On the Hardness of Robust Classification

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Question

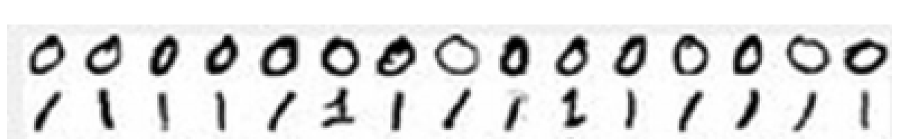
What distributional assumptions are needed and how much power can we give an adversary to ensure efficient robust learning?

Problem Setting

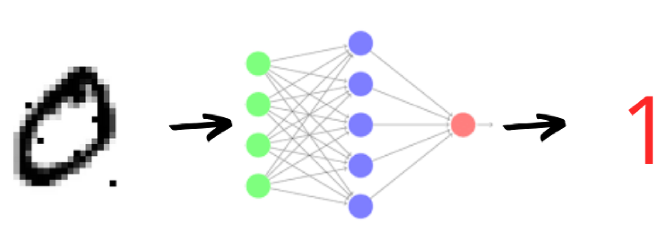
Our paper:

- Binary classification
- Binary feature vectors (input space: $\mathcal{X} = \{0,1\}^n$)
- An adversary can modify input bits after training (*evasion attacks*)

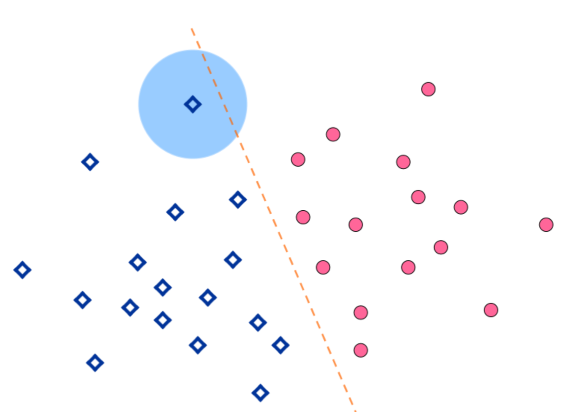
For example, we wish to be able to differentiate between 0's and 1's:



The image of a 0 should not be classified as a 1 if it is slightly perturbed by an adversary:



Efficient Robust Learning:
In general, we want to prove or disprove the existence of an algorithm with *polynomial sample complexity* (in the learning parameters and input dimension n) that will output a hypothesis such that the probability of drawing a new point that can be perturbed by an adversary and resulting in a misclassification to be small:



But what counts as a misclassification?

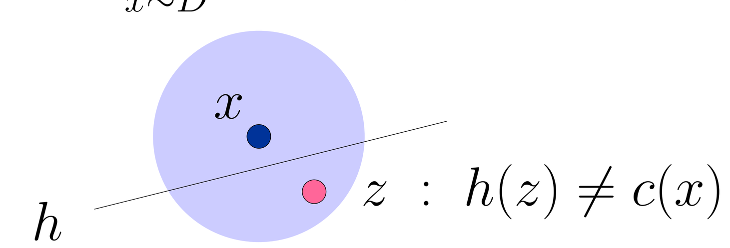
Take Away

- Inadequacies of widely-used definitions of robustness surface under a learning theory perspective.
- It may be possible to only solve robust learning problems with strong *distributional assumptions*.
- Easy proof for computational hardness of robust learning.

Robust Risk Definitions

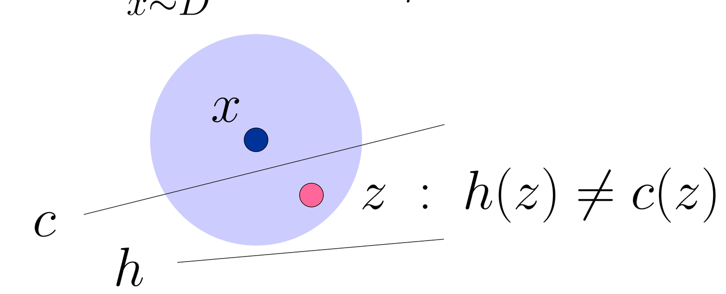
Constant-in-the-ball:

$$\mathsf{R}^C_\rho(h,c) = \mathbb{P}_{x\sim D}\left(\exists z \in B_\rho(x) : h(z) \neq c(x)\right) \ .$$

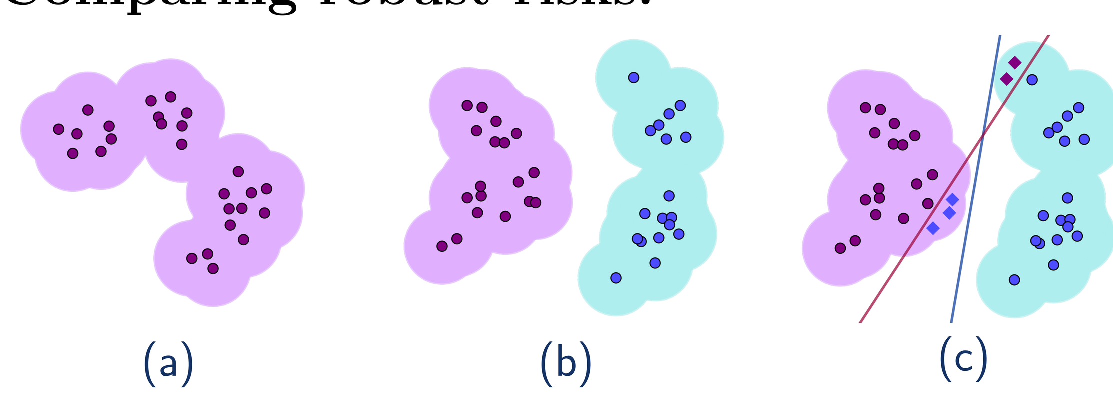


Exact-in-the-ball:

$$\mathsf{R}^E_\rho(h,c) = \mathbb{P}_{x\sim D}\left(\exists z \in B_\rho(x) : h(z) \neq c(z)\right) \ .$$



Comparing robust risks:



(a) $\mathsf{R}^C_\rho(h,c) = 0$ only achievable if c is constant.
(b) There exist h such that $\mathsf{R}^C_\rho(h,c) = 0$.
(c) R^C_ρ and R^E_ρ differ. The red concept is the target, while the blue one is the hypothesis. The dots are the support of the distribution and the shaded regions represent their ρ-expansion. The diamonds represent perturbed inputs which cause $\mathsf{R}^E_\rho > 0$, while $\mathsf{R}^C_\rho = 0$.

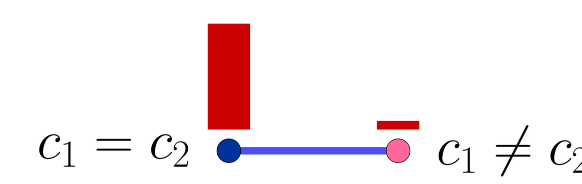
For us, adversary's power: create perturbations that cause the hypothesis and target functions to disagree, so we use the *exact-in-the-ball* definition.

Distribution-Free Robust Learning

Theorem: *Any concept class $\mathcal{C}$ is efficiently distribution-free robustly learnable if and only if it is trivial.*

A class of functions is *trivial* if $\mathcal{C}_n$ has at most two functions, and that they differ on every point.

Distributional assumptions are *essential*:



Monotone Conjunctions

Question: How much power can we give an adversary and still ensure efficient robust learnability?

Monotone conjunctions:

$$\text{thesis} \wedge \text{sleep deprivation} \wedge \text{caffeine}$$

Theorem: *The threshold to robustly learn monotone conjunctions under log-Lipschitz distributions is $\rho(n) = O(\log n)$.*

$\rho = O(\log n)$: PAC algorithm is a robust learner.
$\rho = \omega(\log n)$: no sample-efficient learning algorithm exists.

Log-Lipschitz Distributions:

$$\begin{matrix} x_1 = (0,\dots,1,1,1,\dots,0) \\ x_2 = (0,\dots,1,0,1,\dots,0) \end{matrix} \implies \frac{D(x_1)}{D(x_2)} \leq \alpha \ .$$

For e.g.: uniform distribution, product distribution where the mean of each variable is bounded, etc.

Intuition: input points that are close to each other cannot have vastly different probability masses.

Computational Hardness

- An information-theoretically easy problem can be computationally hard.
- We give a simple proof of the computational hardness of robust learning result of [1].
- We reduce a computationally hard PAC learning problem to a robust learning problem.
- We use the trick from [1] of encoding a point's label in the input for the robust learning problem.

Reduction. Take a PAC learning problem for concept and distribution classes $\mathcal{C}$ and $\mathcal{D}$ defined on $\mathcal{X} = \{0,1\}^n$. Define φ_k as follows:

$$\varphi_k(x) := \underbrace{x_1 \dots x_1 x_2 \dots x_{d-1} x_d \dots x_d}_{2k+1 \text{ copies of each } x_i} c(x) \ ,$$

1. Blow up input space to $\mathcal{X}' = \{0,1\}^{(2k+1)n+1}$.
2. New concept class:
$$\mathcal{C}' = \{c \circ \mathrm{maj}_{2k+1} \mid c \in \mathcal{C}\} \ ,$$
where maj_l returns the majority vote on each subsequent block of l bits, and ignores the last bit.
3. Distribution family $\mathcal{D}'$: for each $c \in \mathcal{C}$, $D \in \mathcal{D}$, we have a new D' as follows for $z \in \mathcal{X}'$:
$$D(z) = \begin{cases} D(x) & z = \varphi_k(x), \\ 0 & \text{otherwise.} \end{cases}$$

Reasoning.

- Any algorithm for learning $\mathcal{C}$ w.r.t. $\mathcal{D}$ yields an algorithm for learning the pairs $\{(c', D')\}$.
- A *robust* learner cannot only rely on the last bit of $\varphi_k(x)$ (it could be flipped by an adversary).
- A *robust* learner can be used to PAC-learn $\mathcal{C}_n$.

References

[1] Sébastien Bubeck, Eric Price, and Ilya Razenshteyn. Adversarial examples from computational constraints. *arXiv preprint arXiv:1805.10204*, 2018.