9. Stable matching

Design and Analysis of Algorithms

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(borrowed heavily from Kevin Wayne's presentation)

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- Stable matching is a simple game-theoretic algorithmic problem
- Multiple applications
- Nobel Prize to Lloyd Shapley and Alvin Roth, 2012

Goal. Given a set of preferences among hospitals and med-school students, design a self-reinforcing admissions process.

Unstable pair. Hospital *h* and student *s* form an unstable pair if both:

- *h* prefers *s* to one of its admitted students
- *s* prefers *h* to assigned hospital.

Stable assignment. Assignment with no unstable pairs.

- Natural and desirable condition.
- Individual self-interest prevents any hospital-student side deal.

Input: A set of n hospitals H and a set of n students S.

- Each hospital $h \in H$ ranks students
- Each student $s \in S$ ranks hospitals

More concretely, the input is a set of 2n permutations of $(1, \ldots, n)$.

Example : input

n = 3

- Hospitals = $\{A, B, C\}$
- Students = $\{X, Y, Z\}$

Preferences of hospitals (left) and students (right)

	1	2	3
Α	Х	Υ	Ζ
В	Υ	Х	Ζ
С	Х	Υ	Ζ

	1	2	3
X	В	А	С
Y	А	В	С
Ζ	А	В	С

Definition

A matching M is a set of ordered pairs h - s with $h \in H$ and $s \in S$, such that

- Each hospital $h \in H$ appears in at most one pair of M
- Each student $s \in S$ appears in at most one pair of M

A matching M is perfect if every member of H (and S) is matched, i.e., appears in M.

Example : perfect matching

n = 3

 $\mathsf{Hospitals} = \{A, B, C\}$

Students = {X, Y, Z}

Preferences of hospitals (left) and students (right)

	1	2	3
Α	Х	Υ	Ζ
В	Υ	Х	Ζ
С	Х	Υ	Ζ



A perfect matching: $\{A - Z, B - Y, C - X\}$.

Definition

Given a perfect matching M, hospital h and student s form an unstable pair if both:

- *h* prefers *s* to matched student
- *s* prefers *h* to matched hospital.

An unstable pair h-s could each improve by joint action.

Example : unstable pair

n = 3

Hospitals = $\{A, B, C\}$

Students = {X, Y, Z}

Preferences of hospitals (left) and students (right)





A - Y is an unstable pair.

Stable matching problem

Definition (Stable matching)

A **stable matching** is a perfect matching with no unstable pairs. Stable matching problem.

Definition (Stable matching problem)

Given the preference lists of n hospitals and n students, find a stable matching (if one exists).

	1	2	3	
A	Х	Υ	Ζ	
B	Υ	Х	Ζ	
С	Х	Y	Ζ	

	1	2	3
X	В	А	С
Y	А	В	С
Ζ	А	В	С

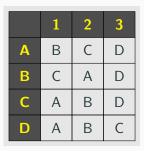
A stable matching $\{A - X, B - Y, C - Z\}$.

Stable roommate problem

- Do stable matchings always exist?
- Not obvious a priori.

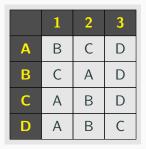
Stable roommate problem:

- 2n people; each person ranks others from 1 to 2n-1
- Assign roommate pairs so that no unstable pairs.



No perfect matching is stable

Stable roomate problem



No perfect roommate matching is stable

A - B, $C - D \Rightarrow B - C$ is unstable A - C, $B - D \Rightarrow A - B$ is unstable A - D, $B - C \Rightarrow A - C$ is unstable

Therefore, stable roommate matchings may not exist.

Gale-Shapley deferred acceptance algorithm

A natural algorithm that guarantees to find a stable matching.

GALE-SHAPLEY(lists of poreferences)

- 1 $M \leftarrow \emptyset$
- 2 while (some hospital h is unmatched and hasn't proposed to every st
- 3 $s \leftarrow$ first student on h's list to whom h has not yet proposed
- 4 if (s is unmatched)
- 5 **then** Add h-s to matching M
- 6 **else if** (*s* prefers *h* to current partner h')
- 7 **then** Replace h'-s with h-s in matching M
- 8 else *s* rejects *h*
- 9 **return** stable matching *M*.

Running time

- Hospitals propose to students in decreasing order of preference
- Once a student is matched, the student never becomes unmatched; only "trades up."

Lemma

Algorithm terminates after at most n^2 iterations of While loop.

Proof.

Each time through the While loop, a hospital proposes to a new student. Thus, there are at most n^2 possible proposals.

Examples with n(n-1) + 1 steps



	1	2	3	4
A	W	Х	Υ	Ζ
В	Х	Y	W	Ζ
С	Υ	W	Х	Ζ
D	W	Х	Υ	Ζ

	1	2	3
X	В	С	А
Y	С	А	В
Ζ	А	В	С

	1	2	3	4
W	В	С	D	А
X	С	D	А	В
Y	D	А	В	С
Ζ	А	В	С	D

Lemma

The Gale-Shapley algorithm finds a perfect matching.

Proof.

- Suppose, for sake of contradiction, that some hospital *h* is unmatched upon termination of the Gale–Shapley algorithm
- Then some student, say s, is unmatched upon termination
- So *s* was never proposed to, because once proposed it becomes matched and remains matched thereafter
- But, h proposes to every student, since h ends up unmatched

Proof of correctness

Lemma

The matching M returned by the Gale-Shapley algorithm is stable.

Proof.

Consider any pair h-s that is not in M. We show that it is not unstable.

h **never proposed to** *s***:** Therefore, *h* prefers its student in *M* to *s*.

h proposed to s: Therefore s rejected h at some point, which means that s ended up with a more preferred hospital.

Theorem (Gale–Shapley 1962)

The Gale–Shapley algorithm guarantees to find a stable matching for any problem instance.

Multiple stable matchings

An instance may have multiple stable matchings. For example: $\{A - X, B - Y, C - Z\}$ and $\{A - Y, B - X, C - Z\}$

	1	2	3
Α	Х	Y	Ζ
В	Y	Х	Ζ
С	Х	Υ	Ζ

	1	2	3
X	В	А	С
Y	А	В	С
Ζ	А	В	С



	1	2	3
X	В	А	С
Y	А	В	С
Ζ	А	В	С

Valid partners

A student s is a valid partner for hospital h if there exists any stable matching in which h and s are matched.

For example

- Both X and Y are valid partners for A.
- Both X and Y are valid partners for B.
- Z is the only valid partner for C.

	1	2	3
Α	Х	Υ	Ζ
В	Υ	Х	Ζ
С	Х	Υ	Ζ

	1	2	3
X	В	А	С
Y	А	В	С
Ζ	А	В	С

Hospital-optimal assignment: Each hospital receives best valid partner.

- Is it a perfect matching?
- Is it stable?

Lemma

The Gale-Shapley algorithm returns the hospital-optimal assignment.

As a corollary, we get that the hospital-optimal assignment is stable.

Student-pessimal assignment: Each student receives worst valid partner.

Lemma

The Gale-Shapley algorithm returns the student-pessimal assignment.

As a corollary, we get that the student-pessimal assignment is stable.

Is the Gale-Shapley algorithm truthful? That is, can participants gain by misrepresenting their preferences?

- A hospital cannot get a better solution by lying about their preference
- But a student may gain by lying about their preferences