Design and Analysis of Algorithms

Part 8

Matroids

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Matroids [CLRS 16.4]

- The theory of matroids is a rich mathematical theory, which is tightly connected to the greedy algorithm. Many, but not all, applications of the greedy algorithm involve a matroid.
- Matroids are abstract combinatorial structures that generalize the notion of spanning trees (or spanning forests).

Definition of matroids

Definition 1 (Matroid)**.** *A matroid is a pair* $(E, \mathcal{I})$*, where* E *is a finite set and* $\mathcal{I}$ *a family of subsets of* E*, that satisfies the following two properties*
Hereditary property: *If* $A \in \mathcal{I}$ *and* $B \subseteq A$*, then* $A \in \mathcal{I}$
Exchange property: *If* $A, B \in \mathcal{I}$ *and* $|B| < |A|$*, there exists* $a \in A - B$ *such that* $B \cup \{a\} \in \mathcal{I}$*.*
The set E *is called the* ***ground set*** *and the elements of* $\mathcal{I}$ *are called* ***independent sets***.

Example of a matroid:
$E = \{1, 2, 3, 4\}$
$\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}\}$

Bases

Due to the hereditary property, a matroid can be defined by the set $\mathcal{B}$ of the ***maximal independent sets***. The elements of $\mathcal{B}$ are called ***bases***.
The following is an immediate consequence of the exchange property.

Lemma 1. *All bases of a matroid have the same cardinality.*

$E = \{1, 2, 3, 4\}$
$\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}\}$
$\mathcal{B} = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}\}$

Basis exchange property

We can get an ***equivalent definition of matroids*** by replacing the exchange property with the following property:

Basis exchange property: If $A, B \in \mathcal{B}$ and $a \in A - B$, then there exists an element $b \in B - A$ such that $(A \cup \{b\}) - \{a\} \in \mathcal{B}$.

or even with the following stronger property:

Double basis exchange property: If $A, B \in \mathcal{B}$ and $a \in A - B$, then there exists an element $b \in B - A$ such that $(A \cup \{b\}) - \{a\} \in \mathcal{B}$ and $(B \cup \{a\}) - \{b\} \in \mathcal{B}$.

One can easily verify the basis exchange property for the example:

$E = \{1, 2, 3, 4\}$
$\mathcal{B} = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}\}$

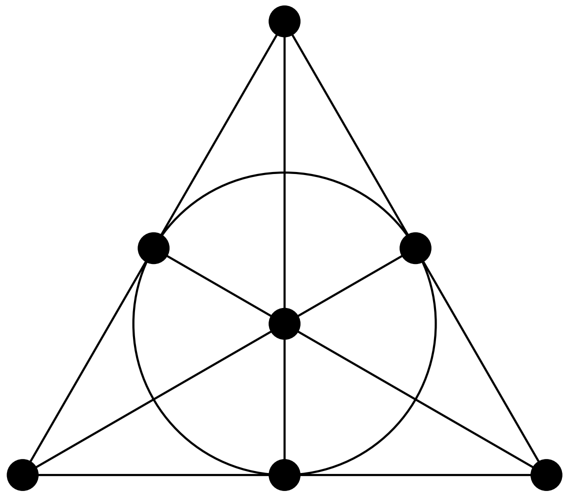
Example: graphic matroids

Let $G(V, E)$ be a graph and let $\mathcal{I}$ be the set of spanning forests of the graph. The elements of$\mathcal{I}$ are subsets of edges. Then $(E, \mathcal{I})$ is a matroid. Such matroids are called ***graphic matroids***.
When the graph is connected, the basis set $\mathcal{B}$ is the set of spanning trees of G.

Let's use the basis exchange property to argue that this is a matroid. Consider two distinct spanning trees A and B and some edge $x \in A - B$. If we remove edge x from A, we get two trees L and R. We want to show that there is an edge $y \in B$ with one vertex in L and one vertex in R, so that when we remove x and add y, we get a $(A \cup \{y\}) - \{x\}$, which is a spanning tree because it is connected and has the same number of edges with A.
There is a unique path p in B that connects the two vertices of x. One end vertex of p belongs to L and the other to R, so there must exist an edge $y \in p$ with one vertex in L and one vertex in R.

Example: Fano matroid

The ground set consists of the 7 vertices.
The set of bases consists of every triple of points that do not belong to a line or the circle.



Independent columns of a matrix

Let M be a matrix with entries real numbers. Let E be its sets of columns and $\mathcal{I}$ the subsets of E that are linearly independent. Then $(E, \mathcal{I})$ is a matroid.

Transversal matroids

Let $G(L, R, E)$ be a bipartite graph.
A set M of disjoint edges is called a ***matching***.
We will say that a set $A \subset L$ is independent if there is a matching with M whose vertices in L is A. If $\mathcal{I}$ denotes the family of independent sets, then $(L, \mathcal{I})$ is a matroid.
These matroids are called ***transversal matroids.***

Weighted matroids

Given a matroid $(E, \mathcal{I})$, we can define a ***weighted matroid*** by associating a positive weight $w(x)$ to each element x of the ground set E.

The weighted matroid problem has

Input: A weighted matroid

Output: An independent set of maximum total weight

Usually the input is given implicitly. For example, a weighted graph defines a weighted graphic matroid. Similarly, a bipartite graph $G(L, R, E)$ with weights on its left part L defines a transversal matroid.

The greedy algorithm for the weighted matroid problem

GREEDY$(E, \mathcal{I}, w)$

$A = \emptyset$
Sort E in decreasing order by weight w
for each $x \in E$
    if $A \cup \{x\} \in \mathcal{I}$
        $A = A \cup \{x\}$
return A

Running time: For sorting: $O(n \log n)$ (where $n = |E|$).
For the main loop: $O(nf(n))$, where $f(n)$ is the time to check $A \cup \{x\} \in \mathcal{I}$.
Total time $O(n \log n + nf(n))$.

Greedy and matroids

Theorem 1. *For every weighted matroid* $(E, \mathcal{I}, w)$*, the Greedy algorithm returns an optimal solution.*

Its proof is based on the following lemma.

Lemma 2. *At every point during the execution of the Greedy algorithm, there exists an optimal basis* B *that consists of the current set* A *and unprocessed elements.*

Greedy and matroids

Lemma 3. *At every point during the execution of the Greedy algorithm, there exists an optimal basis B that consists of the current set A and unprocessed elements.*

Proof. By induction on the number of processed elements. The basis of the induction, when $A = \emptyset$, is trivial. Suppose now that it is true before processing element x.
If x is not added to A or $x \in B$, where B is an optimal basis, there is nothing to prove. Suppose now that $A \cup \{x\} \in \mathcal{I}$ and x is added to A, but $A \cup \{x\}$ is not part of an optimal solution B. By induction, there is an optimal solution B that contains A and unprocessed elements, i.e., elements with weight at most equal to $w(x)$. Let A^* be the final solution produced by the Greedy algorithm. Note that $A + \{x\} \subseteq A^*$.
We apply the double basis exchange property to B, A^* and $x \in A^* - B$: there exists an element in $y \in B - A^*$ such that $B + \{x\} - \{y\} \in \mathcal{I}$. But the weight of this basis differs from the weight of B by $w(x) - w(y) \geq 0$. Therefore $B + \{x\} - \{y\}$ is optimal and contains $A \cup \{x\}$, a contradiction. $\square$

Greedy ⇒ matroid

Theorem 2. *Let E be a set and $\mathcal{I}$ a family of its subsets that satisfies the hereditary property. If $(E, \mathcal{I})$ is not a matroid, there exist weights for which the Greedy algorithm fails to return an optimal solution.*

Proof. Since $(E, \mathcal{I})$ is not a matroid, there exist sets $A, B \in \mathcal{I}$ and $a \in A - B$ such that for every $b \in B - A$: $A + \{b\} - \{a\} \notin \mathcal{I}$. Note, that this implies that A and B differ in at least two elements. Consider the weights, where $\epsilon > 0$ is very small:

$$w(x) = \begin{cases} 1 + \epsilon & x \in (A \cap B) \cup \{a\} \\ 1 & x \in B - A \\ 2\epsilon & x \in A - B - \{a\} \\ \epsilon & \text{otherwise} \end{cases}$$

The Greedy algorithm will return the suboptimal solution A, with weight $w(A) = (|A \cap B| + 1)(1 + \epsilon) + (|A - B| - 1)2\epsilon \approx |A \cap B| + 1$. However, for small ϵ, the optimal solution is B with total weight $w(B) \geq |B|$. For small ϵ, $w(B) > w(A)$, because $|A \cap B| + 1 \leq |B| - 1$ since A and B differ in at least two elements. □

Partition matroids and matchings

Let $G(L, R, E)$ be a bipartite graph.
Let

$$\mathcal{I}_L = \{A\colon A \subseteq E \text{ and no vertex of } L \text{ is in two or more edges of } A\}$$

Then $(L, \mathcal{I}_L)$ is a matroid (a partition matroid).
We can define similarly the matroid $(R, \mathcal{I}_R)$.
The intersection of the two matroids $(L, \mathcal{I}_L)$ and $(R, \mathcal{I}_R)$ is

$$\mathcal{M} = \{A\colon A \subseteq E \text{ edges of } A \text{ are disjoint}\},$$

which is the set of matchings of the bipartite graph.
The algorithmic problem of finding the ***maximum intersection of two matroids*** —and its special case of the ***maximum matching problem***— can be solved in polynomial time (but we will not discuss it here).
But when we ask for the intersection of three matroids, the problem becomes NP-hard.