Design and Analysis of Algorithms

Part 7

Greedy Algorithms

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The greedy approach [CLRS 16.2]

Greedy algorithms are typically used to solve optimisation problems.

The solution is constructed step by step.
At each step, the algorithm makes the choice that offers *the greatest immediate benefit* (also called the *greedy choice*).
A choice made at one step is *not* reconsidered at subsequent steps.

Example: Dijkstra's algorithm

Optimization problem: find all shortest paths from the source.
Construction of the solution: shortest paths built vertex by vertex.
Greedy choice: at each step, choose the closest reachable vertex.

The greedy approach does *not* always work: for some problems, it fails to produce an optimal solution. But *when it does work*, it is attractive:

- It is conceptually simple.
- It does not require us to compare candidate solutions, or to keep a record of them.

Warm up: Coin Changing [CRLS problem 16-1]

Suppose we are in a country with the following coin denominations:
quarters (25 cents), dimes (10 cents),
nickels (5 cents), and pennies (1 cent).

Problem: *Assuming an unlimited supply of coins of each denomination, find the minimum number of coins needed to make change for n cents.*

Example. n = 89 cents. What is the optimal solution?

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Example. n = 89 cents. What is the optimal solution?
Answer: 8 coins (3 quarters, 1 dime and 4 pennies).

A greedy algorithm

- Construct the solution coin by coin, reducing the amount at each step.
- Greedy choice: at each step, choose the coin of the largest denomination that does not exceed the remaining amount.

Exercise. Prove that in this case the greedy algorithm yields the optimal solution, and find a choice of coin denominations for which the greedy algorithm does *not* yield the optimal solution.

Minimum spanning trees [CLRS 23]

Example problem:

- A gas company undertakes to supply gas to all villages within a region.
- The cost of laying a pipeline between two villages depends on the distance between them, ease of access, etc.

Task: *Find the cheapest way to lay pipelines to reach every village.*

Graph theoretic formulation

Input: Connected undirected graph $G = (V, E)$ with *weights* $w : E \longrightarrow \mathbb{R}_{\geq 0}$.
Task: Find a connected subgraph that has minimum weight and connects all the vertices of G.

Since the weights are nonnegative, there exists an optimal subgraph without cycles, i.e., a spanning tree. Such a tree is called a ***minimum spanning tree***.

Clarifications

- Since the graph is *undirected*, it is assumed that the weight function w is ***symmetric***, namely $w(u, v) = w(v, u)$ for every $(u, v) \in E$.
- A ***tree*** is an ***undirected*** graph that is ***connected and has no cycles***.
- We will often identify a ***tree*** by its ***set of edges*** $T \subseteq E$.

Basic facts about trees [CLRS Appendix B.5]

Lemma 1. *An undirected graph is a tree iff each pair of vertices are connected by a* **unique** *(simple) path.*

Proof. A connected undirected graph has a cycle iff there exist two vertices connected by distinct paths. □

Basic facts about trees [CLRS Appendix B.5]

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Proof. A connected undirected graph has a cycle iff there exist two vertices connected by distinct paths. □

Lemma 2. *If a graph $G = (V, E)$ is a tree, then $|E| = |V| - 1$.*

The proof is divided in two parts:

- □ G connected $\implies |E| \geq |V| - 1$
- □ G acyclic $\implies |E| \leq |V| - 1$

(continues on the next slide)

G **connected** $\Longrightarrow |E| \geq |V| - 1$

Proof

Set $n = |V|$.
A graph of n vertices and no edges has n connected components.
Adding an edge reduces the number of connected components by at most 1.
To reduce the number of components to 1,
we need to add at least $n - 1$ edges. $\square$

G acyclic $\Longrightarrow |E| \leq |V| - 1$

Proof By induction on $n = |V|$.

Base case. For $n = 1$, an acyclic graph must have $|E| = 0$.

Inductive step. Suppose that $|E| \leq |V| - 1$ for every acyclic graph with $|V| \leq n$, and consider a graph G with $|V| = n + 1$.
There are two possibilities:

1. G has more than one connected component. Then, the induction hypothesis implies $|E_i| \leq |V_i| - 1$ for each component i, and therefore, $|E| \leq |V| - 1$.
2. G has only one connected component.
   Since G is acyclic, removing one edge cuts the graph into two connected components, each satisfying $|E_i| \leq |V_i| - 1$.
   In total:

$$\begin{aligned} |E| &= |E_1| + |E_2| + 1 \\ &\leq (|V_1| - 1) + (|V_2| - 1) + 1 \\ &= |V| - 1\,. \end{aligned}$$

□

Spanning trees

A ***spanning tree*** of a graph $G = (V, E)$ is a subgraph with edge-set $T \subseteq E$ such that

- T is a tree.
- T reaches all the vertices of G:
  for each $u \in V$, there is some $v \in V$ such that (u, v) or (v, u) is in T

Lemma 3. *Every connected graph has a spanning tree.*

Proof. Start from $T = \emptyset$.
Take edges from E and add them to T so long as no cycles are formed.
This procedure constructs a maximal acyclic subgraph T of G.
Now, T must be connected, for if not, since G is connected it would be possible to add another edge to T without making a cycle.

Since T is acyclic and connected, it is a tree. $\square$

Definition: Minimum Spanning Tree (MST)

Let $G = (V, E)$ be a weighted graph, i.e. a graph equipped with a function $w : E \to \mathbb{R}$, assigning each edge $e \in E$ its weight $w(e)$.

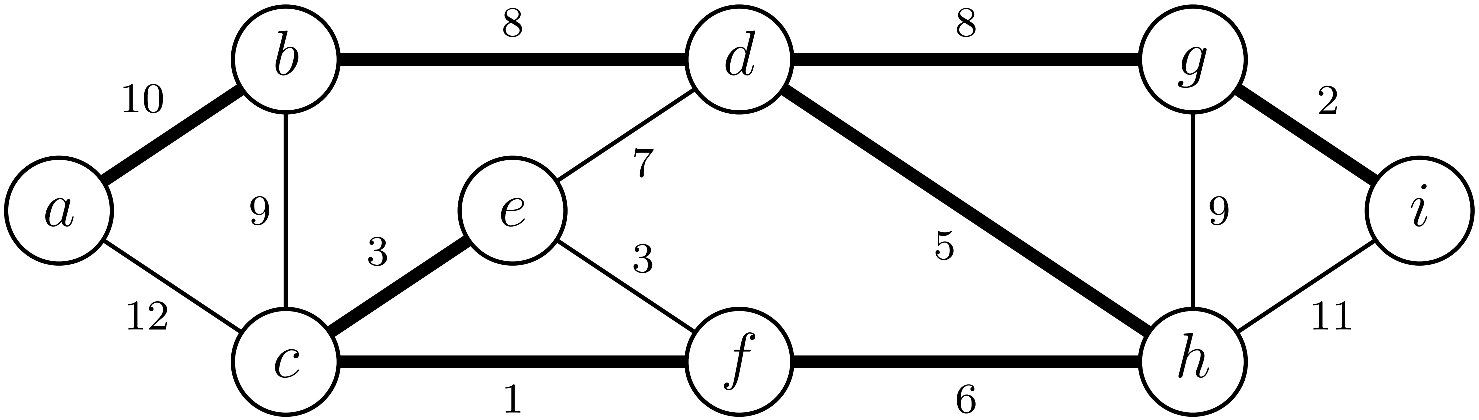
If $T \subseteq E$ is a set of edges, the ***weight of*** T, denoted by $w(T)$, is the sum of the weights of the edges in T.

A ***minimum spanning tree*** (MST) is a spanning tree of minimum weight i.e. there is no spanning tree T' with $w(T') < w(T)$.

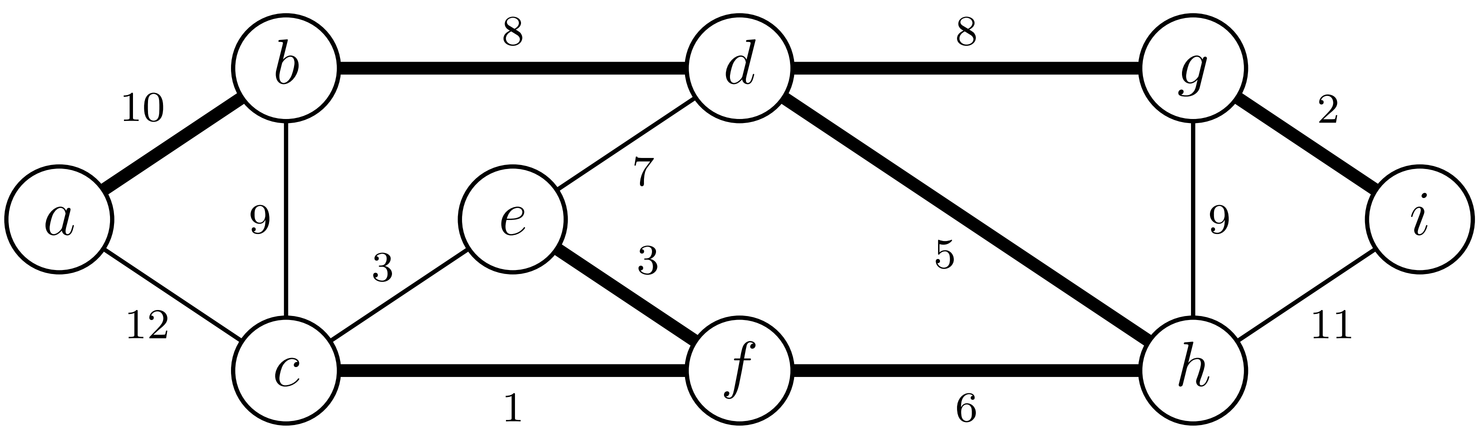
Note: in general, the MST of a graph is *not unique*.
But all MSTs have the *same number of edges*, equal to $|V| - 1$.

Example: Two MSTs

The tree indicated by thick edges is an MST.



Replacing (c, e) by (e, f) gives a different MST.



How to build an MST?

Idea: build the MST edge by edge.

- ☐ Start from $A = \emptyset$.
  By definition A is a (trivial) subset of an MST
- ☐ Add edges to A,
  maintaining the property that A is a subset of some MST.
- ☐ Stop when no edge can be added to A anymore.
  At this point, A will be an MST.

Definition. Let $A \subseteq E$ be a subset of an MST T.
We say that an edge (u, v) is ***safe for A*** iff
$A \cup \{(u, v)\}$ is a subset of *some* MST (not necessarily T).

To build an MST, we start from $A = \emptyset$ and we add a safe edge at each step.

Generic MST algorithm [CLRS 23.1]

GENERIC-MST(V, E, w)

```
A = ∅
while A is not a spanning tree
    find an edge (u, v) that is safe for A
    A = A ∪ { (u, v) }
return A.
```

Loop invariant: *A is safe i.e. a subset of some MST.*

Initialization: The invariant is trivially satisfied by $A = \emptyset$.

Termination: All edges added to A are in an MST, so upon termination, A is a spanning tree that is also an MST.

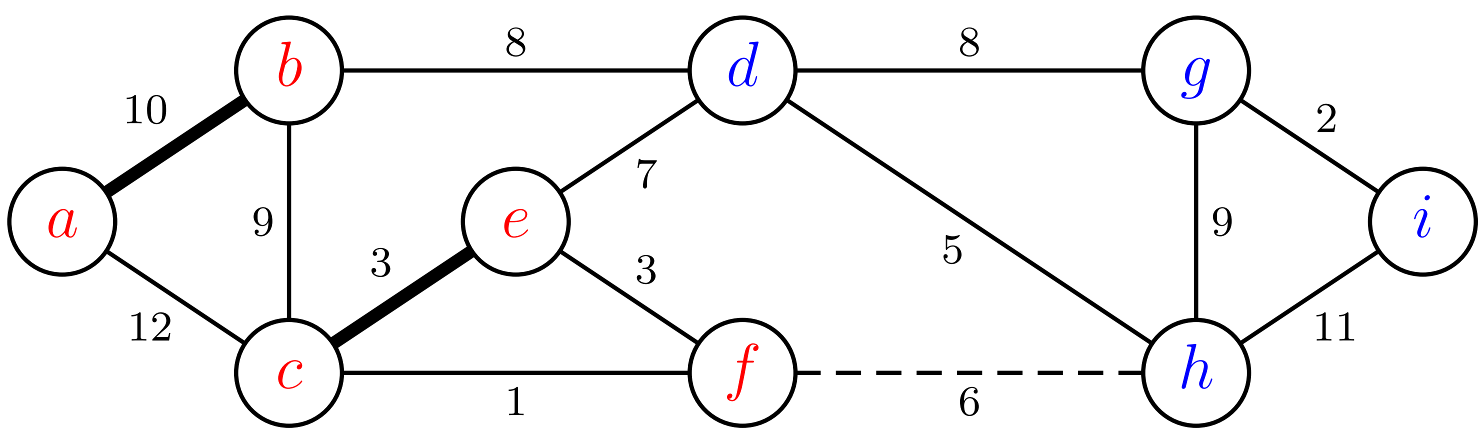
Maintenance: Since only safe edges are added,
A remains a subset of some MST.

How to find safe edges: preliminary definitions

Let $G = (V, E)$ be an undirected graph.

- A ***cut*** is a partition of the vertex-set into two subsets S and $V \setminus S$.
- An edge $(u, v) \in E$ ***crosses*** a cut $(S, V \setminus S)$ if one endpoint is in S and the other in $V \setminus S$.
- A cut ***respects*** $A \subseteq E$ if no edge in A crosses the cut.
- An edge is a ***light edge crossing a cut*** if its weight is minimum over all edges that cross the cut.

Example



$S = \{a, b, c, e, f\}$ $\quad A = \{(a, b)\ ,\ (c, e)\}$ $\quad$ light edge: (f, h)

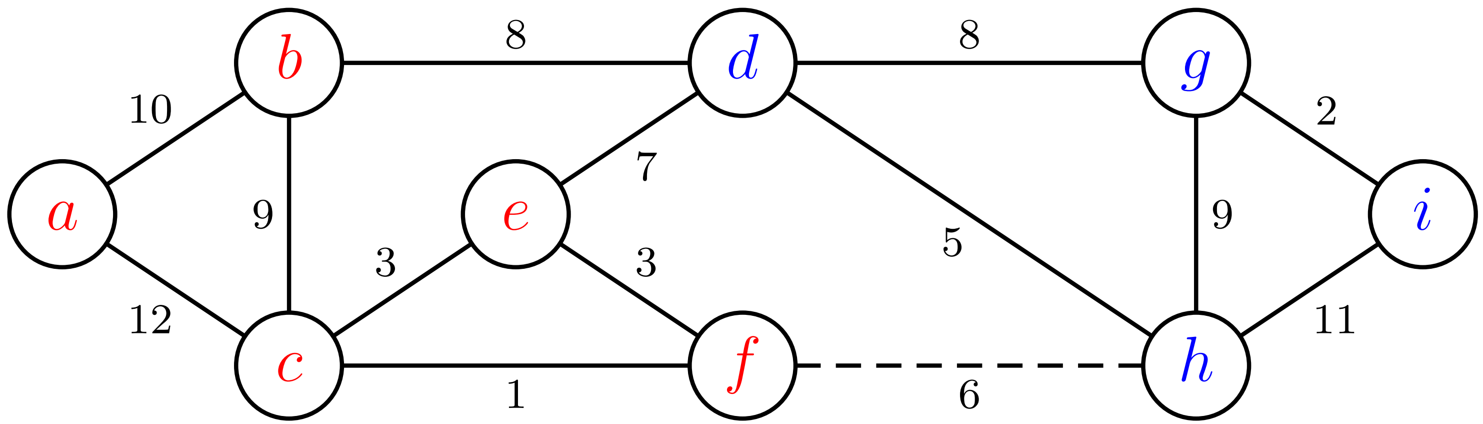
How to find safe edges: the Cut Lemma

Lemma 4 (Cut)**.** *Let* A *be a subset of some MST.*
If $(S, V \setminus S)$ *is a cut that respects* A*, and* (u, v) *is a light edge crossing the cut, then* (u, v) *is safe for* A*.*

How to find safe edges: the Cut Lemma

Lemma 4 (Cut). *Let A be a subset of some MST.*
If $(S, V \setminus S)$ is a cut that respects A, and (u, v) is a light edge crossing the cut, then (u, v) is safe for A.

Example: Assume that $A = \{(a, b), (c, e)\}$ is included in some MST.



(f, h) is a light edge crossing the cut.
Hence, $A' = \{(a, b), (c, e), (f, h)\}$ is included in an MST.

Proof of the Cut Lemma

Let T be a MST that includes A. Since T is a tree, it contains a **unique** path p between u and v.

Path p must cross the cut $(S, V - S)$ at least once. Let (x, y) be an edge of p that crosses the cut.

Adding (u, v) and removing (x, y) creates a tree T'.

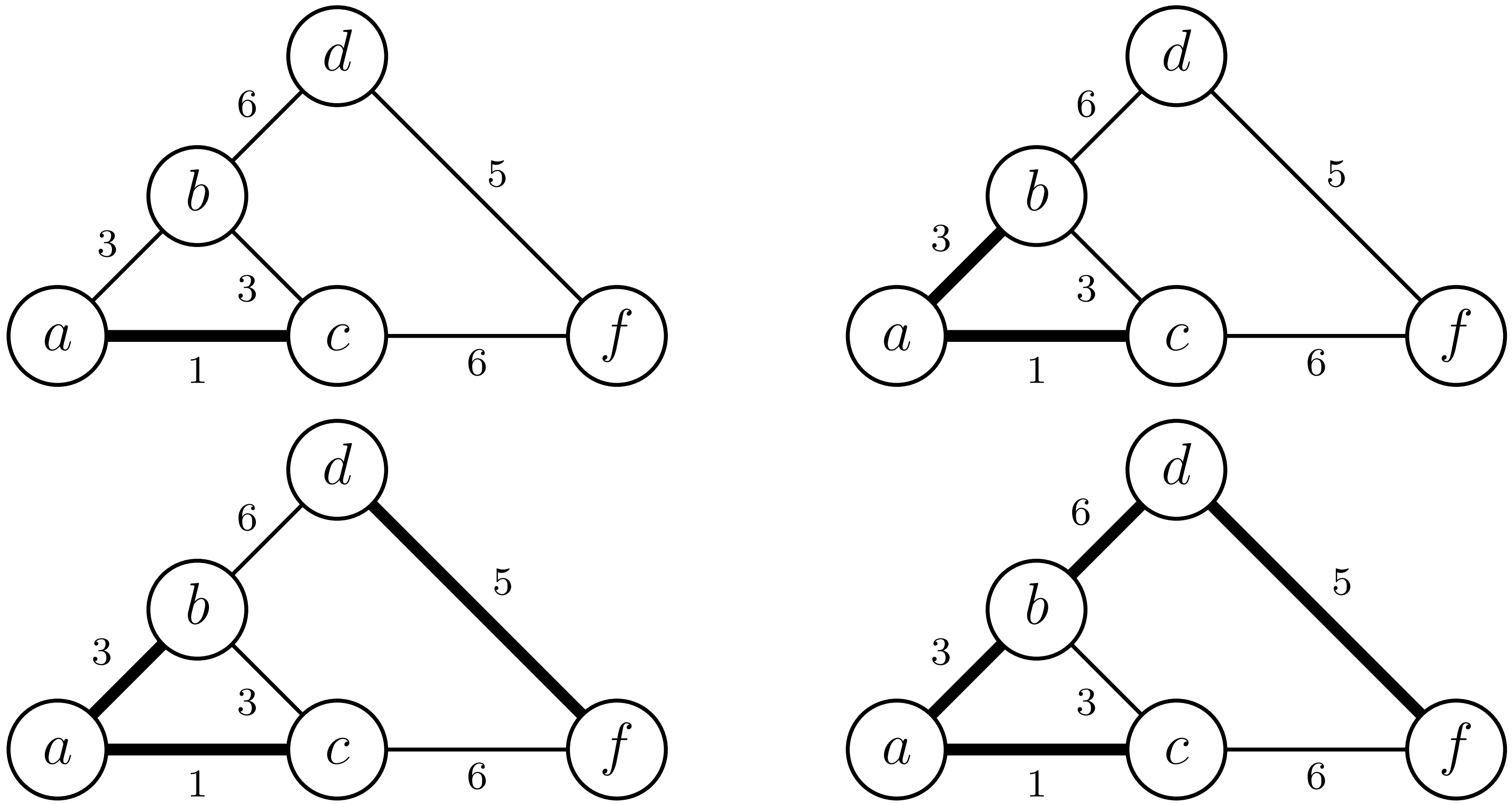
- Why is T' a tree? Because by adding (u, v), we create a cycle that consists of p and (u, v). This cycle contains edge (x, y), which when removed leaves the graph connected with $|V| - 1$ edges.
- Why is it minimum? Because the weight of T' is $w(T') = w(T) - w(x, y) + w(u, v) \leq w(T)$.
- Does T' include A? Yes, because A was included in T, and A did not contain (x, y), the only edge we removed from T.

Kruskal's algorithm [CLRS 23.2]

Idea: Start from $A = \emptyset$

At every step, pick the edge with the smallest weight and add it to A, *if it does not create cycles.*

Example:



How to avoid cycles

Kruskal's algorithm is simple and intuitive, but how does the computer check whether adding an edge introduces a cycle?

Idea: keep track of the connected components.
At each step, the set $A \subseteq E$ divides V into connected components.
We can add an edge only if it connects two distinct components.

To implement Kruskal's algorithm we need a data structure that

- tells us whether two vertices u and v are in the same connected component
- merges two components when we put an edge between them.

This data structure is the ***disjoint-set data structure.***

Disjoint-set data structure [CLRS 21]

Disjoint-set data structure

- Maintains a collection $\mathcal{S} = \{ S_1, \cdots, S_k \}$ of disjoint *dynamic* sets (i.e. disjoint sets changing over time).
- Each set is identified by a *representative*, a member of the set.

  It does not matter which member is the representative.

Three basic operations:

1. $\text{Make-Set}(x)$: Makes a new set $\{ x \}$ and add it to $\mathcal{S}$.
2. $\text{Union}(x, y)$: Removes S_x and S_y from $\mathcal{S}$,
   and adds the new set $S_x \cup S_y$ to the collection $\mathcal{S}$.
3. $\text{Find-Set}(u)$: Returns the representative of the set containing u.

Example: Consider the following sequence of operations:
$\text{Make-Set}(a), \text{Make-Set}(b), \text{Union}(a, b), \text{Make-Set}(c), x = \text{Find-Set}(a), \text{Union}(x, c)$. After these operations, $\mathcal{S}$ is $\{ \{ a, b, c \} \}$.

Running times of different implementations

Running time analysis: given in terms of two numbers, m and n.

- m = total number of operations
- n = number of MAKE-SET operations.

Running times of different implementations

1. *Linked-list*: $O(m + n^2)$ time.
2. *Weighted linked-list*: $O(m + n \log n)$ time.
3. *Disjoint-set forest*: $O(m\,\alpha(n))$ time,
   where $\alpha(n)$ is an **extremely slow-growing function**
   (for all practical purposes, $\alpha(n)$ can be treated as a constant).

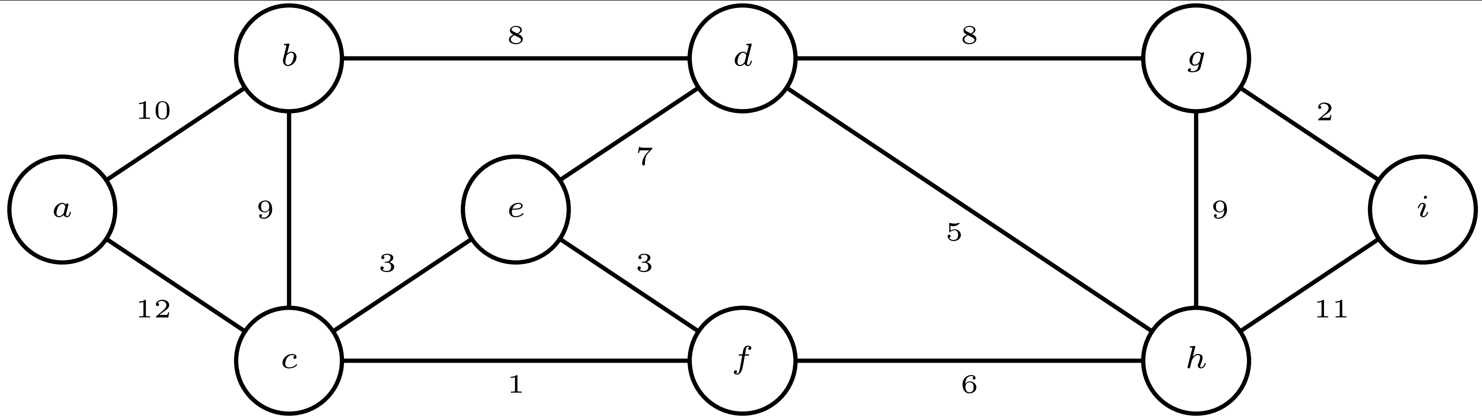
n	$\alpha(n)$
from 0 to 2	0
3	1
from 4 to 7	2
from 8 to 2047	3
from 2048 to $A_4(1) \gg 10^{80}$	4

Kruskal's algorithm

```
KRUSKAL(V, E, w)

A = ∅
for each v ∈ V
    MAKE-SET(v)
Sort E into increasing order by weight w
for each edge (u, v) taken from the sorted list
    if FIND-SET(u) ≠ FIND-SET(v)
        A = A ∪ { (u, v) }
        UNION(u, v)
return A.
```

Example



Iter.	Edge	Add to A?	Connected Components
			$\mid a \mid b \mid c \mid f \mid d \mid e \mid g \mid h \mid i \mid$
1	(c, f)	yes	$\mid a \mid b \mid c, f \mid d \mid e \mid g \mid h \mid i \mid$
2	(g, i)	yes	$\mid a \mid b \mid c, f \mid d \mid e \mid g, i \mid h \mid$
3	(c, e)	yes	$\mid a \mid b \mid c, e, f \mid d \mid g, i \mid h \mid$
4	(e, f)	no	$\mid a \mid b \mid c, e, f \mid d \mid g, i \mid h \mid$
5	(d, h)	yes	$\mid a \mid b \mid c, e, f \mid d, h \mid g, i \mid$
6	(f, h)	yes	$\mid a \mid b \mid c, d, e, f, h \mid g, i \mid$
7	(d, e)	no	$\mid a \mid b \mid c, d, e, f, h \mid g, i \mid$
8	(b, d)	yes	$\mid a \mid b, c, d, e, f, h \mid g, i \mid$
9	(d, g)	yes	$\mid a \mid b, c, d, e, f, g, h, i \mid$
10	(b, c)	no	$\mid a \mid b, c, d, e, f, g, h, i \mid$
11	(g, h)	no	$\mid a \mid b, c, d, e, f, g, h, i \mid$
12	(a, b)	yes	$\mid a, b, c, d, e, f, g, h, i \mid$

Running time of Kruskal's algorithm

- Initializing A takes $O(1)$.
- First for-loop uses $|V|$ MAKE-SET operations.
- Sorting E takes $O(|E| \cdot \log |E|)$.
- Second for-loop takes $2|E|$ FIND-SET and $|V| - 1$ UNION operations.

Hence, $n = |V|$ and $m = \Theta(|V| + |E|) = \Theta(|E|)$ (because the graph is connected).

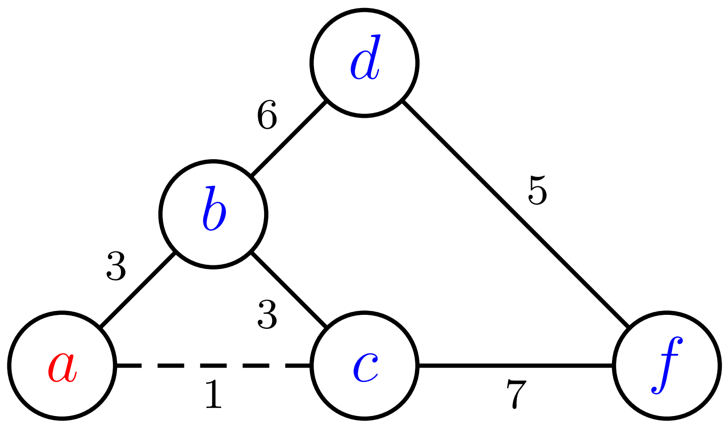
Overall running time:

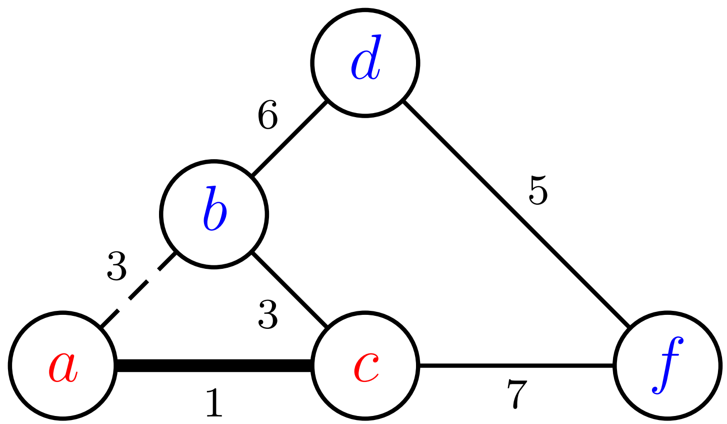
- $O(|E| \log |E| + |V|^2)$ linked-list implementation
- $O(|E| \log |E|)$ weighted linked-list implementation
- $O(|E| \log |E|)$ disjoint-set forest implementation.

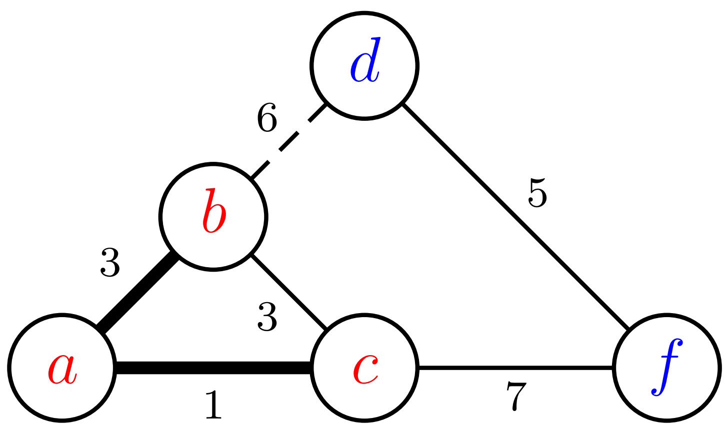
Prims's algorithm [CLRS 23.2]

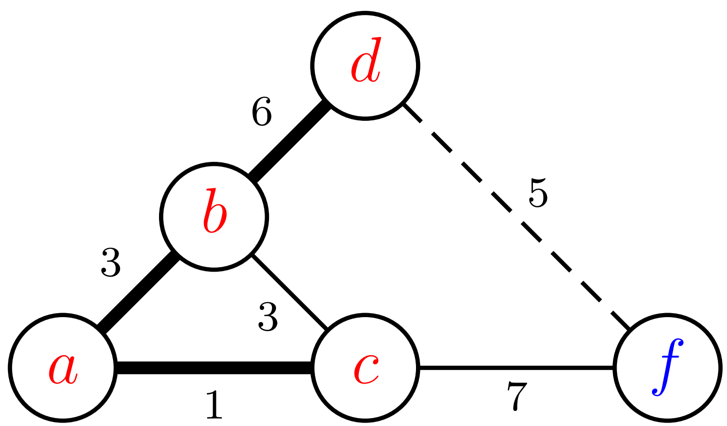
Idea: Pick a vertex $r \in V$ and grow the tree from that vertex.
Set $S = \{ r \}$ and $A = \emptyset$.
At every step, find a light edge (u, v) connecting $u \in S$ to $v \in V \setminus S$.
Update S to $S \cup \{ v \}$ and A to $A \cup \{ (u, v) \}$.

Example:









How to find the light edge?

Construct a **priority queue** Q, such that

- □ $Q = V \setminus S$.
- □ The key of v is the minimum weight of any edge (u, v) where $u \in S$ (If v is not adjacent to any vertex in S, set $key[v] = \infty$).

To find a light edge crossing the cut $(S, V \setminus S)$,
extract the minimum from the queue.
If $v = \textsc{Extract-Min}(Q)$, then exists a light edge (u, v) for some $u \in S$.

The vertex u can be retrieved by a **backpointer**:
when the key of v is set to $key(v) = w(u, v)$, we define $\pi[v] = u$.

Prim's algorithm

PRIM(V, E, w, r)

```
Q = ∅
for each u ∈ V  // Initializes key values and backpointers
    key[u] = ∞
    π[u] = NIL
    INSERT(Q, u)
DECREASE-KEY(Q, r, 0)
while Q ≠ ∅
    u = EXTRACT-MIN(Q)      // finds light edge for cut (Q, V \ Q)
    for each v ∈ Adj[u]     // updates keys and backpointers
        if v ∈ Q and w(u, v) < key[v]
            π[v] = u
            DECREASE-KEY(Q, v, w(u, v))
```

Running time

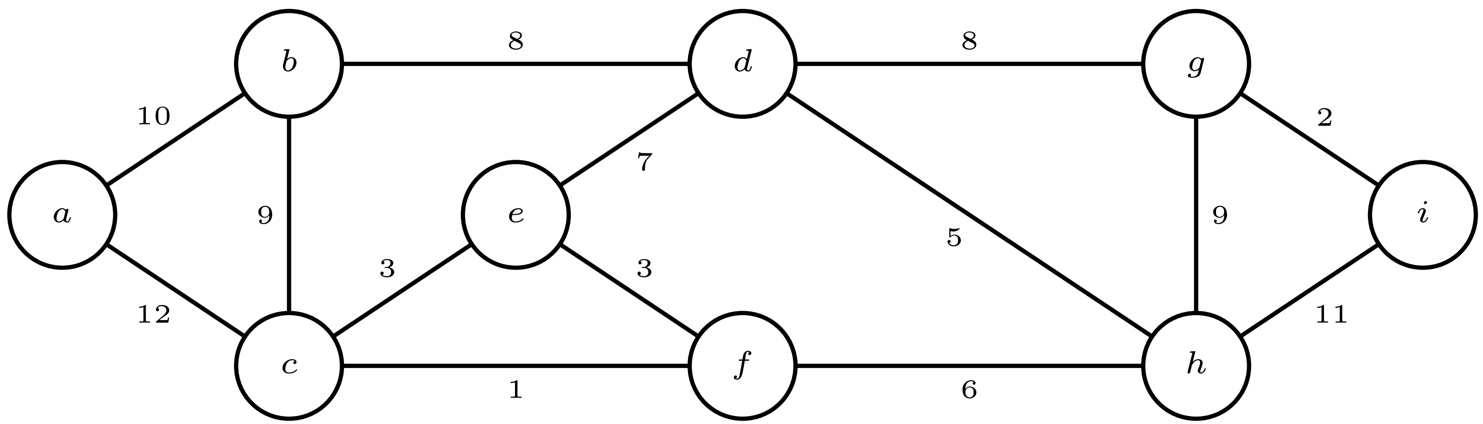
- Initializing Q to $\emptyset$ takes $O(1)$.
- Initializing $key[v]$ and $\pi[v]$ for every vertex takes $O(|V|)$.
- Each DECREASE-KEY operation takes $O(\log |V|)$ (assuming a min-heap implementation of the min-priority queue).
- While-loop takes $|V|$ EXTRACT-MIN and at most $|E|$ DECREASE-KEY operations.

Hence overall running time is $O(|E| \cdot \log |V|)$.

Note. Since $\log |E| = \Theta(\log |V|)$ for a connected graph, PRIM and KRUSKAL have the *same asymptotic running time*.

Note. The running time of PRIM can be improved to $O(|E| + |V| \log |V|)$ using a Fibonacci heap implementation of the min-priority queue.

Example



Iter.	u	Contents of Q								
		a	b	c	d	e	f	g	h	i
		∞, NIL	0, NIL	∞, NIL	∞, NIL	∞, NIL	∞, NIL	∞, NIL	∞, NIL	∞, NIL
1	b	$10, b$	–	$9, b$	$8, b$	∞, NIL	∞, NIL	∞, NIL	∞, NIL	∞, NIL
2	d	$10, b$	–	$9, b$	–	$7, d$	∞, NIL	$8, d$	$5, d$	∞, NIL
3	h	$10, b$	–	$9, b$	–	$7, d$	$6, h$	$8, d$	–	$11, h$
4	f	$10, b$	–	$1, f$	–	$3, f$	–	$8, d$	–	$11, h$
5	c	$10, b$	–	–	–	$3, f$	–	$8, d$	–	$11, h$
6	e	$10, b$	–	–	–	–	–	$8, d$	–	$11, h$
7	g	$10, b$	–	–	–	–	–	–	–	$2, g$
8	i	$10, b$	–	–	–	–	–	–	–	–
9	a	–	–	–	–	–	–	–	–	–

Columns have $key[u], \pi[u]$ for $u \in Q$, and – for $u \notin Q$.

Epilogue: greedy algorithms vs dynamic programming

The ***property of optimal substructure*** —that an optimal solution to a problem is composed of optimal solutions to some of its subproblems — must hold in order to solve an optimisation problem using a dynamic programming algorithm or a greedy algorithm.

In a **dynamic programming algorithm** we typically solve the problem bottom-up: we solve smaller subproblems first, and use their solutions to obtain an optimal solution to a larger subproblem.

In a **greedy algorithm** we typically solve the problem top-down:
we make a greedy choice at each step and then solve the resulting smaller subproblem.