\mathbf{v} \in V$ we have a phase point operator $A_{\mathbf{v}}$ such that

$$\boxed{W_\rho(\mathbf{v}) = \frac{1}{d^n}\mathrm{Tr}\left(A_{\mathbf{v}}\rho\right), \ \ \forall \rho.}$$

$$\boxed{\rho = \sum_{\mathbf{v}\in V} W_\rho(\mathbf{v}) A_{\mathbf{v}}.}$$

- To define the Wigner function W, we need to define the phase point operators.

The phase point operators $A_{\mathbf{v}}$

Consider the qudit Pauli operators ($d \times d$-matrices)

$$X = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & & \\ 1 & & & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & & \\ & & & \omega^{d-1} \end{pmatrix},$$

and introduce the "translators" $T_{\mathbf{a}}$, $\forall \mathbf{a} = (\mathbf{a}_X, \mathbf{a}_Z) \in \mathbb{Z}_d^n \times \mathbb{Z}_d^n$,

$$\boxed{T_{\mathbf{a}} = \omega^{\gamma(\mathbf{a})} \bigotimes_{j=1}^{n} \left(X_j\right)^{a_X(j)} \left(Z_j\right)^{a_Z(j)}}$$

Remark:

- Choice $\gamma(\mathbf{a}) := \mathbf{a}_Z^T \mathbf{a}_X / 2$ ensures that $T_{\mathbf{a}+\mathbf{b}} = T_{\mathbf{a}} T_{\mathbf{b}}$, $\forall$ commuting $T_{\mathbf{a}}$, $T_{\mathbf{b}}$
- That choice works only when d is odd.

The phase point operators $A_{\mathbf{v}}$

Recall: $T_{\mathbf{a}} = \omega^{\gamma(\mathbf{a})} \bigotimes_{j=1}^{n} \left(X_j\right)^{a_X(j)} \left(Z_j\right)^{a_Z(j)}.$

The phase point operator at the origin is

$$A_{\mathbf{0}} = \frac{1}{d^n} \sum_{\mathbf{a} \in \mathbb{Z}_d^2 \times \mathbb{Z}_d^n} T_{\mathbf{a}}.$$

All phase point operators are

$$A_{\mathbf{v}} = T_{\mathbf{v}} A_{\mathbf{0}} T_{\mathbf{v}}^{\dagger}.$$

Now use this in:

$$W_\rho(\mathbf{v}) = \frac{1}{d^n} \mathrm{Tr}\left(A_{\mathbf{v}} \rho\right)$$

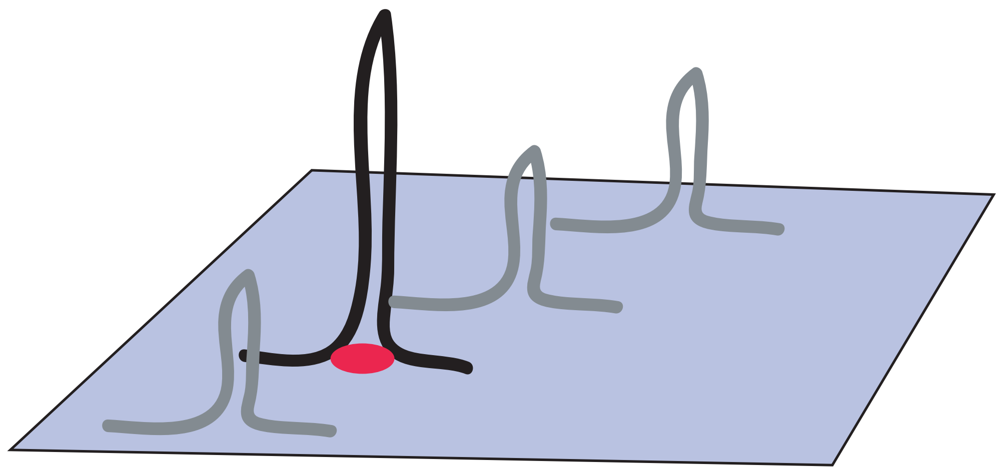
D. Gross, PhD Thesis, Imperial College London, 2005.

Pauli measurement & the qudit Wigner function

If the local dimension is odd, then Gross' n-qudit Wigner function *preserves* positivity under all Pauli measurements.

Denote $P_{\mathbf{a},s}$ the projector corrresponding to the measurement of the observable $T_{\mathbf{a}}$ with eigenvalue ω^s. Then,

$$W_\rho > 0 \Rightarrow W_{P_{\mathbf{a},s}\rho P_{\mathbf{a},s}} > 0, \quad \forall \mathbf{a}, \forall s.$$



Quantum speedup requires $W < 0$

In the case of odd prime local Hilbert space dimension:

Theorem [*]: *Quantum computation with magic states can have a quantum speedup only if the Wigner function of the initial magic states is negative.*

*: V. Veitch *et al.*, New J. Phys 14 (2012).

Proof idea

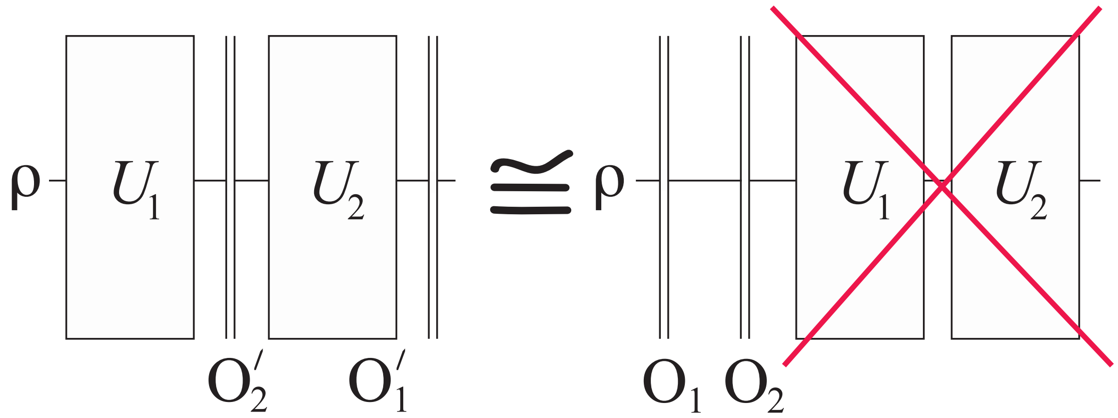
We will show that:
If $W_{\rho_{\text{magic}}} \geq 0 \Rightarrow$ efficiently classical simulation $\Rightarrow$ no speedup.

Simulation algorithm:

1. $W_{\rho_{\text{magic}}} \geq 0$ is a probability distribution. $\longrightarrow$ Sample from it! Each sample is a point in phase space.

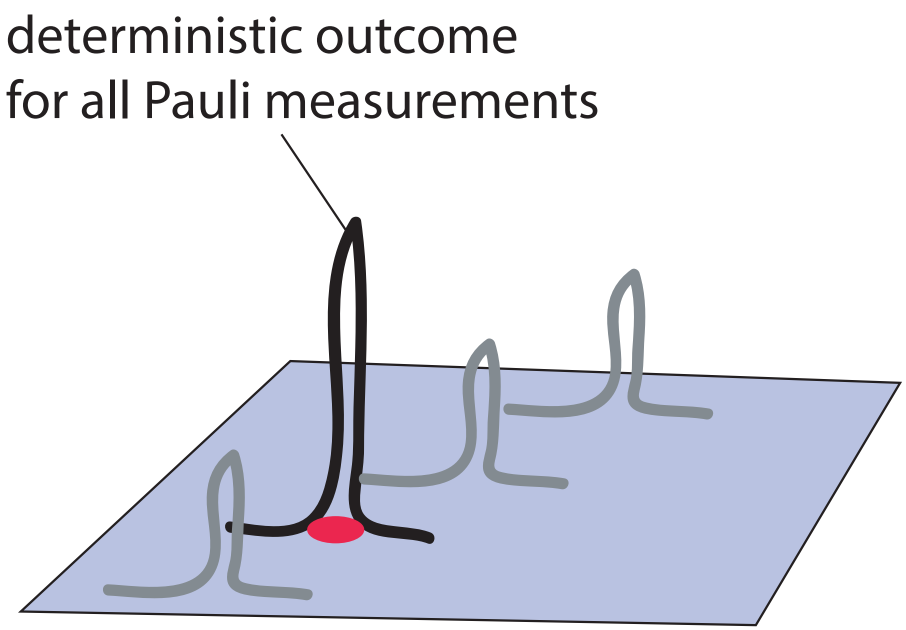
2. Update the phase space points under Clifford gates and Pauli measurement.

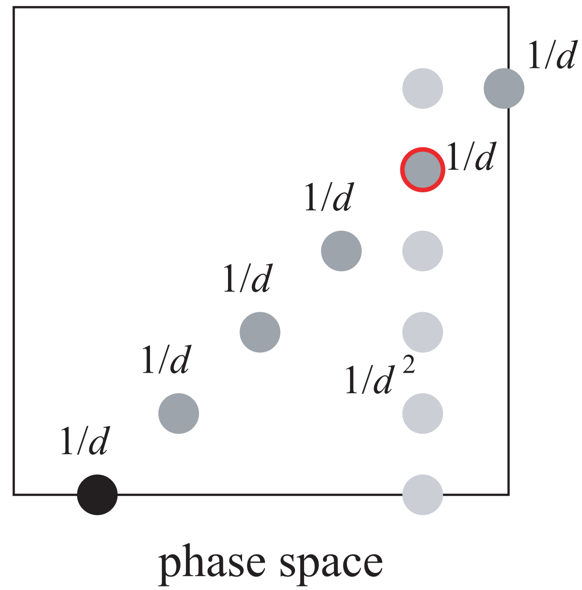
Eliminating Clifford unitaries



- Only the measurement statistics matters
- All Clifford unitaries can be propagated forward in time past the last measurement, and then discarded.

Update under Pauli measurements



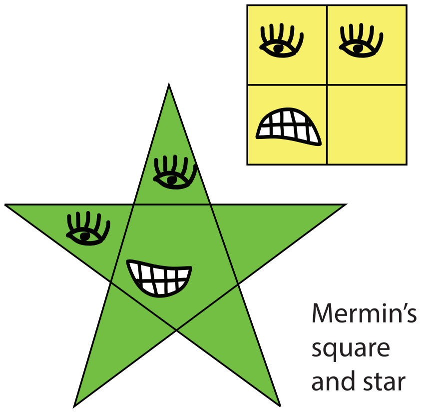


Use positivity-preservation under Pauli measurement

The trouble with

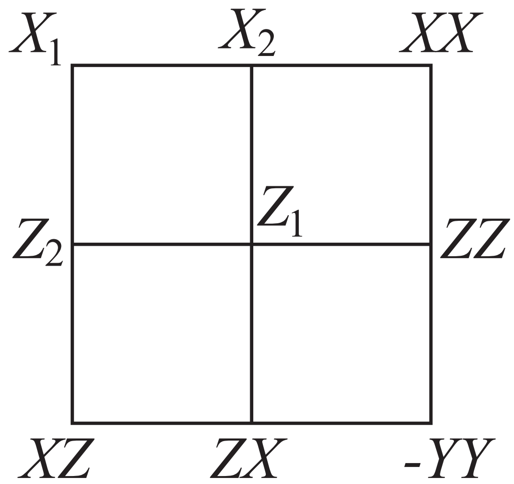
Local Hilbert space dimension $d = 2$

The trouble with $d = 2$



Standard Wigner function does not preserve positivity under Pauli measurement

The trouble with $d = 2$



In at least one context it must hold that

$$T_{\mathbf{a}+\mathbf{b}} = -T_{\mathbf{a}} T_{\mathbf{b}},$$

otherwise we could consistently assign the value $\lambda(T_{\mathbf{a}}) = 1$, $\forall \mathbf{a}$.

The trouble with $d = 2$

Assume we use an analogous Wigner function for qubits, with phase point operators

$$A_{\mathbf{0}} = \frac{1}{2^n} \sum_{\mathbf{a} \in \mathbb{Z}_2^n \times \mathbb{Z}_2^n} T_{\mathbf{a}}, \quad A_{\mathbf{v}} = T_{\mathbf{v}} A_{\mathbf{0}} T_{\mathbf{v}}^{\dagger} \tag{1}$$

Now consider $W_\rho(\mathbf{0})$ for the stabilizer state

$$\rho = \frac{(I - T_{\mathbf{a}})(I - T_{\mathbf{b}})}{2^n} = \frac{I - T_{\mathbf{a}} - T_{\mathbf{b}} - T_{\mathbf{a}+\mathbf{b}}}{2^n}$$

We find

$$W_\rho(\mathbf{0}) = \frac{1}{2^n} \mathrm{Tr}(A_{\mathbf{0}} \rho) = \frac{1 - 1 - 1 - 1}{4^n} < 0.$$

Starting from Eq. (**??**), whatever the phase convention for the $T_{\mathbf{a}}$, there are stabilizer states with negative W.

Local Hilbert space dimension $d = 2$

Goals

- Construct a Wigner function for multi-qubit systems that preserves positivity under all Pauli measurement.

- Efficient classical simulation of QC with magic states for $W_{\rho_{\text{init}}} \geq 0$.

We obtain that, and in addition:

- Our construction applies to all d, & reproduces Gross' Wigner function if d is odd.

$\Rightarrow$ Unified method for classical simulation based on phase space.

$+$ Also contains simulation of stabilizer mixtures as a special case.

Phase point operators for $d = 2$

The multi-qubit Wigner function is defined through

$$\rho = \sum_{\Omega,\gamma} W(\Omega,\gamma) A_\Omega^\gamma,$$

where the phase point operators are given by

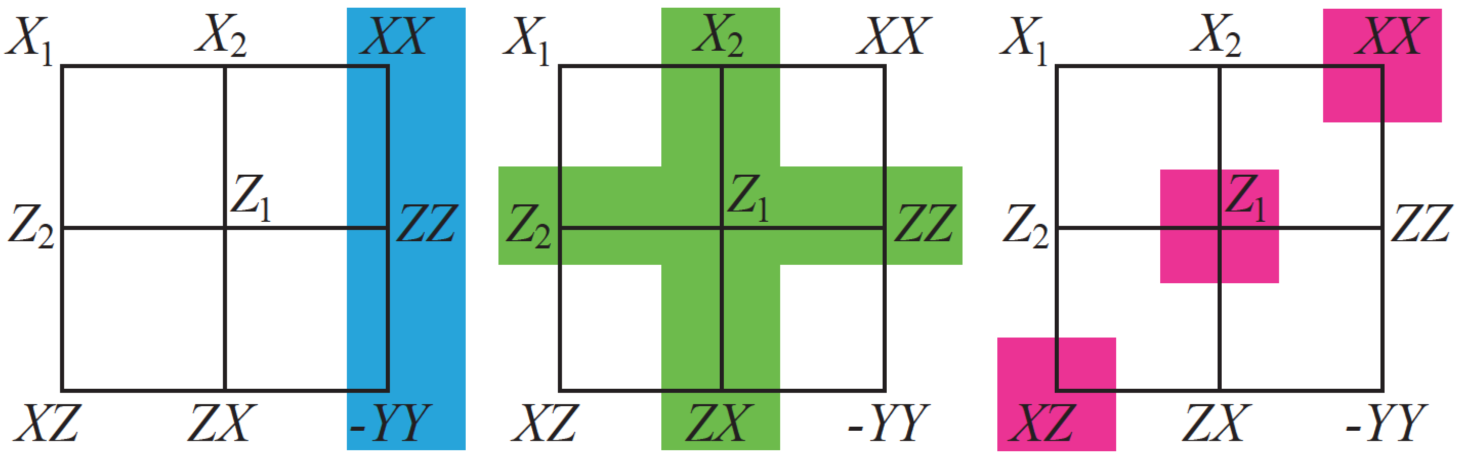
$$A_\Omega^\gamma = \frac{1}{2^n} \sum_{a\in\Omega} (-1)^{\gamma(a)} T_a, \qquad \Omega \subset V = \mathbb{Z}_2^n \times \mathbb{Z}_2^n.$$

The sets $\Omega \subset V$ and the functions $\gamma : \Omega \longrightarrow \mathbb{Z}_2$ satisfy the following constraints:

- Ω is free of parity-based Kochen-Specker proofs.
- Ω is closed under inference.

  If $a, b \in \Omega$ and $[T_a, T_b] = 0$ then $a + b \in \Omega$.
- γ is a consistent value assignment.

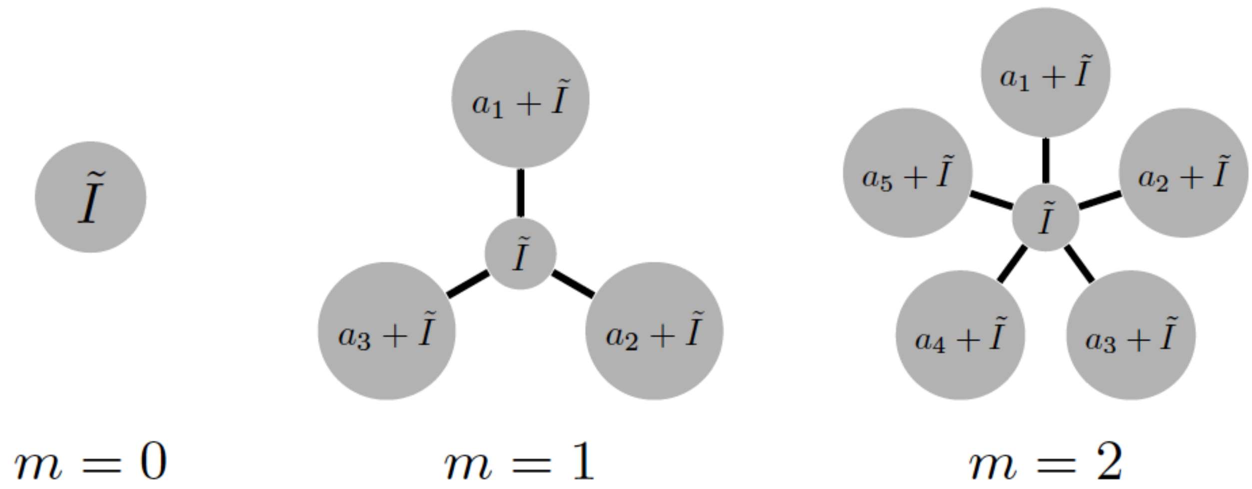
Phase point operators for Mermin's square



Example for phase point operator (middle):

$$A_{\Omega}^{\gamma} = \frac{1}{4}\left(I + Z_1 + Z_2 + Z_1 Z_2 + X_2 + Z_1 X_2\right).$$

Phase point operators are classified for $d = 2$



- The classification of phase point operators is related to Majorana fermions

See: arXiv:1905.05374.

Properties of phase point operators

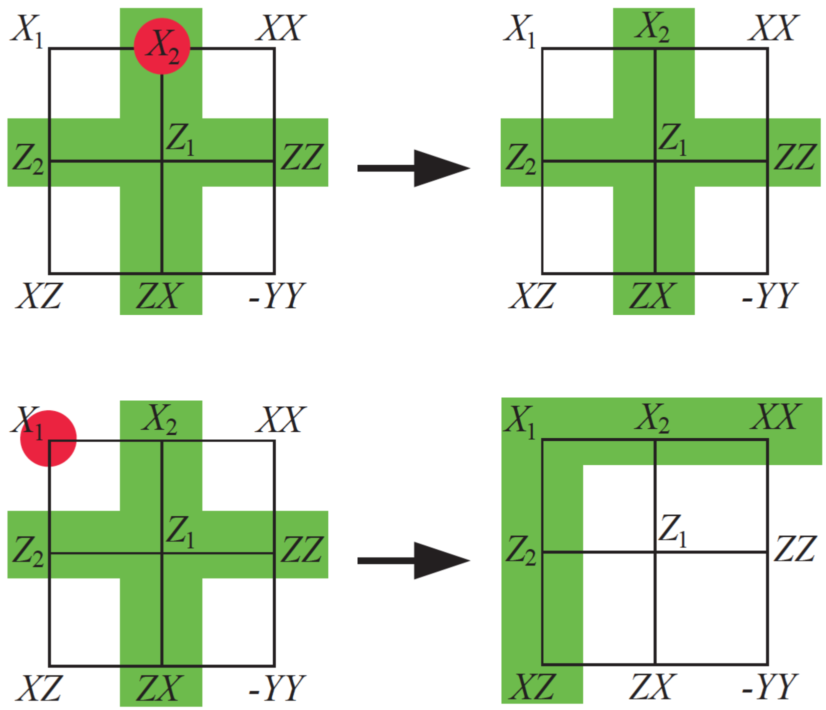
- Phase point operators map to probabilistic mixtures of phase point operators under measurement

- Phase point operators map to phase point operators under Clifford unitaries (covariance)

$\Rightarrow$ Positivity is preserved in either case

Update of phase point operators

Example of Mermin's square (update of Ω):



- The sets Ω and the functions γ change under the evolution by Pauli measurement.

Efficient classical simulation for $W_\rho \geq 0$

Theorem. If the Wigner function $W_{\rho_\text{init}} \geq 0$ and can be efficiently sampled from, then all magic state quantum computation on ρ_init can be efficiently classically simulated.

$W_{\rho_\text{init}} < 0$ is a quantum computational resource!

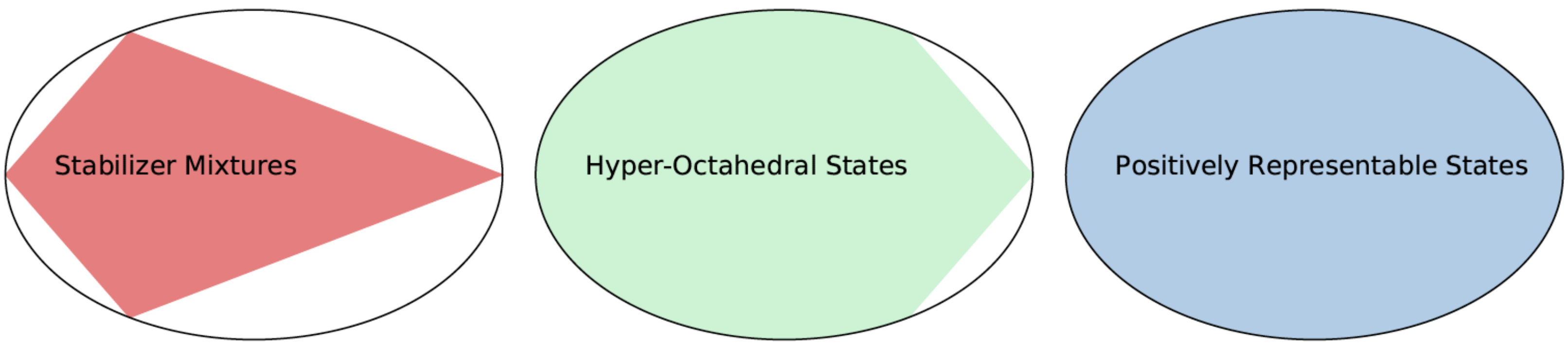
The classical simulation algorithm is as follows:

1. Sample phase space points (Ω, γ) according to the positive W_{ρ_init}.
2. Propagate phase space point (Ω, γ) through circuit, one measurement at a time.
   - For the measurement of the Pauli observable T_a:
     If $a \in \Omega$, then output $\gamma(a)$
     If $a \notin \Omega$, then flip a coin.
   - Update Ω, γ depending on a.

Positively representable states

Portion of positively representable states for the cross section of the 2 qubit Bloch Sphere:

$$\rho(x, y) = \frac{1}{4} I_1 I_2 + x(X_1 X_2 + Z_1 Z_2 - Y_1 Y_2) + y(Z_1 + Z_2)$$



- For any number of qubits: The set of positively W-representable states is strictly larger than stabilizer mixtures.

But what if $W < 0$?

- When $W_{\rho_{\text{init}}} < 0$, classical simulation using W provides amplitude estimation.

- Number of samples required scales as $\mathfrak{R}(\rho)^2/\epsilon^2$, where

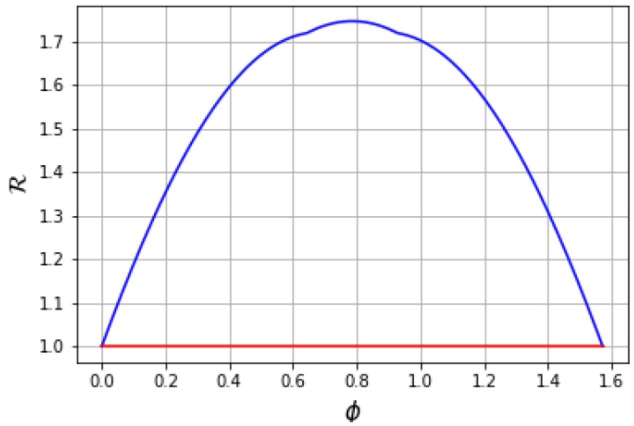
$$\mathfrak{R}(\rho) = \min_W \left(\|W\| : \ \rho = \sum_{\Omega,\gamma} W(\Omega,\gamma) A_\Omega^\gamma \right).$$

- For all n, for all n-qubit states ρ it holds that

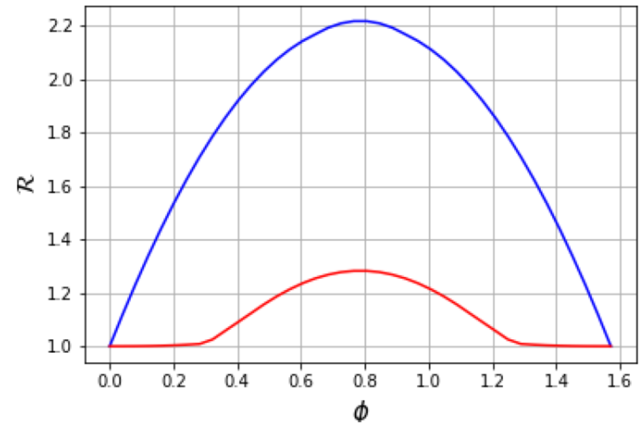
$$\mathfrak{R} \leq \mathfrak{R}_S,$$

  with $\mathfrak{R}_S$ the robustness of magic.

Robustness $\mathfrak{R}$



(a) 2 Qubits



(b) 3 Qubits

$$|H(\phi)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{-i\phi}|1\rangle\right)$$

- In addition to governing the hardness of classical simulation, the robustness $\mathfrak{R}$ is also a *monotone* under Clifford gates and Pauli measurements.

Results

We have constructed a Wigner function for qubits which:

- Is positivity-preserving under all Pauli measurements
- Is Clifford covariant
- Provides a simulation algorithm for quantum computation with magic states on qubits, for $W_{\rho_{\text{init}}} \geq 0$.

We extend/unify the results of

- Veitch et al., New J Phys (2012) (odd dimension)
- Howard and Campbell. Phys Rev Lett (2017) (Simulation based on stabilizer mixtures)
- Wallman and Bartlett, Phys Rev A (2012) (Eight state model for one qubit)

[arXiv:1905.05374]