Contextuality as a Resource for Shallow Circuits

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Introduction

Bravyi, Gosset, König: Quantum Advantage using Shallow Circuits.

The *2D Hidden Linear Function problem* is solved by a bounded depth and fan-in (shallow) family of quantum circuits, any shallow classical circuit fails with probability at least $1/8$.

ABM: The Contextual Fraction as a Measure of Contextuality

$$\epsilon \geq (1 - \mathsf{CF}(e)) \cdot v(f)$$

ϵ - degree of error, $\mathsf{CF}(e)$ - degree of contextuality, $v(f)$ - hardness of task.

Outline

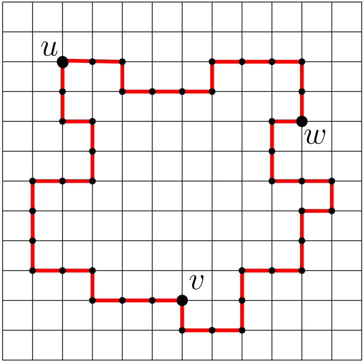
- BGK as a game.
- Measurement scenarios as a framework for non-local games:
  - Strategies.
  - Bell inequalities, no-communication and contextuality.
  - Value passing.
  - Restricted communication strategies.
- Contextuality as a resource.
- Resource sensitive BGK.

2D Graph States

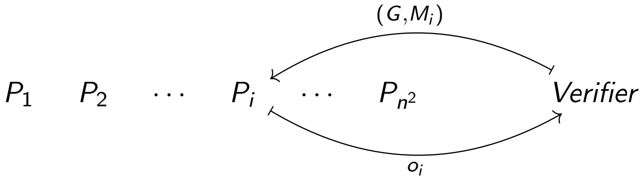
- Let Grid(n, n) be the $n \times n$ grid width nodes labelled by $1, \ldots, n^2$.
- If $G \leq$ Grid(n, n) is a graph, then

$$|G\rangle := \Big(\prod_{\{w,v\} \in E} CZ_{w,v} \Big) |+\rangle^{n^2}$$

where $|+\rangle := (|0\rangle + |1\rangle)/\sqrt{2}$.



The 2D Graph State Game



At each round:

- Verifier picks $G \leq \text{Grid}(n, n)$ and $M \in \bigotimes_{i=1}^{n^2} \{X, Y\}$.
- Each player $i = 1, \ldots, n^2$ returns a value $o_i \in \{+1, -1\}$.
- Players win if:

$$\text{Prob}_{|G\rangle}((o_1, o_2, \ldots, o_{n^2}) \mid M) > 0$$

Bravyi, Gosset, König's Result

- Suppose that each $i \in [1, n^2]$ has a *neighbourhood* $N_i \subset [1, n^2]$.
- A classical/quantum strategy is N-local if it only involves classical/quantum communication from $i \in [1, n^2]$ to $j \in [1, n^2]$ if $i \in N_j$.

Theorem (Bravyi-Gosset-König)

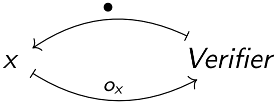
- *The 2D-Graph State Game of any size n is solved by an N-local quantum strategy with every $\#N_i \leq k$, where k is independent of n.*
- *For all sufficiently large n any classical N-local strategy with every $\#N_i \leq n^{1/4}$ fails on some instance (G, M) with probability at least $1/4$.*

Non-local games

A game over a measurement scenario $(X, \mathcal{M}, O)$ is a tuple (Q, C, A) where

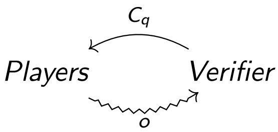
- Q is a set of questions.
- $C_q \in \mathcal{M}$ for every $q \in Q$.
- $A_q \subset O^{C_q}$ for every $q \in Q$.

At each round:

- Verifier picks $q \in Q$ and for each $x \in C_q$: 

- The Players win if $(x \mapsto o_x) \in A_q$.

Communication and Contextuality



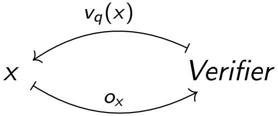
- The *behaviour* of the players: $b = \{b_C \in \mathcal{D}(O^C)\})_{C \in \mathcal{M}}$.
- $\text{Prob}(\text{Players win} \mid q, b) := b_{C_q}(A_q)$.
- No-signalling behaviours = Empirical models.
- Classical no-communication strategies $\leftrightarrow$ Non-contextual behaviours

Value passing

Suppose that (Q, C, A) is a game over $(X, \mathcal{M}, O)$ and furthermore

- V_x is a set of values for each $x \in X$.
- $v_q(x) \in V_x$ is a value for every $q \in Q$ and $x \in X$.

The game



can be modelled as a game on a different scenario $(Y, \mathcal{N}, P)$.

N-local strategies

Let $\mathcal{G} = (Q, C, A)$ be a game on $(X, \mathcal{M}, O)$ and $N = \{N_x \subset X\}_{x \in X}$.

Definition

A classical strategy Cl is *N-local* if:

$$(x \xrightarrow{\mathrm{m}} y) \in \mathrm{Cl} \implies x \in N_y$$

for all players $x, y \in X$.

We show:

Classical N-local strategies for $\mathcal{G} \leftrightarrow$ Non-contextual behaviours for $\mathcal{G}[N]$

where $\mathcal{G}[N]$ is constructed from $\mathcal{G}$ by adding the value

$$v_q(x) := N_x \cap C_q$$

Contextuality as a Resource

- Empirical models are resources.
- Quality of a resource e is proportional to the degree of contextuality as measured by the *contextual fraction* $CF(e)$
  - Completely (strongly) contextual: $\mathrm{CF}(e) = 1$.
  - Non-contextual: $\mathrm{CF}(e) = 0$.

Abramsky et al.:

$$\epsilon \geq (1 - \mathrm{CF}(e)) \cdot v(f)$$

ϵ - degree of error, $\mathrm{CF}(e)$ - degree of contextuality, $v(f)$ - hardness of task.

Example: $l2 - MBQC$

Theorem (Raussendorf)

If an l2-MBQC deterministically computes a non linear function (mod 2) using a resource e then $NCF(e) = 0$.

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Theorem (Abramsky et al.)

If an l2-MBQC probabilistically computes a function $f : \mathbb{B}^m \to \mathbb{B}^n$ then

$$p_F \geq NCF(e) \cdot v(f)$$

where $v(f) \in [0, 1]$ measures the distance from f to the closest mod 2 linear function.

Example: Games for measurement scenarios

Theorem (Abramsky et al.)

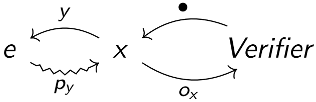
Let e be a no-signalling behaviour for a non-local game $\mathcal{G}$. For some question q

$$Prob(Players\ lose \mid q, e) \geq NCF(e) \cdot v(\mathcal{G})$$

where $v(\mathcal{G})$ is a measure of the hardness of $\mathcal{G}$.

Measurement Aided Strategies

- Suppose that the players share a resource $e : (Y, \mathcal{N}, P)$.
- If Verifier picks $q \in Q$ then for all $x \in C_q$:



- We show:

$$\text{Classical } N\text{-local strategies with resource } e \text{ for } \mathcal{G}.$$
$$\updownarrow$$
$$\text{No-signalling behaviours } e' \text{ for } \mathcal{G}[N] \text{ with a simulation } s : e \to e'.$$

A resource sensitive BGK theorem

Theorem

For all sufficiently large n:

- *For every N-local measurement aided strategy for* $\mathcal{G}_{2D}^n$ *there is a question q such that*

$$Prob(Players\ lose \mid q) \geq NCF(e) \cdot \frac{1}{4}$$

where e is the resource and $\#N_i \leq n^{1/4}$ *for every* $i \in [1, n^2]$.

Proof sketch:

Let Cl be a classical N-local strategy with resource $e : (Y, \mathcal{N}, P)$ for $\mathcal{G}^n_{2D}$.

- $p_F(\mathcal{G}^n_{2D}, Cl) = p_F(\mathcal{G}^n_{2D}[N], e')$ where $s : e \to e'$.
- By Abramsky et al.: $p_F(\mathcal{G}^n_{2D}[N], e') \geq \text{NCF}(e') \cdot v(\mathcal{G}^n_{2D}[N])$.
- Because s is a simulation: $\text{NCF}(e') \geq \text{NCF}(e)$.

$$\begin{aligned} p_F(\mathcal{G}^n_{2D}, Cl) &= p_F(\mathcal{G}^n_{2D}[N], e') \\ &\geq \text{NCF}(e') \cdot v(\mathcal{G}[N]) \\ &\geq \text{NCF}(e) \cdot v(\mathcal{G}[N]) \end{aligned}$$

For all sufficently large n: $\#N_i \leq n^{1/4} \; \forall i \implies v(\mathcal{G}[N]) \geq \frac{1}{4}$

Summary

- Framed *BGK* as a theorem about games and N-local strategies.
- Considered N-local classical with a resource.
- Refined *BGK*'s bound using the inequality

$$p_F \geq \mathrm{NCF}(e) \cdot v(\mathcal{G})$$