DESIGN AND ANALYSIS OF ALGORITHMS — HT 2022

Problem Sheet 1

Answers for questions marked $*$.

Big-O and other asymptotic notations

Answer to question 2

If $f = O(n^k)$ then there are $c > 0$ and n_0 such that for all $n \geq n_0$ we have $f(n) \leq cn^k$. Take $a = c$ and $b = 1 + \max\{\, f(n) : n < n_0 \,\}$ where $\max \emptyset = 0$, then $f(n) \leq an^k + b$ for all $n \geq 0$.

Recurrences

Answer to question 5

(a) By induction on k. The base case is given in the definition, i.e. $f_0 = O(1)$. For $k > 0$, by induction hypothesis ($f_{k-1} = O(n^{k-1})$ and Question 2 there are constants $a, b > 0$ such that $f_k(n) \leq f_k(n-1) + an^{k-1} + b$. So

$$f_k(n) \;\leq\; f_k(0) + \Sigma_{i=1}^{n}(ai^{k-1} + b) \;=\; O(n^k)$$

since $i^{k-1} \leq n^{k-1}$ for $1 \leq i \leq n$.

(b) By induction on k. The base case is given in the definition, i.e. $g_0 = \Omega(1)$. For $k > 0$, by induction hypothesis ($g_{k-1} = \Omega(n^{k-1})$ there are $a > 0$ and n_0 such that $g_{k-1}(n) \geq an^{k-1}$ for $n \geq n_0$. Then for $n \geq n_0$,

$$g_k(n) \;\geq\; g_k(0) + \Sigma_{i=1}^{n} ai^{k-1} \;\geq\; a\Sigma_{i=(n/2)+1}^{n} i^{k-1} \;\geq\; a(n/2)(n/2)^{k-1} \;=\; \Omega(n^k)\,.$$

Comparison problems: Searching, sorting, selection

Answer to question 10

Various straightforward approaches. For example, first use binary search to determine the position of the minimal element. If at position k, then start using binary search from position $(k + n/2)$ mod n.