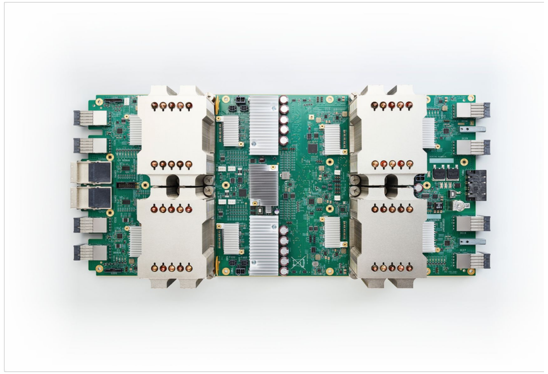
Getting to the Point.

Safe Parallel Programming for Scientific Applications

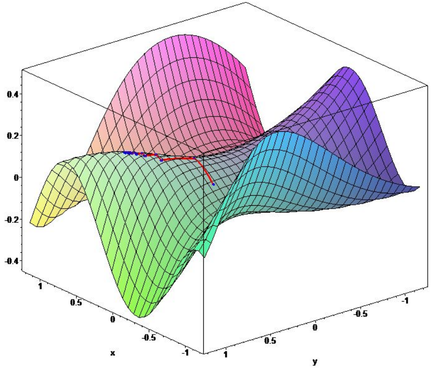
       

Background: the success of first-order array libraries

Accelerators



Autodiff



Autodiff only sees *and outputs* a sequential composition of *opaque* parallel programs.

🤯 Standard sequential autodiff *gives us efficient parallel programs out of the box!*

👀 What users see

Function types, dually

	Function	Array
Type	`a -> b`	`a=>b`
Introduction	`\x:ty. expr`	`for x:ty. expr`
Elimination	`f expr`	`f.expr`
Reduction	`(\x. e) u ↦ e[x/u]`	`(for x:ty. e).u ↦ e[x/u]`
Construction	Cheap	Expensive
Application	Expensive	Cheap
Domain	Arbitrary	Finite (ordered)

Potential deja'vu if you've heard of representable functors

Quick examples

```
3d     : (Fin 3)=>Float

vector : (Fin n)=>Float                      (assuming n:Int in scope)

matrix : (Fin n)=>(Fin m)=>Float             (assuming n:Int and m:Int in scope)

sum    : n:Type ?-> n=>Float -> Float

intIndexed : Int=>Float
> Type error! Couldn't synthesize (Ix Int)!
```

Syntax benchmark: matrix multiply

SOAC

```
combinator_matrix_multiply = \x y.
  yt = transpose y
  dot = \x y. sum (map (uncurry (*)) (zip x y))
  map (\xr. map (\yc. dot xr yc) yt) x
```

NumPy

```
matmul = lambda x, y: np.einsum('ik,kj->ij', x, y)
```

SaC

```
{ [i,j] -> sum ({ [k] -> A[i,k]* B[k,j] }) }
```

Dex

```
for i:(Fin n). for j:(Fin m). sum (for k:(Fin q). x.i.k * y.k.j)
for i:(Fin n) j:(Fin m). sum (for q:(Fin k). x.i.k * y.k.j)
for i j. sum (for q. x.i.k * y.k.j)
for i j. sum for q. x.i.k * y.k.j
```

By the way: you can be as pointfree as you'd like!

```
def uncurry {a b c} (f:a -> b -> c) : (a & b) -> c = \(x, y). f x y
def zip {n a b} (x:n=>a) (y:n=>b) : n=>(a & b) = for i. (x.i, y.i)
def map {n a b} (f:a -> b) (x:n=>a) : n=>b = for i. f x.i
def transpose {n m a} (x:n=>m=>a) : m=>n=>a = for i j. x.j.i

def combinator_matrix_multiply {n k m}
    (x:n=>k=>Float) (y:k=>m=>Float) : n=>m=>Float =
  yt = transpose y
  dot = \x y. sum (map (uncurry (*)) (zip x y))
  map (\xr. map (\yc. dot xr yc) yt) x
```

A pointful foundation doesn't make pointfree programming harder!

Rank polymorphism

Not supported!

In the vast majority of cases used for *batching*.

Have a larger collection? Use a loop!

Some rank polymorphism possible to recover using typeclasses.

```
interface Add a
  (+) : a -> a -> a
instance Add Int …
instance {n a} [Add a] Add (n=>a)
  (+) = \x y. for i. x.i + y.i

matrix : n=>m=>Int = …
matrix + matrix  -- well typed!
```

Type system

```
def broadcast {a} (v:a) (n: Type) [Ix n]: n=>a = for i. v

broadcast 2.0 (Fin 5)
> [2.0, 2.0, 2.0, 2.0, 2.0]
```

Loop bound inferred from *return type annotation*

```
i5  = 2 + 3
i5' = 2 + 3
broadcast 2.0 (Fin i5) + broadcast 2.0 (Fin i5')
> Type error! Expected (Fin i5)=>Float, but got (Fin i5')=>Float!
```

Very limited normalization applied to types

```
-- in lib/prelude.dx
def Fin   (n:Int) : Type = Range 0 n
def Range (low:Int) (high:Int) = …

x : (Fin 5) = …
```

But not entirely trivial!

A quick look under the hood

```
data Atom = Var Name
          | …
          | Pair     Atom Atom
          | PairType Type Type
          | …
          | TypeKind
          | …
          | Lambda Name Type Expr

type Type = Atom  -- statically unchecked invariant: should be of TypeKind


data Expr = BinOp BinOpKind Atom Atom
          | For Atom
          | …
```

Sum and (dependent) product types

```
data Maybe a =
  Just a
  Nothing

data List a =
  MkList (length:Int) (elements:(Fin length)=>a)

def filter {n a} (f:a -> Bool) (x:n=>a) : List a = …

MkList _ validData = filter isValid data
sum validData
```

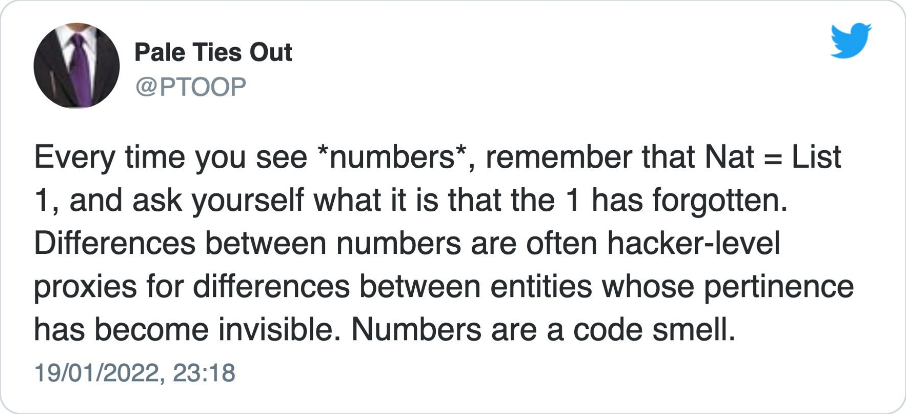
🤑 What does this buy us?

Can tensor programming be liberated from integer indices?

Arrays are predominantly indexed *by integers*, but:

- static reasoning about integers is difficult;
- integers erase lots of structure that's often useful.

"Parse, don't validate.[1]"



Pale Ties Out
@PTOOP

Every time you see *numbers*, remember that Nat = List 1, and ask yourself what it is that the 1 has forgotten. Differences between numbers are often hacker-level proxies for differences between entities whose pertinence has become invisible. Numbers are a code smell.

19/01/2022, 23:18

[1] https://lexi-lambda.github.io/blog/2019/11/05/parse-don-t-validate/

Rich index sets

In Dex, any type *conforming to Ix* can be an array index:

```
interface Ix n where
  size n             : Int
  toOrdinal          : n -> Int
  unsafeFromOrdinal : Int -> n
```

size

& isomorphism with a prefix of natural numbers

```
def fromOrdinal {n} [Ix n] (o:Int) : n =
  case 0 <= o && o < size n of
    True  -> unsafeFromOrdinal o
    False -> error …
```

Basic shape arithmetic can be done using standard type constructors:

Products	`(n & m)`
Sums	`(n | m)`
Exponentials	`(n=>m)`

Basic examples

Reshapes

```
reshape (2, -1, 4) x
```

(n & m)-typed binder

```
for i (j, k) l. x.i.j.k.l
```

Concatenation

```
concatenate x y
```

(n | m)-typed binder

```
for ci. case ci of
  Left  xi -> x.xi
  Right yi -> y.yi
```

Named axes

`image[h, w]` or `image[w, h]`?

```
image.{height=h, width=w}
image.{width=w, height=h}
```

Boundary conditions

```
x: (Fin (1 + n))=>a
x[0] vs x[1 + i]
```

```
x: (Unit|n)=>a
x.(Left ()) vs x.(Right i)
```

Index sets for compilers

Integer-based indexing

```
nmp = n + m + p
for i in range(nmp).
  if i < n
    then x[i]
    else if i - n < m
      then y[i - n]
      else z[i - n - m]
```

Sum-type-based indexing

```
for i in (n|(m|p)).
  case i of
    Left ni -> x.ni
    Right i' -> case i' of
      Left  mi -> y.mi
      Right pi -> z.pi
```

A loop with a sum-typed index set either never inspects the index, or is a very good candidate for loop splitting!

Indexing lemmas

Array reversal

```
def reflect {n} (i:n) : n =
  unsafeFromOrdinal n (size n - 1 - ordinal i)
```

```
sequence : (Fin s)=>Int = …
for i in range(len(sequence)).
  sequence[len(sequence) - 1 - i]
```

Correctness reasoning requires non-local context (e.g. range of i)

```
sequence : n=>Int = …
for i.
  sequence.(reflect i)
```

Dynamic programming

```
def prev (i:n) : (Unit|n) =
  unsafeFromOrdinal _ (ordinal i)
```

```
x : (Fin s)=>Int = …
sumWithPrev = for i in range(len(x)).
  if i == 0
    then x[i]
    else x[i - 1] + x[i]
```

Easy to forget about the base case and read out of bounds!

```
x : (Unit|n)=>Int = …
sumWithPrev = for i.
  case i of
    Left  () -> x.i
    Right i' -> x.(prev i') + x.i
```

Index sets are user-definable

```
data RGB = Red | Green | Blue
instance Ix RGB
  size = 3
  toOrdinal = \x. case x of
    Red   -> 0
    Green -> 1
    Blue  -> 2
  unsafeFromOrdinal = ...

data HSV = Hue | Saturation | Value
instance Ix HSV ...

Image = \h w colorSpace. { height: (Fin h) & width: (Fin w) }=>colorSpace=>UInt8

imgRGB : Image 200 200 RGB = loadKnownSizeJPG "doggo.jpg"
imgHSV : Image _   _   HSV = RGBtoHSV imgHSV
hues = for h w. imgHSV.{height=h, width=w}.Hue
```

Arrays can function as *named tuples*

Array type zoo

🤔 If we have dependent functions... why don't we try dependent arrays?

Homogeneous ↕ Heterogeneous

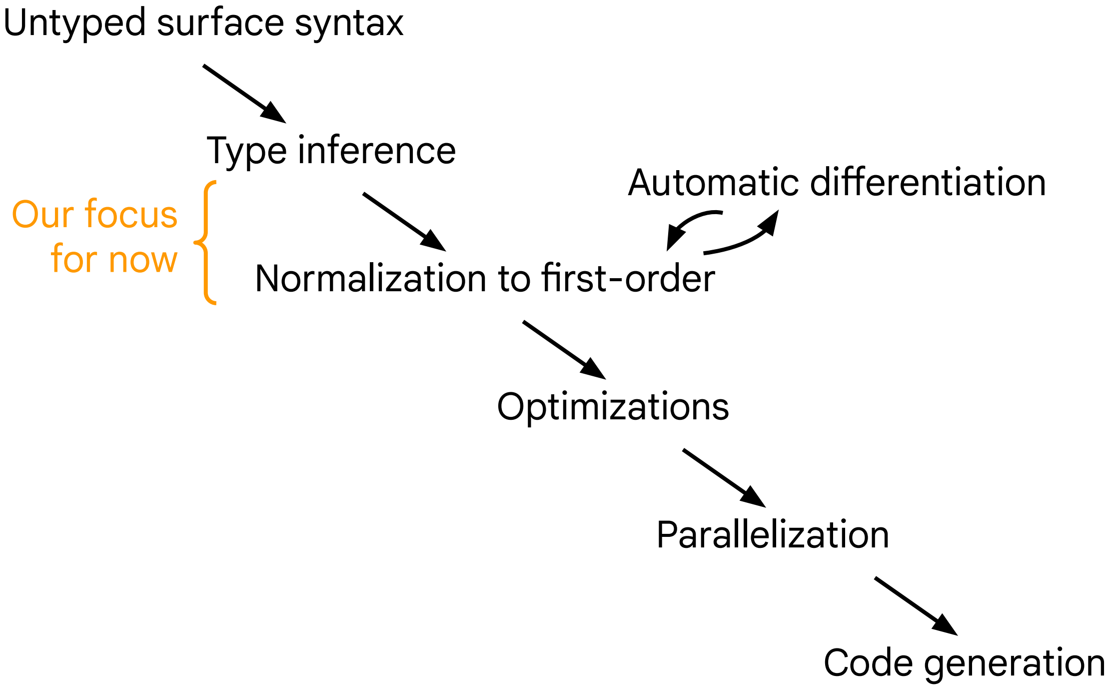
Array kind	Example type
Static	`(Fin 10)=>(Fin 20)=>Float`
Dynamic	`(Fin n)=>(Fin m)=>Float`
Structured ragged	`(i:Fin 10)=>(...i)=>Float`
Ragged	`(i:Fin 10)=>(Fin lengths.i)=>Float`
Jagged	`(Fin 10)=>List Float`

Pushing the limits of our type system here

Also:
Position-dependent arrays and their application for high performance code generation, F. Pizzuti et al.
Generating High Performance Code for Irregular Data Structures using Dependent Types, F. Pizzuti et al.

🤫 What users don't see

Going deeper



Also: High-Performance Defunctionalisation in Futhark, A. K. Hovgaard et al.

Zooming into AD

forward-mode AD ≈ linearize

```
linearize : (a -> b) -> a -> (b, a -o b)
```

But, we often want a *representation* of the derivative mapping.

If a is a high-dimensional vector space, then this evaluation is expensive!

But, we also know that every linear transform has a *transpose*.

```
transpose : (a -o b)      -> (b -o a)
```

reverse-mode AD = linearize + transpose[1]

[1]Decomposing reverse-mode automatic differentiation, R. Frostig et al.

Implementing linearization

Multiplication `linearize \x. x * y` ↦

```
\x. (x * y,
     \xt. x * xt + xt * y)
```

Composition `linearize \x. f (g x)` ↦

```
\x. (t, glin) = linearize g x
    (y, flin) = linearize f t
    (y, \xt. flin (glin xt))
```

For loops `linearize \x. for i. f x i` ↦ **???**

(rematerialize)

```
\x. (for i. f (x, i),
     \xt. for i.
            snd (linearize f (x, i)) xt.i)
```

(arrays of functions)

```
\x. (ys, flins) = unzip (for i. linearize f (x, i))
    (ys, \xt. for i. flins.i xt.i)
```

Normalizing arrays of functions

First-order context

Arbitrary atoms (incl. lambdas!)

```
toFirstOrder : Nest Decl -> (Nest Decl, Substitution Name Atom)
```

Lambda for table type

```
toFirstOrder ( x = for i.        ↦  ( tmp = for i.        ,  x ->                                        )
                 v1 = …                  fo1 = …                view i.
                 …                       …                        atom[reconSubst][a1,…,an/tmp.i]
                 vn = …                  fom = …
                 atom )                  (a1, …, ak)
```

```
((fo1 = …; …; fom = …), reconSubst) = toFirstOrder (v1 = …; …; vn = …)
```

Normalize block

```
(a1, …, ak) = intersect (freeVars atom[reconSubst]) (fo1, …, fom)
```

Find first-order variables sufficient for reconstruction

Can only:
(1) reference functions defined outside of for, or
(2) lambda expressions with body FVs.

Similar trick also works (and is needed!) for `case` expressions

Going deeper

Untyped surface syntax

Type inference

Automatic differentiation

Normalization to first-order

This is why we have
to loop here

Optimizations

Parallelization

Code generation

Efficiency issues loom

Scaling

```
\xt       . zt = xt * c        ↦    \zt. xt = zt * c
            zt                            xt
```

Addition

```
\(xt, yt). zt = xt + yt        ↦    \zt. xt = zt
            zt                            yt = zt
                                          (xt, yt)
```

Duplication

```
\xt       . zt = (xt, xt)      ↦    \zt. xt = fst zt
            zt                            xt = xt + snd zt
                                          xt
```

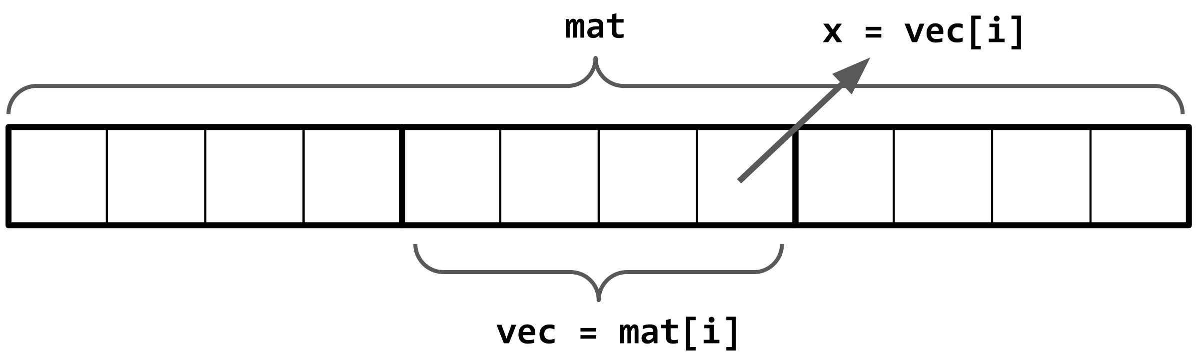
Broadcast

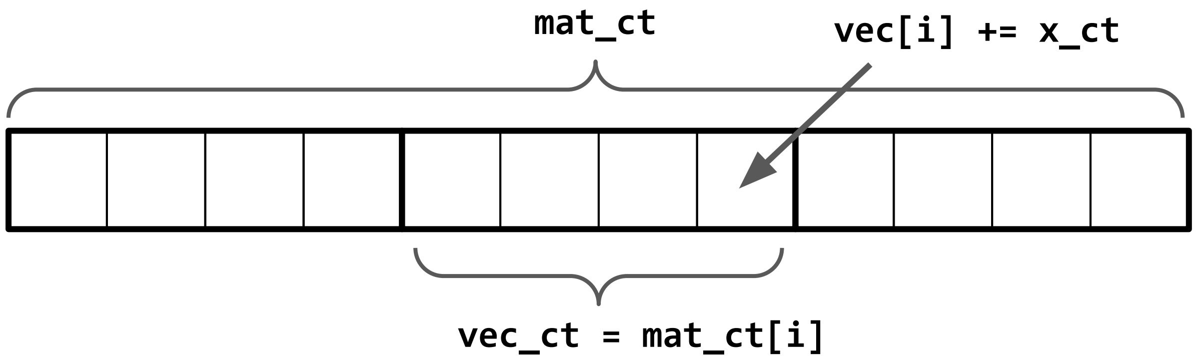
```
\xt       . zt = for i. xt     ↦    \zt. xt = sum zt
            zt                            xt
```

Indexing

```
\xt       . xt.i               ↦    ???xt[i] += zt"
```

FP's unstated cost model: indexing is aliasing





We need to alias writes like we alias reads!

Transposition of indexing

1️⃣ Imperative AD

```
store x_ct[i] ((load x_ct[i]) + y_ct)
```

❌ Unconstrained heap mutation

2️⃣ Dense updates

```
x_ct2 = x_ct + one_hot(y_ct, i)
```

❌ Lots of wasted work, wrong asymptotics

3️⃣ Sparse updates

```
x_ct2 = x_ct + sparse_one_hot(y_ct, i)
```

❌ Unacceptable constant factors, difficult on GPUs

3️⃣ Functional in-place (linear) updates

```
x_ct2 = consume_and_update(x_ct, i, y_ct)
```

❌ Sequentializes code

5️⃣ Associative accumulation effect

```
accumulate y_ct into x_ct[i]
```

Solution: effects

(Basic) Accumulation

```
def sum {n} (x:n=>Float) : Float =
  (_, total) = withAccum \acc.
    for i.
      acc += x.i
  total
```

Accumulator cannot be read

Final value obtained once the accumulator cannot be modified

State

```
def scan {n i o s eff}
    (f:i -> s -> {|eff} (o, s)) (init:s)
    (x:n=>i) : {|eff} n=>o =
  (result, final) = withState init \ref.
    for i.
      ref := f x.i (get ref)
  result
```

Arbitrary monoidal reductions

```
def reduce {n a} (m:Monoid a) (x:n=>a) : a =
  (_, total) = withAccum m \acc.
    for i.
      acc o= x.i
  total
```

Differentiation through reductions over arbitrary monoids is non-trivial![1]

[1]Parallelism-preserving automatic differentiation for second-order array languages, A. Paszke et al.

Efficient AD as a language design benchmark

There exists a constant c such that for every program P the cost of evaluating P' (P' being derived using forward- or reverse-mode AD from P) is at most c times larger than the cost of evaluating P.

Good reverse-mode autodiff support requires:

1️⃣ Closure under partial evaluation

2️⃣ Closure under data-flow duality

For example, reverse-mode AD of (parallel associative) `scan` is inefficient![1]

[1] Parallelism-preserving automatic differentiation for second-order array languages, A. Paszke et al.

Current / future work

- User-extensible (parallel-friendly) algebraic effects (see PEPM paper[1])
- Scope-correctness of compiler implementation
- Monomorphization without complete inlining
- Typeclass system rework (embracing overlap!)
- Nested data parallelism (see Conal Elliot's earlier presentation[2])
- Make Dex fast!
- ...

[1]Parallel Algebraic Effect Handlers, N. Xie, D. J. Johnson et al.
[2]Can Tensor Programming Be Liberated from the Fortran Data Paradigm?

Thank you!

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