Contextuality of Transformations

Shane Mansfield

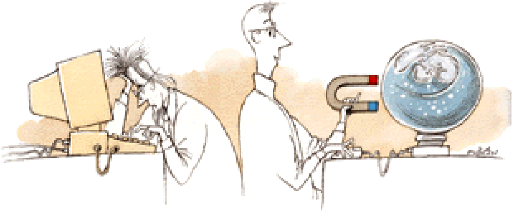
S. Mansfield and E. Kashefi.
"Quantum advantage via sequential transformation contextuality".

Physical Review Letters 121 (23), 230401.

Contextuality Workshop Oxford
June 6th 2019

A Foundational Approach to Quantum Computing

Objective: Understand the **computational advantage** afforded by **quantum resources**



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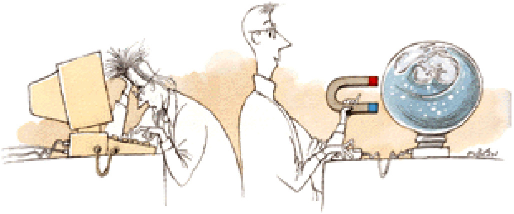

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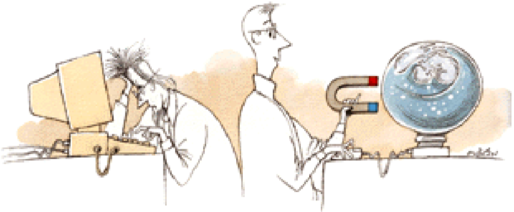

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Questions we begin to have answers to:

- What are these **non-classical behaviours**? *Contextuality, . . .*
- How do they relate to **computational advantage**?
  Specific examples, though not yet a systematic understanding. . .

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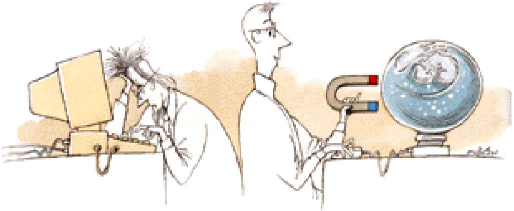

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Where to go with this knowledge:

- **Higher-level reasoning** about contextuality in our resources and how to spend it!
- Systematic, structural path to **quantum superiority**

Non-Classicality

Non-classical behaviours should be established by **no-go theorems**

Behaviour	**Familiar theorems**
Nonlocality	Bell ’64
Contextuality	Kochen, Specker ’76
Non-macrorealism	Leggett, Garg ’85
Generalised contextuality	Spekkens ’05
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Dynamic contextuality	

Ontological Models

- We posit a space of ontic states Λ

	Quantum mechanics	Ontological models
Preparation	ρ	$d_\rho \in P(\Lambda)$
Transformation	U	$f_U : \Lambda \to \Lambda$
Measurement	M	$\xi_M : \Lambda \to P(O)$

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Empirical data $e_{\rho,U,M} : P(O)$ should be reproduced as a **weighted average** over ontic states,

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*No further assumptions about *features* at this stage (cf. Spekkens)

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- ... and are context-independent; e.g. if $M \in C, C'$

$$\xi_{M^{(C)}} = \xi_{M^{(C')}}$$

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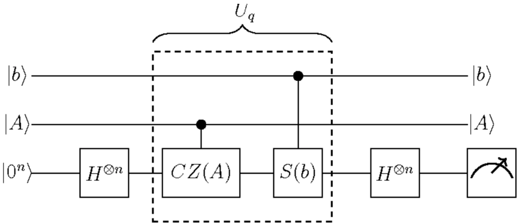
"Contextuality in our sense implies that the system of study cannot have an ontology in which transformations correspond to modular, composable operations on ontic states, such that they are well-defined independently of which transformations may have been performed previously or will be performed subsequently."

Example: Quantum Advantage in Shallow Circuits

Bravyi, Gossett, König, *Science*, 2017

There exists a task for which

Circuit-depth complexity	
Classical	log
Quantum	constant



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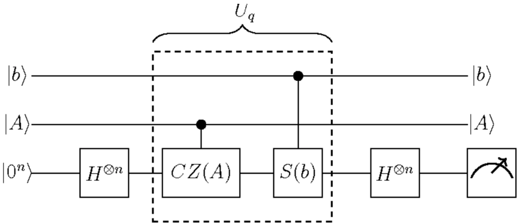

Contextuality analysis

Suppose **sequential noncontextuality**:

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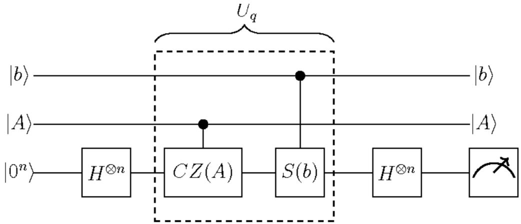

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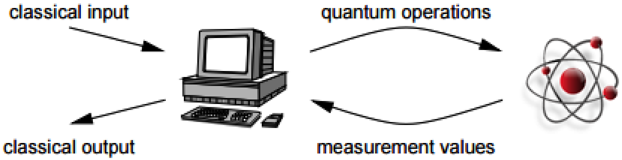
- Sequential contextuality is *necessary* for advantage

(BKS) Contextuality is Useful



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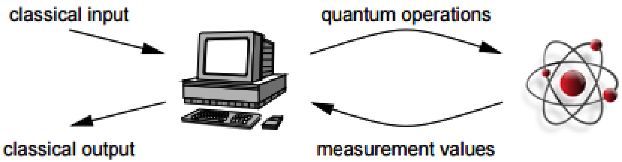

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  - Determines the sequence of measurements
  - In *this example* can perform $\mathbb{Z}_2$-linear computations only
- Power to compute non-linear functions may reside in certain (quantum) resources

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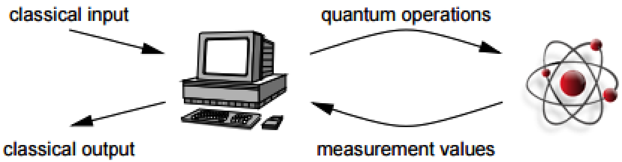

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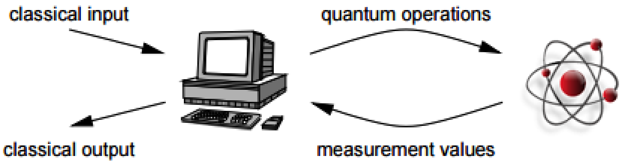
Raussendorf, *PRA*, 2013

- ℓ2-MBQC *deterministically* computes a non-linear Boolean function $f : 2^m \longrightarrow 2^l$ **iff** the resource is *maximally contextual*
- Bounded error non-linear computations require *contextuality*

(BKS) Contextuality is Useful



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Abramsky, Barbosa, M, *PRL*, 2017

$$\overbrace{1-\bar{p}_S}^{\text{error}} \geq \underbrace{[1-\mathsf{CF}(e)]}_{\text{classicality}} \overbrace{\nu(f)}^{\text{hardness}}$$

quantifiable relationship!

Is Dynamic Contextuality Useful?

Dunjko, Kapourniotis, Kashefi, *QIC*, 2016.

Classical control ($\oplus L$):

- Classical inputs $a, b \in \mathbb{Z}_2$
- Controls transformations
- Announces meas. outcome $\{+1 \mapsto 0, -1 \mapsto 1\}$

Quantum resource:

- Prepare qubit in state $|+\rangle$
- Transformations

$$U_0 = V_0 = W_0 = I$$
$$U_1 = V_1 = W_1 = R_z(\pi/2)$$

$$\boxed{f(a,b) = a \otimes_2 b}$$

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- Boosts computational power: $\oplus L \longrightarrow P$
- Contextuality in the traditional sense cannot arise with a single qubit!
- So what, if anything, is the non-classical behaviour?

An Appropriate Ontology

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Commutative $\oplus L$-ontological models:

Ontic states	$\Lambda = (\mathbb{Z}_2)^n$
Transformations	$f_U(\boldsymbol{\lambda}) = (I \oplus A_U)\boldsymbol{\lambda} \oplus \boldsymbol{u}$
Measurements	$\xi_M(\boldsymbol{\lambda}) = [(I \oplus A_M)\boldsymbol{\lambda} \oplus \boldsymbol{u}] \cdot \boldsymbol{\delta}$

Parity Proof of Contextuality

- $\oplus L$-ontological realisation of the protocol requires the following equations to be satisfied:

$$
\begin{aligned}
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- System of equations is not jointly satisfiable
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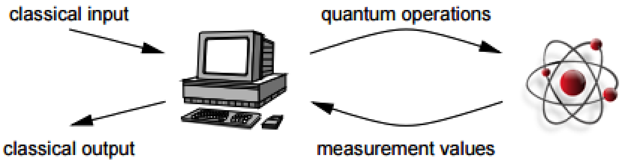
- Such a realisation is necessarily **dynamically contextual**

l2-TBQC



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- Fixed preparation and 2-outcome measurement
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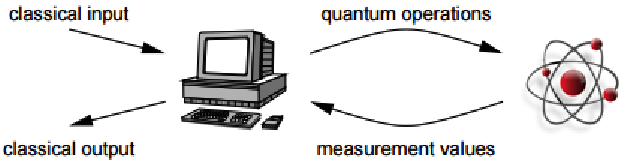

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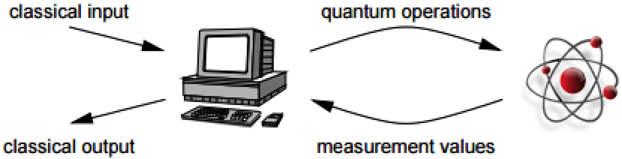

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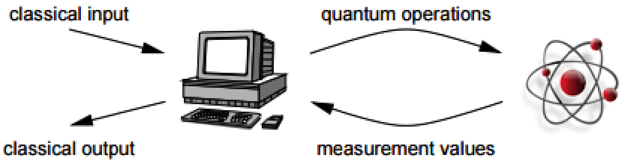

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- l2-MBQC $\neq$ l2-TBQC
- *In general* we could shift perspective and express any MBQC as a TBQC

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$$e = \{e_C\}$$

Cf. Abramsky, Brandenburger, *NJP*, 2011.

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context			outcome	
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C_0	0	0	$3/4$	$1/4$
C_1	0	1	$1/4$	$3/4$
C_2	1	0	$1/4$	$3/4$
C_3	1	1	0	1

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$$e_{C_0} = (^3/_4, ^1/_4)$$
$$e_{C_1} = (^1/_4, ^3/_4)$$
$$e_{C_2} = (^1/_4, ^3/_4)$$
$$e_{C_3} = (0, 1)$$

Quantifying Contextuality

Noncontextual fraction $\mathsf{NCF}(e)$

$\mathsf{NCF}(e)$ is max ω over all decompositions

$$e = \omega e^{\mathsf{NC}} + (1-\omega)e'$$

s.t. e^{NC} is noncontextual

Contextual fraction $\mathsf{CF}(e)$

$$\mathsf{CF} := 1 - \mathsf{NCF}(e)$$

- $\mathsf{CF}(e), \mathsf{NCF}(e) \in [0,1]$

Quantifying Hardness

Distance on functions: Given $f, g : (\mathbb{Z}_2)^r \to \mathbb{Z}_2$,

$$d(f,g) := 2^{-r} \left| \{ \boldsymbol{i} \mid f(\boldsymbol{i}) \neq g(\boldsymbol{i}) \} \right|$$

Fraction of inputs for which outputs differ

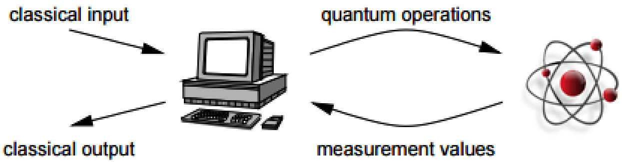
Non-linearity of a function $f : (\mathbb{Z}_2)^r \to \mathbb{Z}_2$,

$$\nu(f) := \min \{ d(f,g) \mid g : (\mathbb{Z}_2)^r \to \mathbb{Z}_2 \text{ linear} \}$$

Distance to nearest linear function

Cf. Abramsky, Barbosa, M, *PRL*, 2017.

Advantage in $l2$-TBQC



m input bits
1 output bit

$l2$-TBQC

$\oplus L$ control computer with access to a (quantum) resource

- Fixed preparation and 2-outcome measurement
- Controlled unitary operations

$$\overbrace{1-\bar{p}_S}^{\text{error}} \geq \underbrace{[1-\mathsf{CF}(e)]}_{\text{classicality}} \overbrace{\nu(f)}^{\text{hardness}}$$

quantifiable relationship!

Example: The CHSH* Game

Henaut, Catani, Browne, Pappa, M, *PRA*, 2018.

- Task: compute $a \otimes_2 b$
- Maximise success probability in various regimes
- Tsirelson bound for qubits
- Similar for qutrits, with $\otimes_3$, etc.
- Dimensional witness!

	$p^{\text{max}}_{\text{success}}$
bit	0.75
Spekkens toy bit	0.75
stabiliser qubit	0.75
qubit	**0.85**...
qutrit	1

Classical Erasure

Henaut, Catani, Browne, Pappa, M, *PRA*, 2018.

- Classically, can compute $a \otimes_2 b$ with $l2$-operations and **erasure**

$$U_0 = I \quad U_1 = \mathsf{NOT} \quad V_0 = \mathsf{RESET}_0 \quad V_1 = I$$

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- Expected *erasure cost* per run, averaged over pairs of inputs, to compute a function g, with 1- and 2-bit gates coincides with non-linearity measure:

$$\nu(g)$$

Contextuality-Erasure Tradeoff

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The Combined Perspective

If an $l2$-TBQC is run n times with uniformly random inputs and the overall change in environmental entropy is ΔS, then

$$\overline{\varepsilon} \geq \left[\mathsf{NCF}(e) - \frac{\Delta S}{n\,kT \ln 2} \right] \tilde{\nu}(g)$$

Equivalently,

$$\mathsf{CF}(e) \geq 1 - \frac{\overline{\varepsilon}}{\tilde{\nu}(g)} - \frac{\Delta S}{n\,kT \ln 2}$$

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- Circuit contextuality: generalise BKS and dynamic contextuality

Bonus: Proposal for Generalised Noncontextuality

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	Components	**Composition**
Locality	M	$\otimes$
Noncontextuality *(BKS)*	M	$\times$
Measurement NC *(Spekkens*)*	M	$\times$
Preparation NC *(Spekkens*)*	P	$+_\lambda$
Transformation NC *(Spekkens*)*	T	$+_\lambda$
Preparation Independence *(PBR)*	P	$\otimes_{\text{min}}$
Subsystem Condition *(SM)*	P	$\otimes$
Dynamic NC	T	$\circ$