Design and Analysis of Algorithms

Part 3

Data structures as a tool for algorithm design: heaps, heapsort, and priority queues

Elias Koutsoupias
with thanks to Giulio Chiribella

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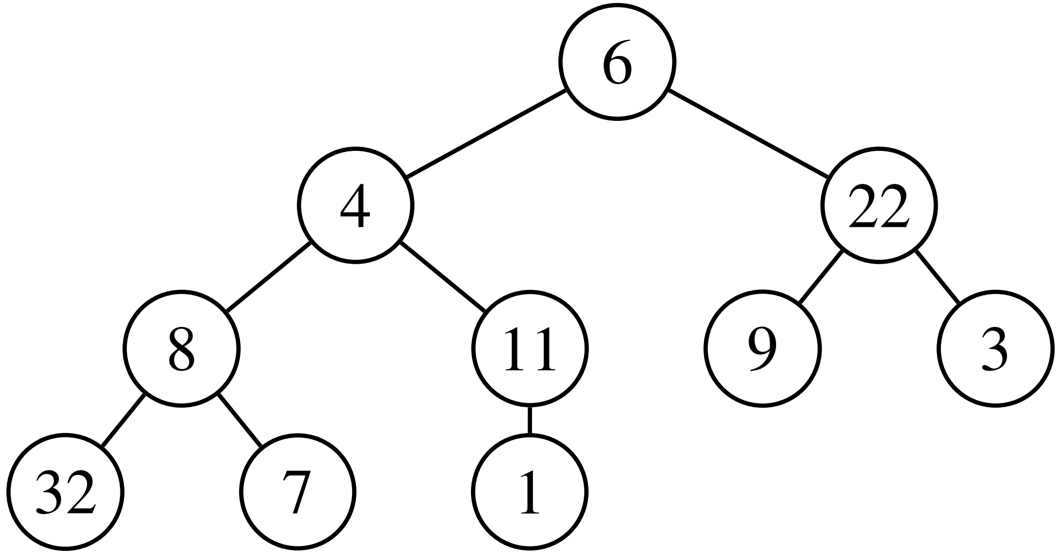
Heaps [CLRS 6.1]

A ***heap*** *is a data structure that organizes data in an* ***essentially complete*** *rooted tree,*
i.e. a rooted tree that is **completely filled on all levels except possibly on the lowest**, *which is* **filled from the left up to a point.**

Example: binary heap, storing numbers (keys) at the nodes of the tree



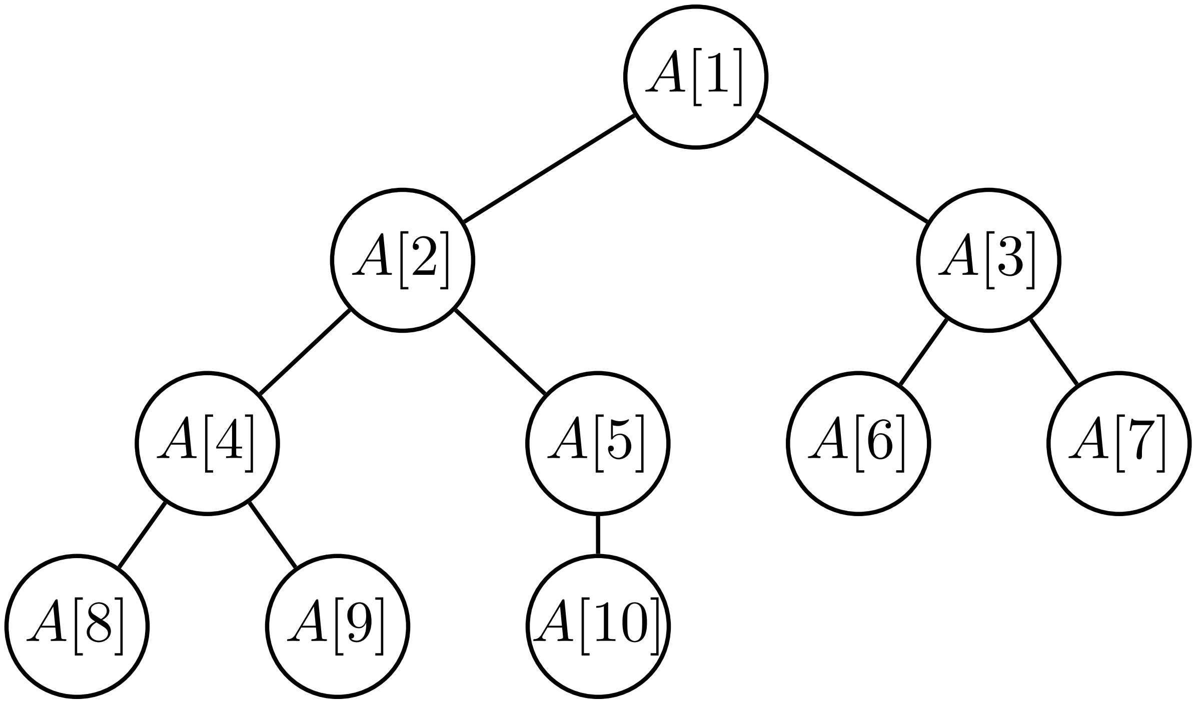
The *height* of a tree is the longest simple path from the root to a leave.
A binary heap with n nodes has height $\lfloor \lg n \rfloor$.

Implementation with arrays

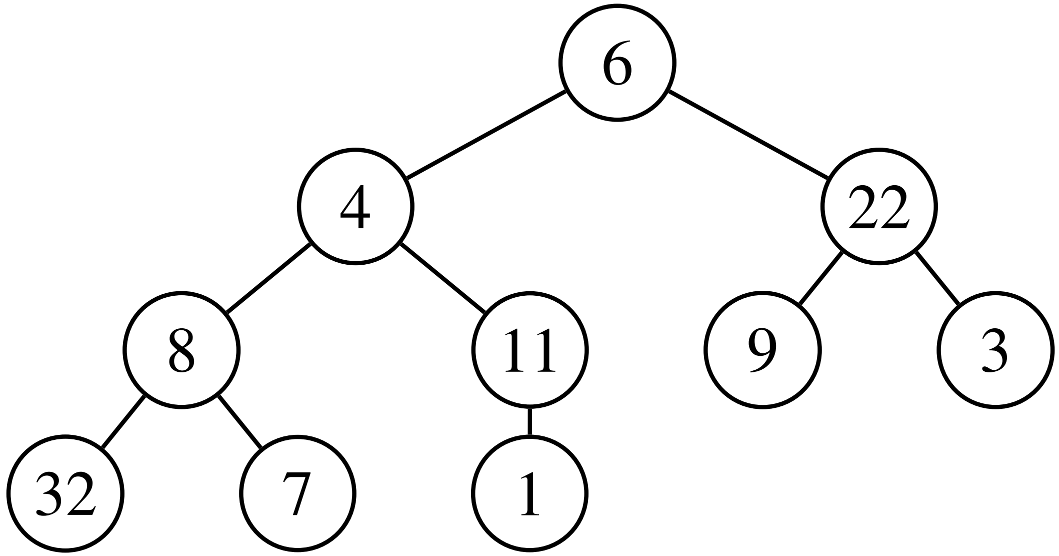
A heap can be implemented by an array without any explicit pointers.

In particular, a *binary* heap can be implemented by an array A as follows:

- Root of the binary tree is $A[1]$
- Left child of $A[i]$ is $A[2i]$.
- Right child of $A[i]$ is $A[2i+1]$.
- Hence, for $i > 0$, the parent of node i is the node $\mathrm{Parent}(i) = \lfloor i/2 \rfloor$.



Example



The heap is stored as the following array:

$$A = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 6 & 4 & 22 & 8 & 11 & 9 & 3 & 32 & 7 & 1 \\ \hline \end{array}$$

Max-heaps

A ***max-heap*** is a heap that satisfies the

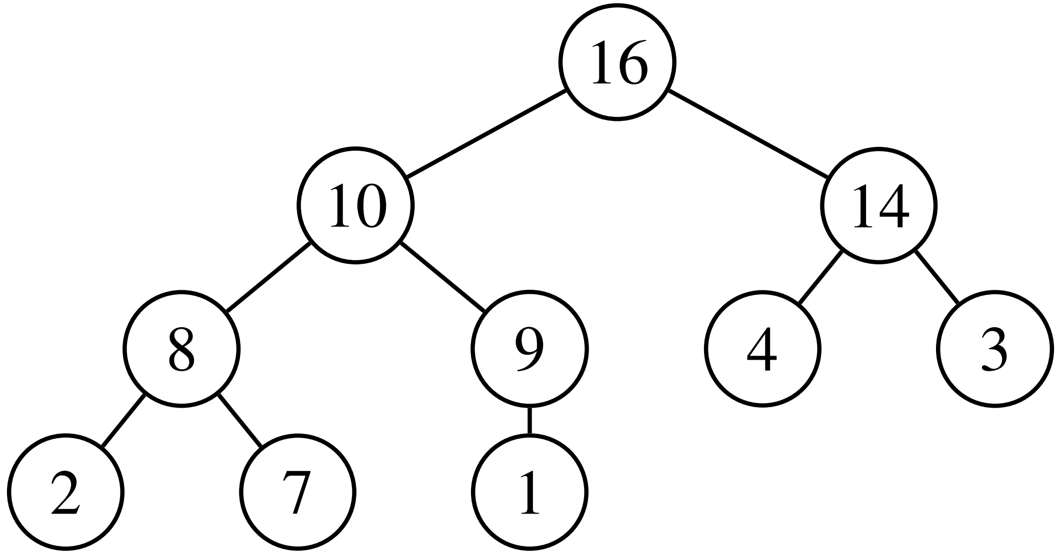
Max-Heap Property: The key of a node (except the root) is less than or equal to the key of its parent.

In the array implementation, the Max-Heap Property Reads:
For all $1 < i \leq A.heap\text{-}size$: $\quad A[i] \leq A[\lfloor i/2 \rfloor]$.

Remarks:

- The maximum element of a max-heap is at the root.
- In the following we will focus on *binary max-heaps*.
  Generally, a max-heap may be k-ary.
- One could also define ***min-heaps***, where the key of each node (except the root) is larger than or equal to the key of its parent.

Example



This is a max-heap. It can be stored in the array

$A =$	16	10	14	8	9	4	3	2	7	1

Note that the array A is *not sorted*:
it does *not* satisfy the property $A[i] \leq A[i-1]$ for every $i > 1$.
However, A satisfies the max-heap property
$A[i] \leq A[\lfloor i/2 \rfloor]$ for every $i > 1$.

Building a max-heap

Given an array A, there is a procedure to turn A into a max-heap:

MAKE-MAX-HEAP(A)

Takes an array A of n integers and rearranges it into a max-heap of size n.

In turn, MAKE-MAX-HEAP is based on the following procedure:

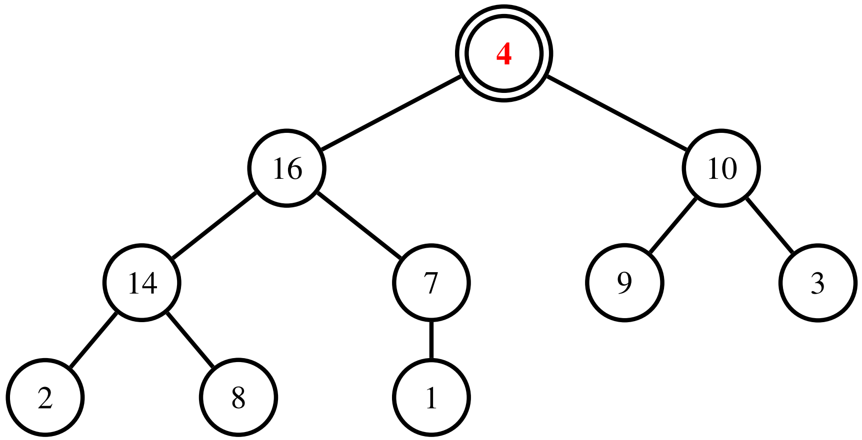
MAX-HEAPIFY(A, i)

Assuming that the left and right subtrees of node i are max-heaps,
MAX-HEAPIFY transforms the subtree rooted at the node i to a max-heap.

MAX-HEAPIFY [CLRS 6.2]

Idea: compare the key at node i with the keys of its children, and rearrange them in order to satisfy the max-heap property.

Example:



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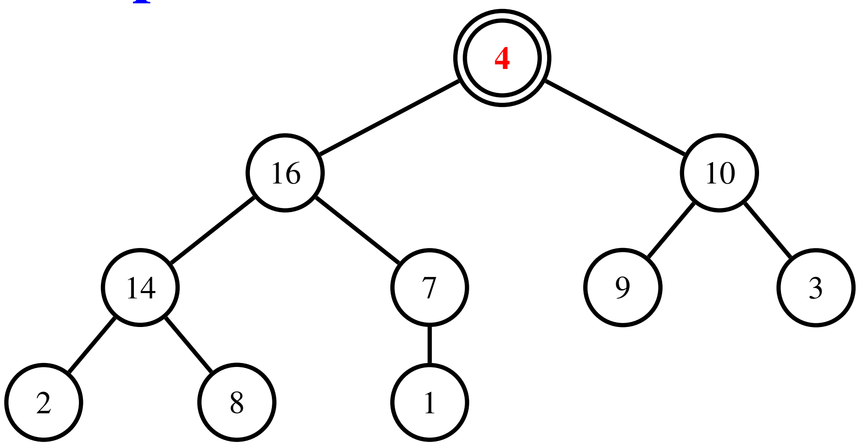

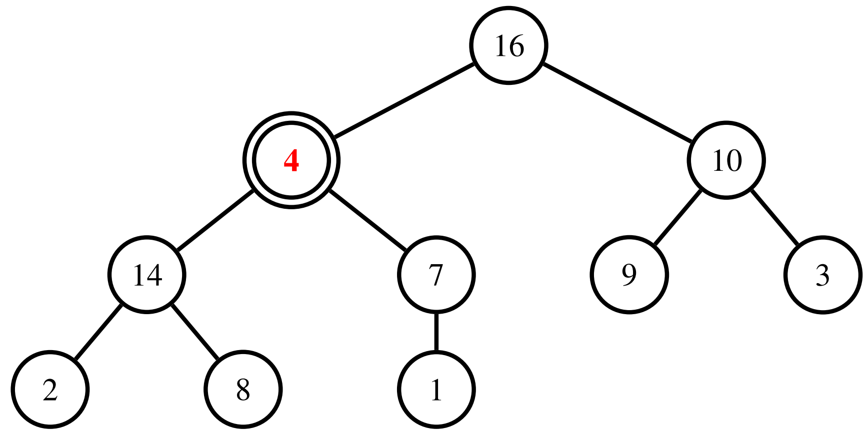

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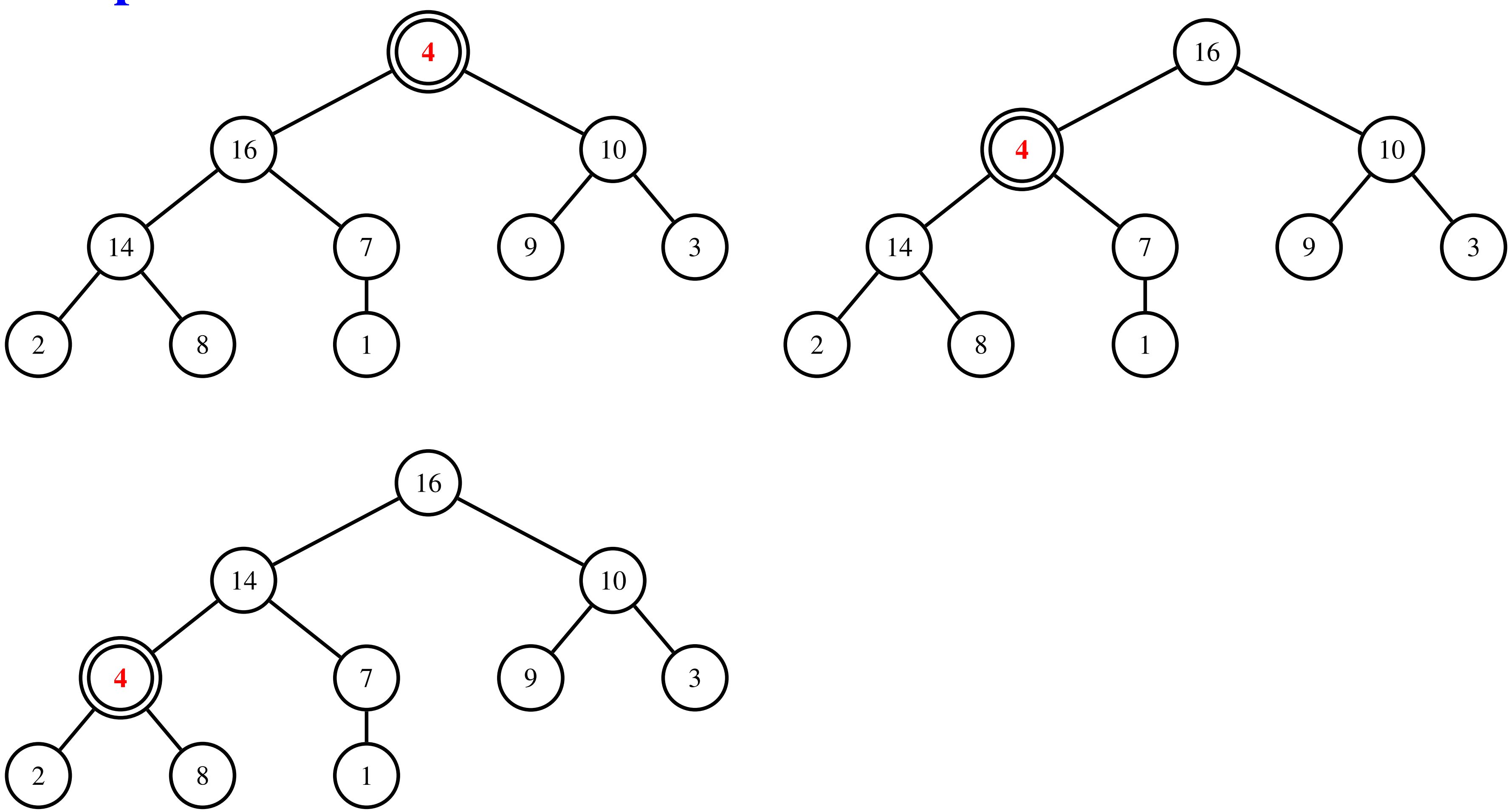

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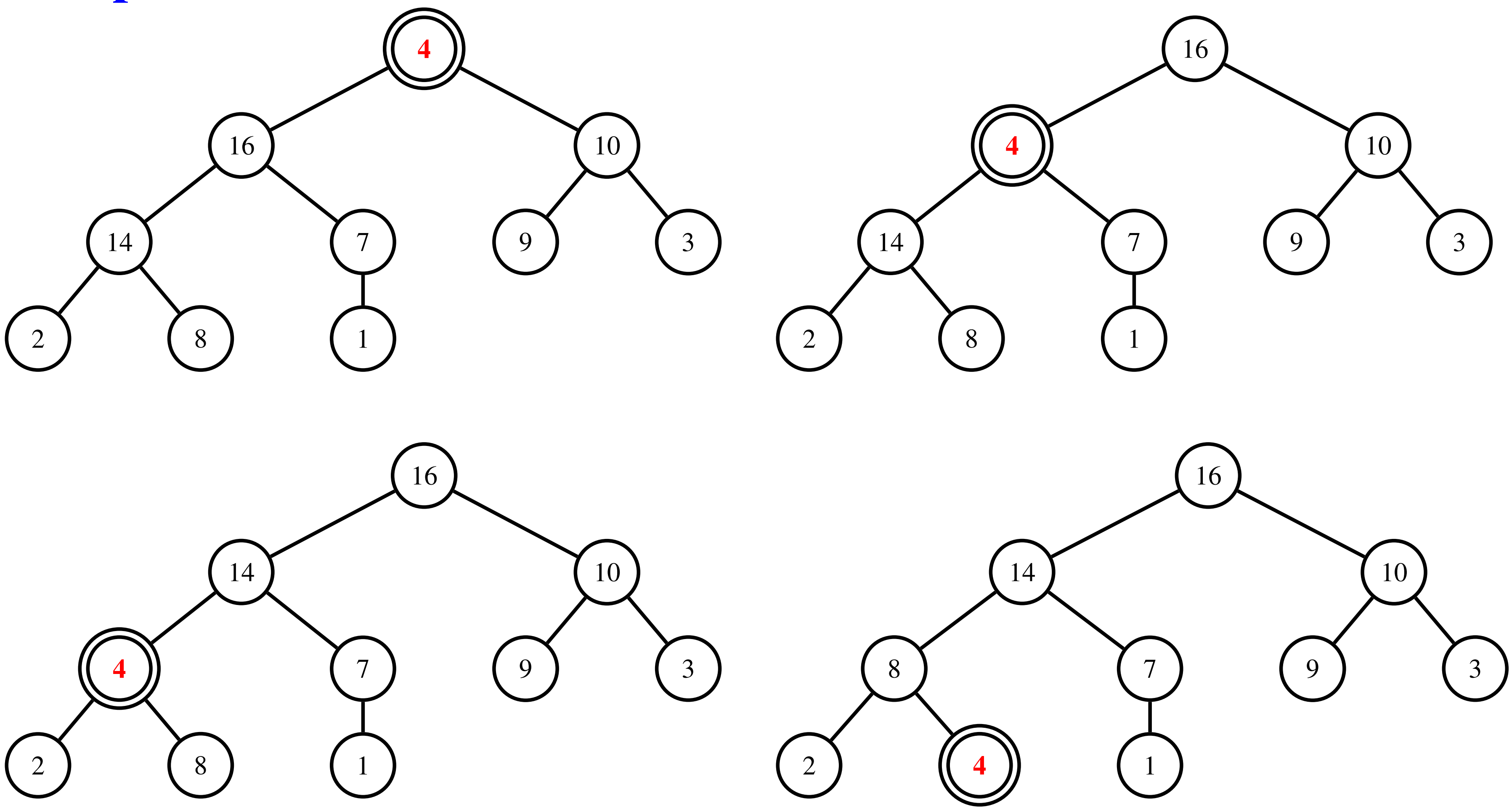

MAX-HEAPIFY **in pseudocode**

MAX-HEAPIFY(A, i)

Input: Assume left and right subtrees of i are max-heaps.
Output: Subtree rooted at i is a max-heap.

```
1   n = A.heap-size
2   l = 2i                            // A[l] is the left-child of A[i]
3   r = 2i + 1                        // A[r] is the right-child of A[i]
4   if l ≤ n and A[l] > A[i]          // Lines 4-8: Determine
5       largest = l                   // largest among A[i], A[l] and A[r].
6   else largest = i
7   if r ≤ n and A[r] > A[largest]
8       largest = r
9   if largest ≠ i
10      exchange A[i] with A[largest]
11      MAX-HEAPIFY(A, largest)
```

Running time of MAX-HEAPIFY

MAX-HEAPIFY a subtree of size n at node i

- $\Theta(1)$ to find the largest among $A[i]$, $A[2i]$ and $A[2i+1]$.
- The subtree rooted at a child of node i has size upper bounded by $2n/3$ (**Exercise.** Prove this fact.
  Proof idea: the worst case is when last row of tree is exactly half full).
- Thus $T(n) \leq T(2n/3) + \Theta(1)$.
- By the Master Theorem, we have

$$T(n) \;=\; O(n^0 \log n) \;=\; O(\log n).$$

Alternative reasoning:

Define the ***height*** of a node to be the number of edges on the longest simple downward path from the node to a leaf.

On a node of height h, MAX-HEAPIFY runs for $O(h)$ time at most.

The height of the root of a heap of size n is $\lfloor \lg n \rfloor$, so $T(n) = O(\log n)$.

MAKE-MAX-HEAP [CLRS 6.3]

Idea: starting from the last *non-leave* node,
apply MAX-HEAPIFY to the subtree based at that node.
Repeat the same procedure for all the previous nodes.

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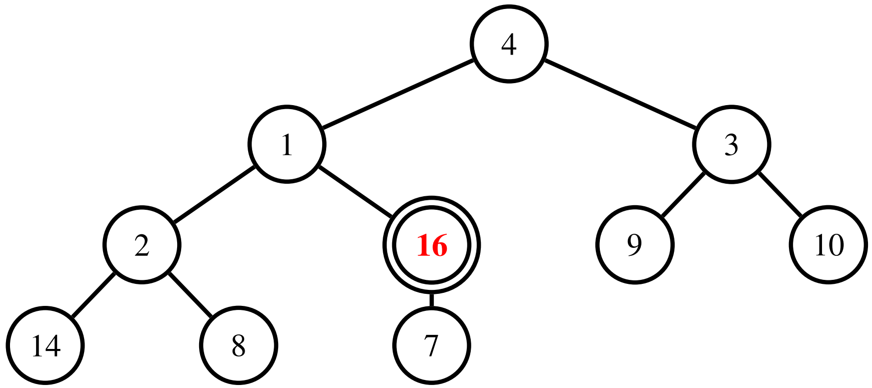

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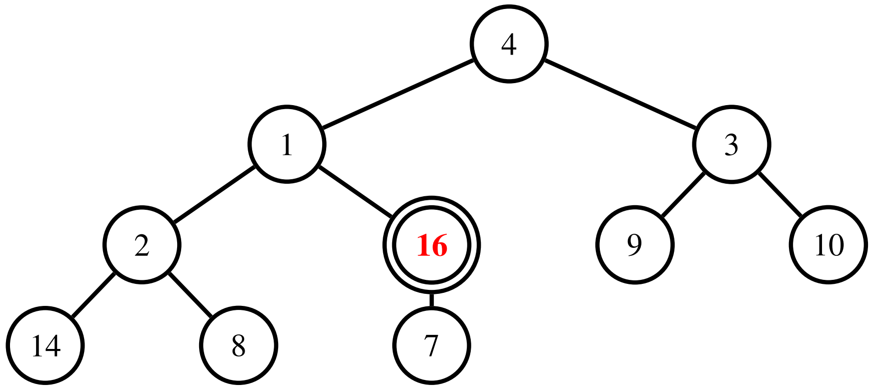

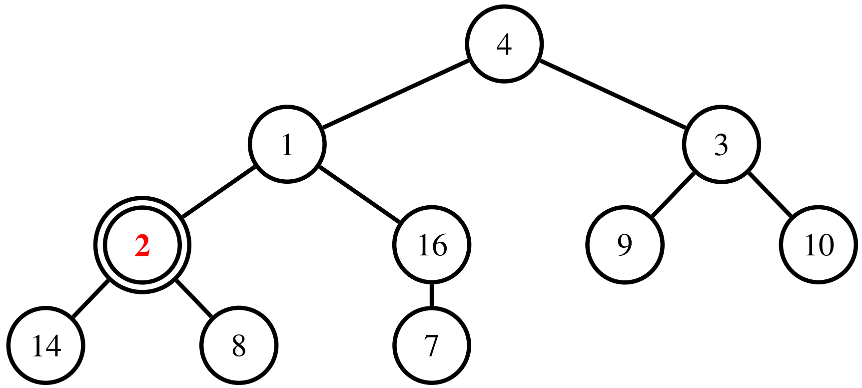

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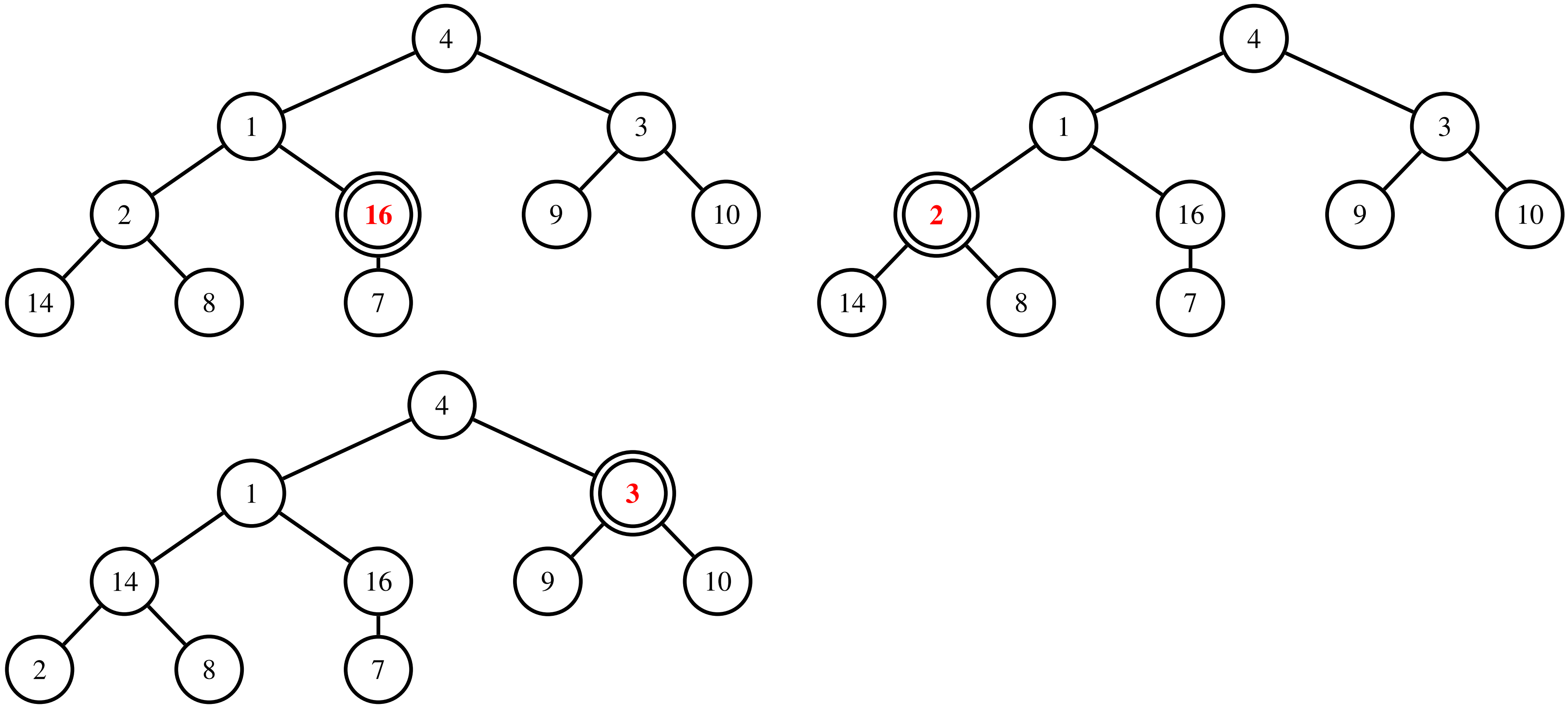

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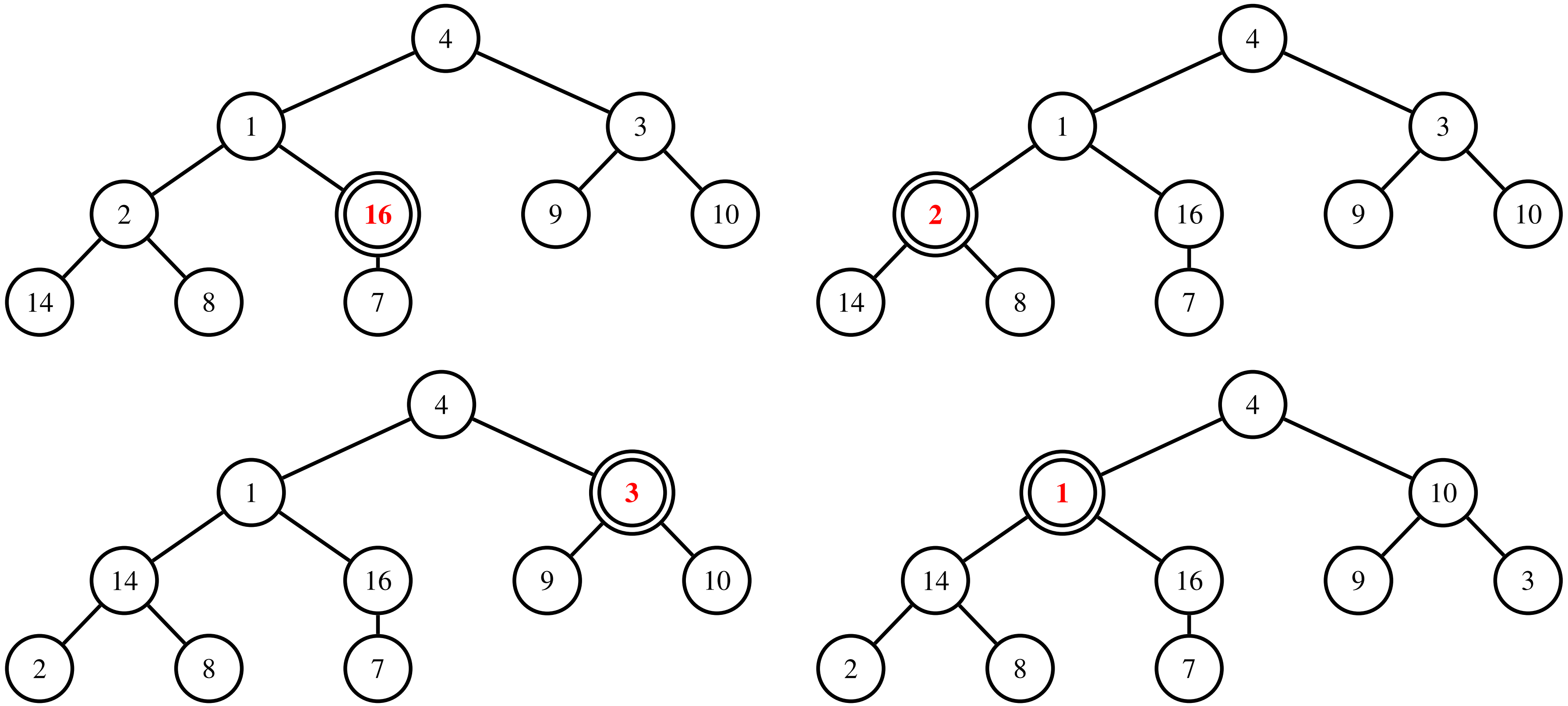

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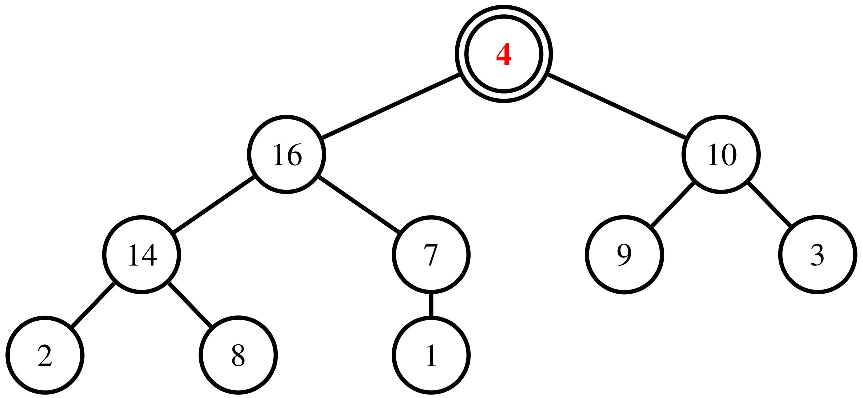

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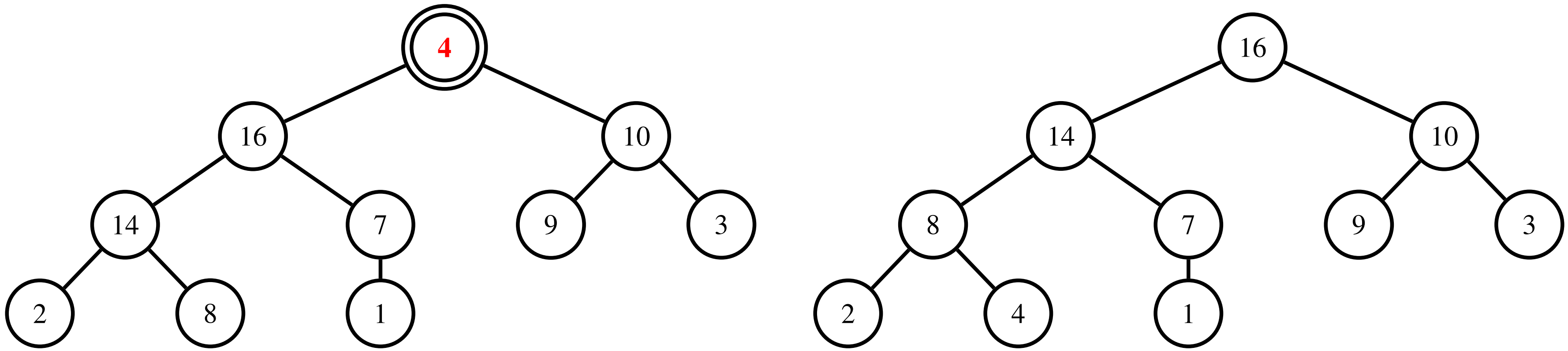


Note that the procedure works because at every step the left and right subtrees are max-heaps.

Pseudocode

Recall that the leaves are the array elements indexed by $\lceil \frac{n+1}{2} \rceil, \lceil \frac{n+1}{2} \rceil + 1, \cdots, n$.

MAKE-MAX-HEAP(A)

Input: An (unsorted) integer array A of length n.
Output: A heap of size n.

$A.heap\text{-}size = A.length$
for $i = \lceil \frac{n+1}{2} \rceil - 1$ **downto** 1
    MAX-HEAPIFY(A, i)

Correctness

Loop invariant: *Each node $i+1, i+2, \cdots, n$ is the root of a max-heap.*

Initialization

Each node $\lceil \frac{n+1}{2} \rceil, \lceil \frac{n+1}{2} \rceil + 1, \cdots, n$ is a leaf, which is the root of a trivial max-heap. Since $i = \lceil \frac{n+1}{2} \rceil - 1$ before the first iteration, the invariant is initially true.

Maintenance

Suppose $i = i_0 \geq 1$ and assume each node $i_0 + 1, i_0 + 2, \cdots, n$ is the root of a max-heap. Executing MAX-HEAPIFY(A, i) causes i_0 to be the root of a new max-heap. Hence each node $i_0, i_0 + 1, \cdots, n$ is now the root of a max-heap, meaning that the loop invariant holds after i has been decremented from i_0 to $i_0 - 1$.

Termination

When $i = 0$ (i.e. after the counter becomes less than 1) the loop terminates. By the loop invariant, each node, in particular node 1, is the root of a max-heap.

Running time analysis

Simple (but loose) bound: $O(n \log n)$.

We have $O(n)$ calls to MAX-HEAPIFY, each taking $O(\log n)$ time.

Tighter analysis: $O(n)$.

MAX-HEAPIFY takes linear time in the height of the node it runs on, and "most nodes have small heights".

Fact. The number of nodes of height h is upper bounded by $n/2^h$, and the cost of MAX-HEAPIFY on a node of height h is $\leq ch$, for some $c > 0$.

Hence, the cost of MAKE-MAX-HEAP is

$$T(n) \leq \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{n}{2^h} ch \ \leq cn \left(\sum_{h=0}^{\infty} \frac{h}{2^h} \right) = 2cn \,,$$

Note. For $|x| < 1$, one has $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$. Differentiating and multiplying by x, we get $\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$.

Applications of heaps

- Sorting: ***heapsort***, an *in-place* sorting algorithm with worst-case complexity $O(n \log n)$.

- Efficient implementation of ***priority queues***:
  Max-heap $\rightarrow$ max-priority queue.
  Min-heap $\rightarrow$ min-priority queue.
  Max-priority queues can be used to schedule jobs on a shared computer.
  Min-priority queues can be used to simulate events in time.

Remark. Actual implementations often have a ***handle*** in each heap element that allows access to an object in the application, and objects in the application often have a handle (likely an array index) to access the heap element.

Heapsort [CLRS 6.4]

A sorting algorithm based on the heap data structure.

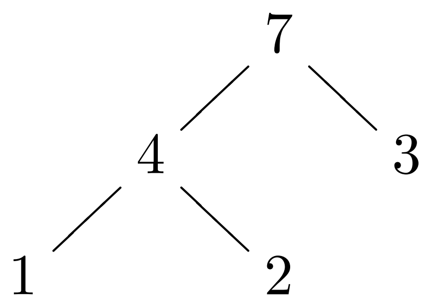
Idea. Given an input array,

- Build a max-heap using MAKE-MAX-HEAP.
- Starting from the root (maximum element), place the maximum element into the correct place in the array by swapping it with the element in the last position in the array.
- "Discard" this last node – decrement the heap size, and call MAX-HEAPIFY on the smaller structure with the possibly incorrectly-placed root.
- Repeat this discarding process until only one node (the minimum) remains, and is therefore in the correct place in the array.

Features:

- $O(n \log n)$ worst case – like merge sort.
- Sorts *in place* – like insertion sort.

Example: heapsort



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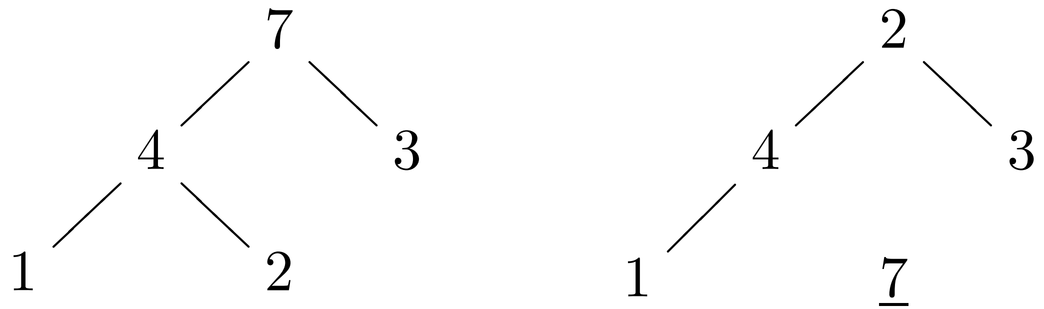

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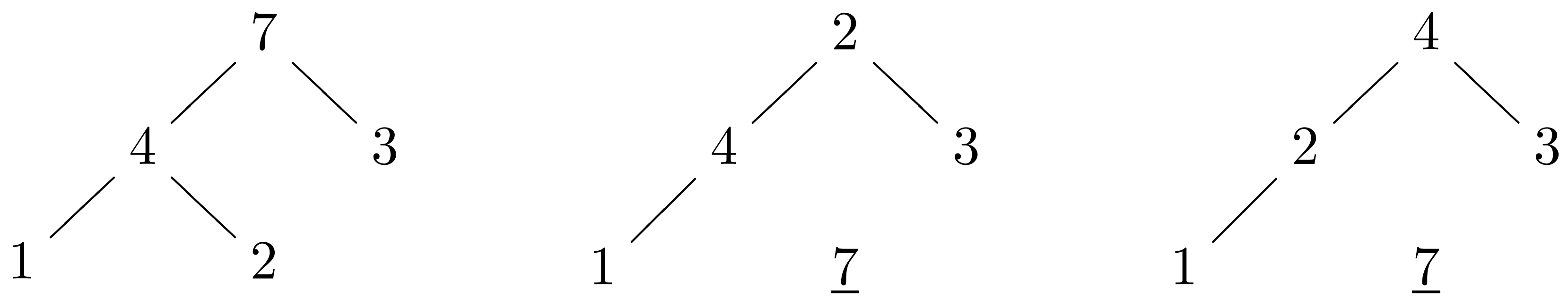

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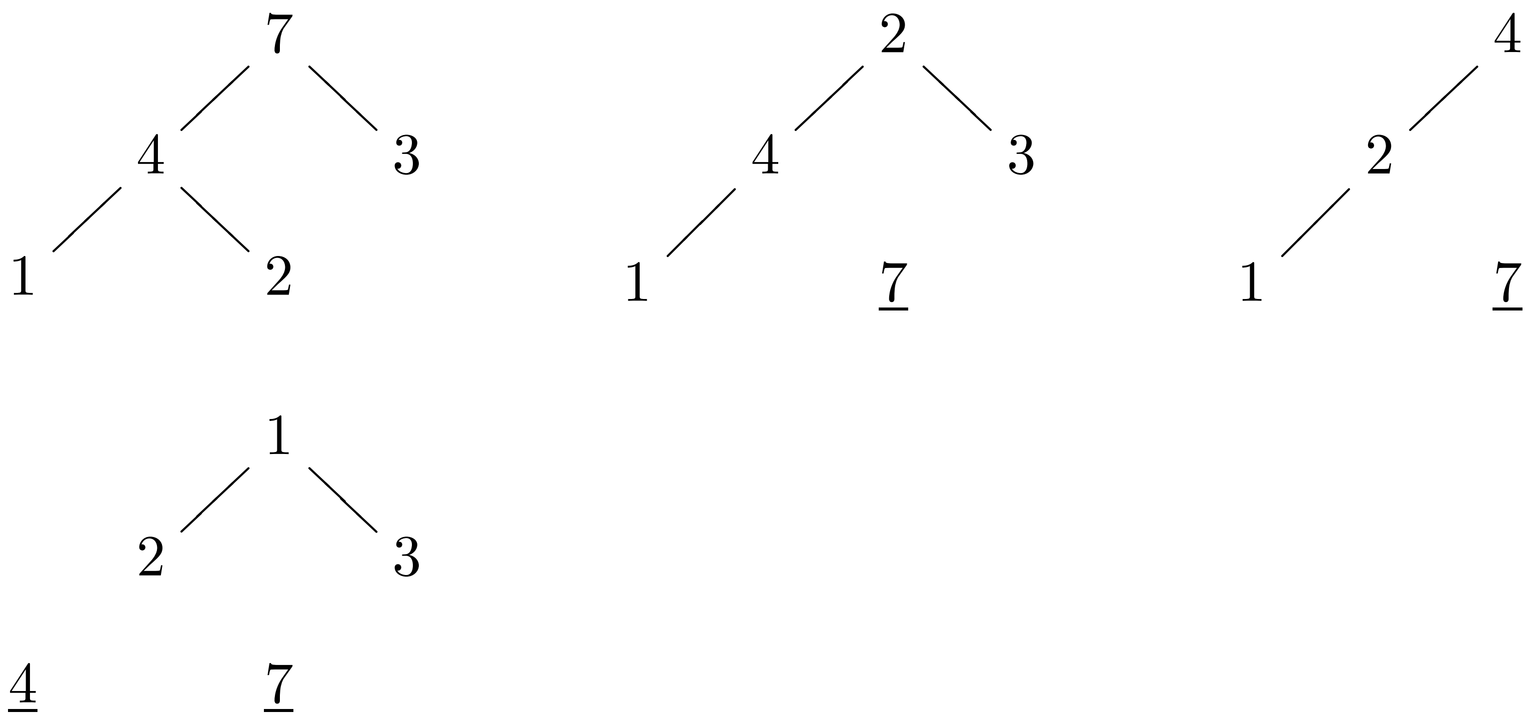

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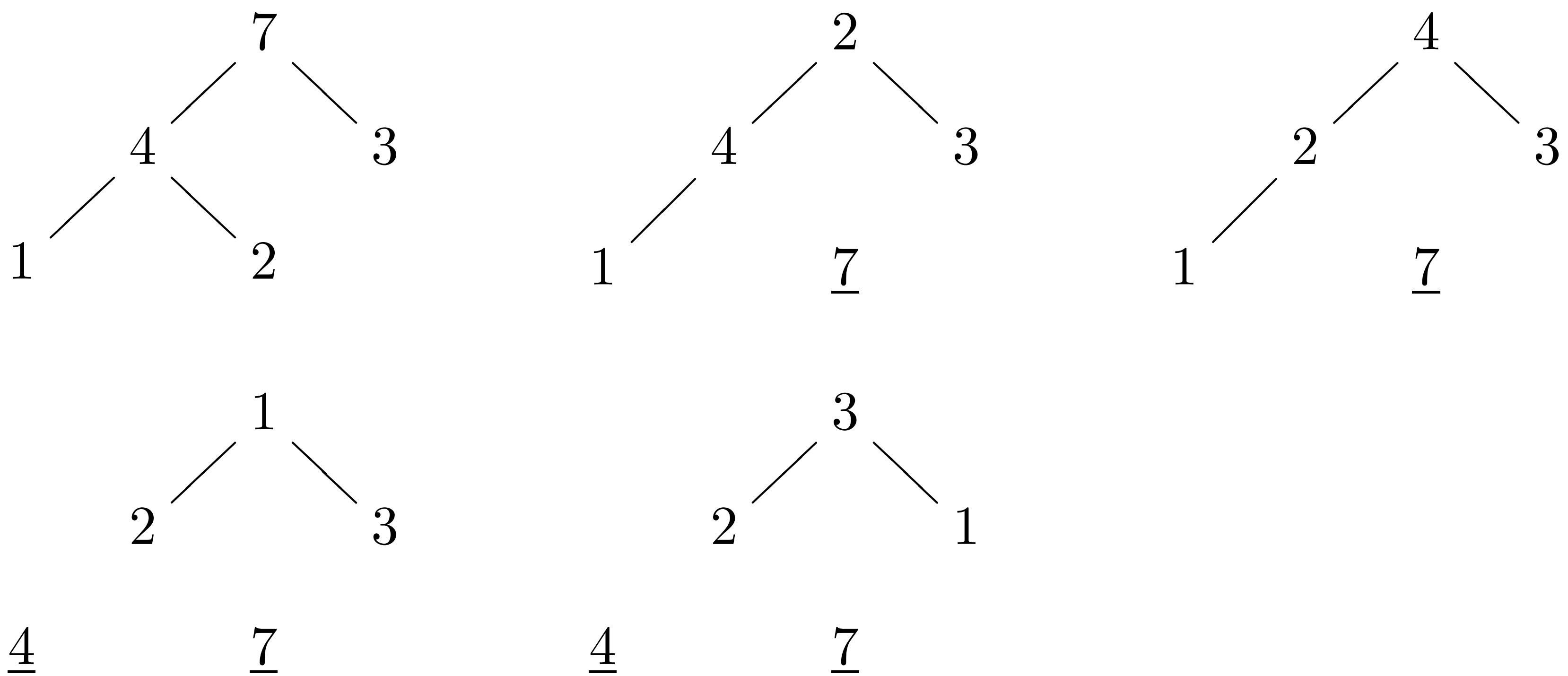

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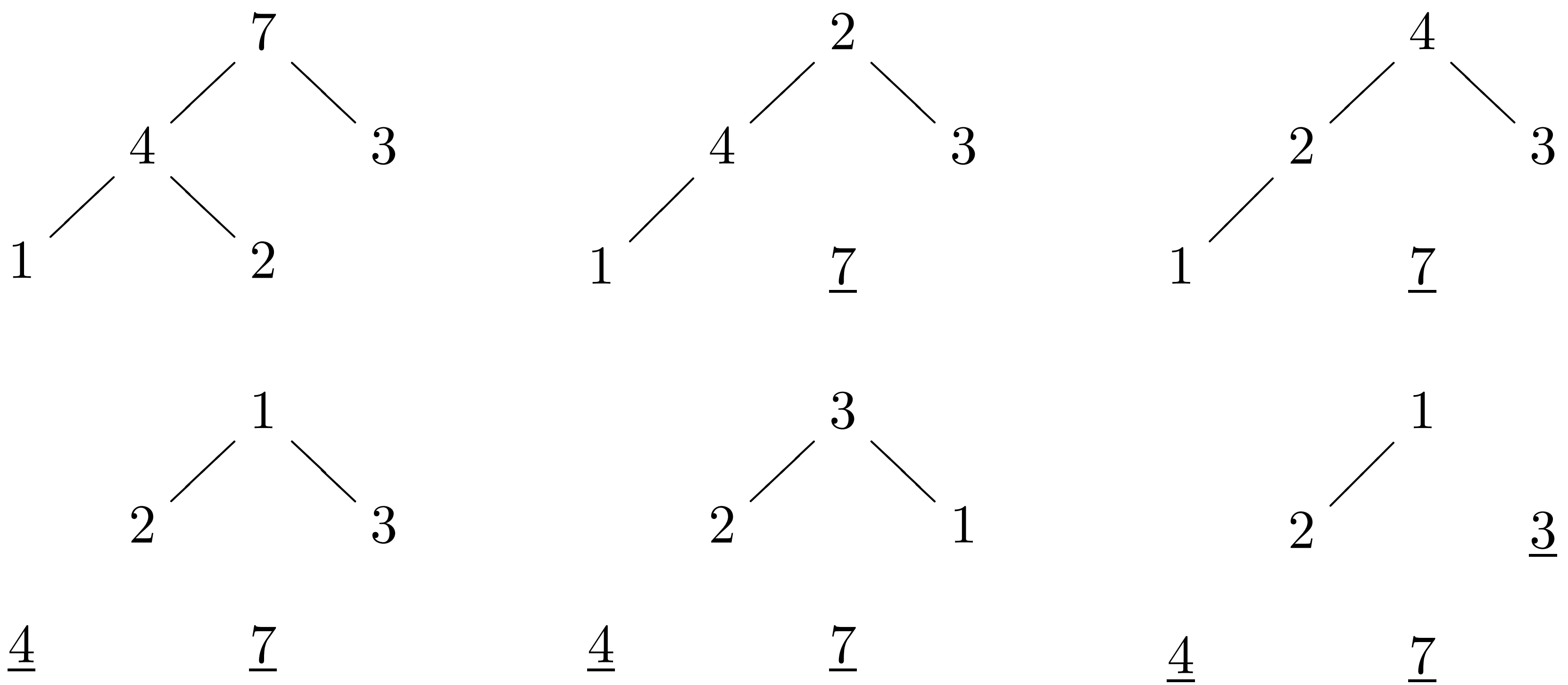

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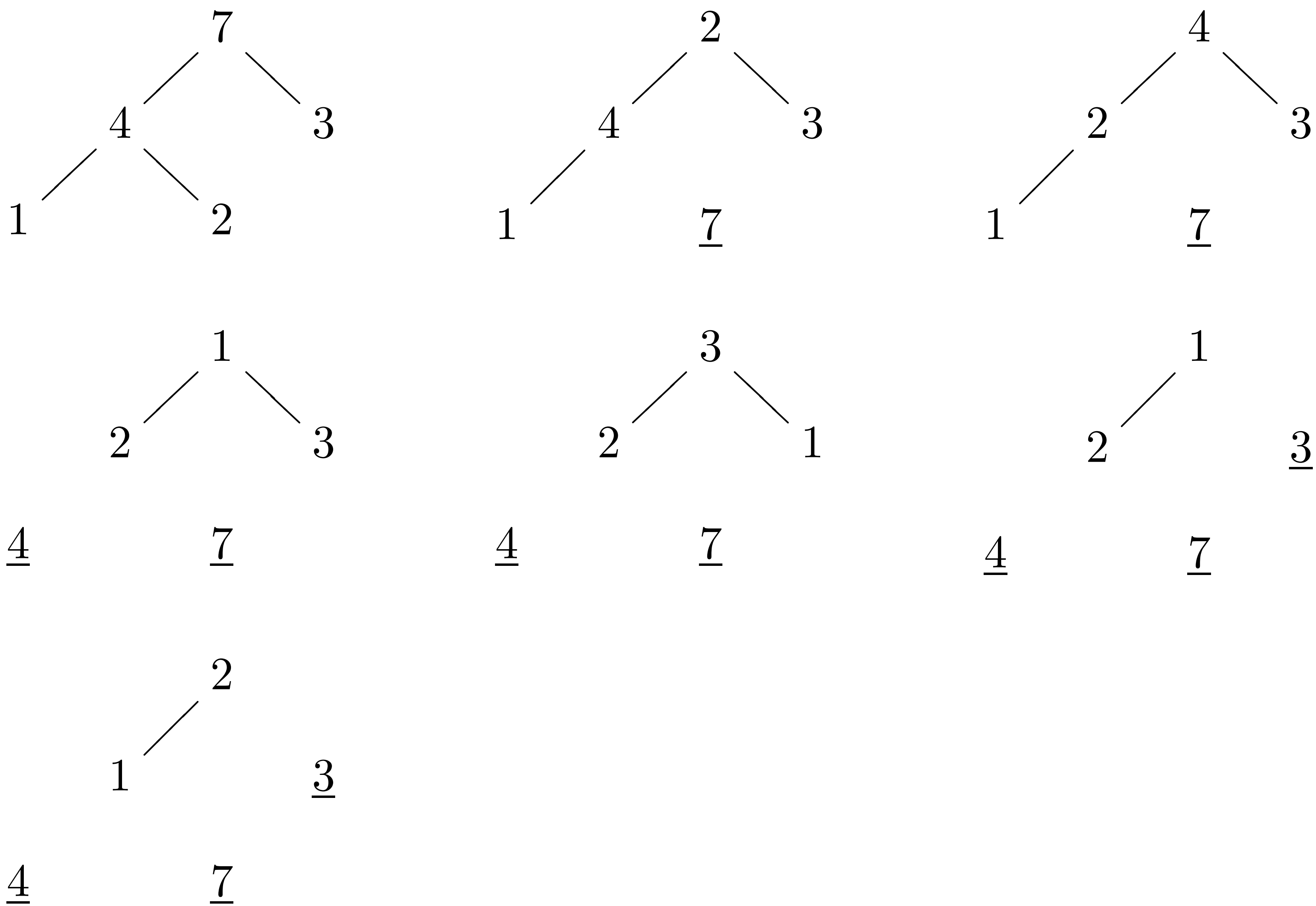

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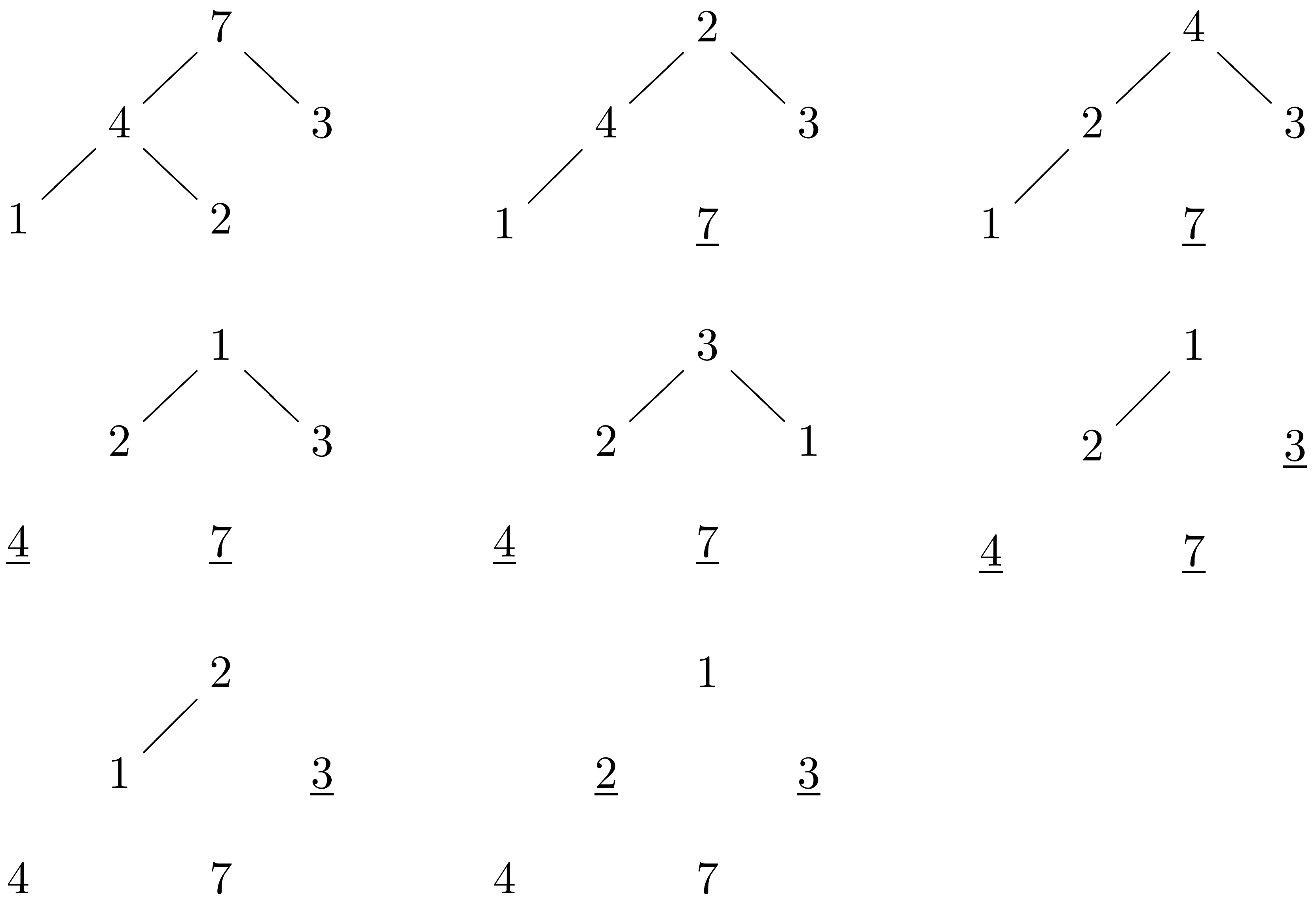

The algorithm

HEAPSORT(A)

```
MAKE-MAX-HEAP(A)
for i = A.heap-size downto 2
    exchange A[1] with A[i]
    A.heap-size = A.heap-size − 1
    MAX-HEAPIFY(A, 1)
```

Loop invariant: subarray $A[i+1\,..\,n]$ is sorted, and the remaining elements in $A[1\,..\,i]$ are $\leq$ than the elements in $A[i+1\,..\,n]$.

Running time

- MAKE-MAX-HEAP takes $O(n)$
- The **for** -loop is executed $O(n)$ times.
- Exchange operation takes $O(1)$.
- MAX-HEAPIFY takes $O(\log n)$.

Total time: $O(n \log n)$.

Priority queues [CLRS 6.5]

A Priority queue is an ***abstract data structure*** for maintaining a set of elements, each with an associated value called a ***key***.
Max-priority queues give priority to the elements with larger keys, min-priority queues give priority to the elements with smaller keys.

Operations supported by a max-priority queue:

1. INSERT(S, x, k) inserts element x with key k into set S.
2. MAXIMUM(S) returns the element of S with the largest key.
3. EXTRACT-MAX(S) removes and returns the element of S with the largest key.
4. INCREASE-KEY(S, x, k) increases value of x's key to k.
   Requires k to be at least as large as x's current key value.

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Operations supported by a min-priority queue supports INSERT(S, x), MINIMUM(S), EXTRACT-MIN(S) and DECREASE-KEY(S, x, k).

Implementation by unordered-sequence

Store the elements e and their keys k (as pairs (e, k)) in an unordered sequence, implemented as an array or a *doubly-linked list*.

- Implement INSERT(S, e, k) by inserting (e, k) at the end of the sequence; takes $O(1)$ time.
- Implement EXTRACT-MAX(S) by inspecting all elements of the sequence and removing the maximum; takes $\Theta(n)$ time.

We can do better with a heap implementation!

Implementation by heap

- A heap offers a good compromise between insertion and extraction. Both operations take $O(\log n)$ time.
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Finding the maximum

HEAP-MAXIMUM(A)

return $A[1]$

Time: $\Theta(1)$

Extracting maximum

- Check that the heap is non-empty.
- Make a copy of the maximum element (root).
- Make the last node in the tree the new root.
- HEAPIFY the array, but *less the last node*.
- Return the copy of the maximum.

HEAP-EXTRACT-MAX(A)

if $A.heap\text{-}size < 1$
    error "heap underflow"
$max = A[1]$
$A[1] = A[A.heap\text{-}size]$
$A.heap\text{-}size = A.heap\text{-}size - 1$
MAX-HEAPIFY$(A, 1)$
return max

Time: $O(\log n)$, where n is the size of the heap.

Increasing key value

Given set S, entry i, and new key value key:

1. Check that key is greater than or equal to i's current value.
2. Update i's key value to key.
3. Traverse the tree upward comparing i to its parent and swapping keys if necessary, until i's key is smaller than its parent's key.

HEAP-INCREASE-KEY(A, i, key)

if $key < A[i]$
  error "new key is smaller than current key"
$A[i] = key$
while $i > 1$ and $A[\text{Parent}(i)] < A[i]$
  exchange $A[i]$ with $A[\text{Parent}(i)]$
  $i = \text{Parent}(i)$

Time. $O(\log n)$

Insertion

Given a key k to insert into the heap:

- □ Insert a new node in the very last position in the tree with key $-\infty$.
- □ Increase the $-\infty$ key to k using HEAP-INCREASE-KEY

HEAP-INSERT(A, key)

1 $A.heap\text{-}size = A.heap\text{-}size + 1$
2 $A[A.heap\text{-}size] = -\infty$
3 HEAP-INCREASE-KEY$(A, A.heap\text{-}size, key)$

Time. $O(\log n)$