Literature Review

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The first axiomatisation of set theory by Zermelo [Zer08, §1 1.] in 1908 allowed "Dinge" (objects) that are not "Mengen" (sets). Such an object a — now known as an urelement or an *atom* — is certainly not the empty set because it is not a set, yet it does not satisfy

$$x \in a$$

for any x. On the other hand, a may still be an element of a set: for example, we have $a \in \{a, \{a\}\}$. Let $\mathbb{A}$ denote the collection of all the atoms, and let G be a group of bijections from $\mathbb{A}$ to itself. Then $\pi \in G$ permutes not only atoms but also sets with atoms: e.g., writing $\pi(a) = a'$ and $\pi(b) = b'$, we get

$$\pi \cdot \{a, \{b\}\} = \{a', \{b'\}\}.$$

A set X is called *finitely supported* if there is a finite $S \subseteq \mathbb{A}$ such that:

$$\pi \cdot X = X \text{ whenever } \pi \in G \text{ fixes every atom in } S.$$

Intuitively, this asserts that X can be described using the finitely many atoms from S. For instance,

- the sets $\{a, b\}$ and $\mathbb{A} \setminus \{a, b\}$ can be described by mentioning just the atoms a and b;
- the function $\mathbb{A} \times \mathbb{A} \to \mathbb{A}; (a, b) \mapsto a$ — i.e., the set $\{((a, b), a) \mid (a, b) \in \mathbb{A} \times \mathbb{A}\} \subseteq (\mathbb{A} \times \mathbb{A}) \times \mathbb{A}$ — can be described without mentioning any atoms at all.

A *hereditarily finitely supported set* is a finitely supported set whose elements are all hereditarily finitely supported. By working with these sets only we arrive at the *permutation model* of set theory with atoms, the introduction of which many attribute to Fraenkel for his 1922 paper "Über den Begriff 'definit' und die Unabhängigkeit des Auswahlaxioms". Fraenkel proceeded to define a permutation model in which the axiom of choice fails in a dramatic manner, namely, for a countable family of pairs [Fra35, p.43]; in the follow-up [Fra37], he attempted to show that the axiom of choice can still fail even if a weaker version — that is, any family of finite sets admits a choice function — holds. However a mistake was noted by Mostowski, who managed to give a correct proof in [Mos39, Korollar 2] by considering, as atoms, the rationals $\mathbb{Q}$ with all order-preserving bijections. (And it was not until 1963 that Cohen, via his famed technique of forcing, proved the axiom of choice can fail in a model of set theory without atoms.) Such classical mathematics was written in very different languages; instead, a modern account of these permutation models can be found in [Jec73, Chapter 4].

Over sixty years had elapsed when, in computer science, Gabbay and Pitts [GP02] dug up the Fraenkel–Mostowski model in search of an elegant representation for abstract syntax trees up to α-renaming. The motivating example they put forward is the untyped λ-calculus, whose terms are either a variable from a countably infinite set like the naturals $\mathbb{N}$, an application of two terms, or an abstraction of a binding variable away from a term; in other words, the set Λ of terms is inductively defined by

$$F(X) = \mathbb{N} \uplus (X \times X) \uplus (\mathbb{N} \times X)$$

which means that proof by structural induction and definition by structural recursion on Λ are mathematically founded. Nonetheless we often wish to identify $(\lambda x.xy)(\lambda x.x)$ with $(\lambda x'.x'y)(\lambda x.x)$ and work with $\Lambda/{=_\alpha}$, the terms modulo α-equivalence. With $\mathbb{N}$ as the atoms, the set Λ becomes a set with atoms; a bijection $\mathbb{N} \to \mathbb{N}$ such as the transposition $(x\ y)$ just renames the variables: e.g., $(x\ y) \cdot (\lambda x.xy)(\lambda x.x) = (\lambda y.yx)(\lambda y.y)$. Accordingly the sets in the Gabbay–Pitts rendition of the permutation model are called *nominal sets*. A first observation is the equivalence of the following statements for two terms $\lambda a.t$ and $\lambda a'.t'$:

1) $\lambda a.t =_\alpha \lambda a'.t'$;
2) $(a\ b) \cdot t =_\alpha (a'\ b) \cdot t'$ for some $b \in \mathbb{N}$ that does not occur (as a binding, free, or bound variable) in t or t';
3) $(a\ b) \cdot t =_\alpha (a'\ b) \cdot t'$ for any $b \in \mathbb{N}$ which does not occur in t or t'.

Thus we can push the global quotient of Λ by $=_\alpha$ down to a local but inductive quotient. To make this concrete, we need the definition of the set $[\mathbb{N}]X$ of *atom abstractions* or *name abstractions* associated with a set X (see [GP02, Definition 5.4] or [Pit13, Definition 4.4]) where, as basic examples,

- $[\mathbb{N}]\mathbb{N}$ comprises the equivalence classes $a.a \stackrel{\text{def}}{=} \{(a,a) \mid a \in \mathbb{N}\}$ and $b.a \stackrel{\text{def}}{=} \{(b,a) \mid b \in \mathbb{N}\setminus\{a\}\}, a \in \mathbb{N}$ corresponding to the α-equivalence classes of $\lambda a.a$ and of $\lambda b.a$, whilst
- $[\mathbb{N}]([\mathbb{N}]\mathbb{N})$ comprises the equivalence classes $\{(b, a.a \mid b \in \mathbb{N}\}$ and $\{(a, b.a)\}$ for $a \in \mathbb{N}$, $\{(c, b.a \mid c \in \mathbb{N} \setminus \{a\}\}$ for $a \in \mathbb{N}$ corresponding to the α-equivalence classes of $\lambda ba.a$, $\lambda ab.a$, and $\lambda cb.a$.

Then $\Lambda/{=_\alpha}$ is in bijection with the set Λ_α inductively defined by

$$F_\alpha(X) = \mathbb{N} \uplus (X \times X) \uplus [\mathbb{N}]X,$$

except the latter supports "α-structural" induction and recursion that perfectly match informal proofs involving "let a be fresh".

The Gabbay–Pitts permutation model was not entirely novel: as explained in [GMM06] and also in [Pit13, Chapter 6], equivalent forms of nominal sets have been known to the concurrency community as named sets [MP05], by category theorists as the Schanuel topos, and by model theorists as continuous $\operatorname{Aut}\mathbb{N}$-sets. The last perspective easily generalises to other choices of atom structures together with their symmetry groups: $(\mathbb{N}, =)$ and arbitrary bijections are to variable names as $(\mathbb{Q}, =, <)$ and monotone bijections are to timestamps. Indeed, sets with atoms provide a convenient setting for studying languages of data words over an infinite alphabet as well as the models of computation that recognise them; this approach has been extensively explored by the Warsaw school beginning with [Boj13]. It is perhaps curious that, like with the axiom of choice almost a century back, the yet unresolved P versus NP problem can be settled in the presence of atoms, negatively [BKLT13, Theorem III.1].

That marks the end of the historical notes, and here begins the mathematical account. A first step towards tackling computation theoretic concerns with atoms is to address the possibly infinite sets involved. Consider for instance $[\mathbb{N}]\mathbb{N} = \{a.a\} \cup \{a+1.a \mid a \in \mathbb{N}\}$ from above, which is infinite but consists of just two orbits — $\operatorname{Aut}\mathbb{N} \cdot a.a$ and $\operatorname{Aut}\mathbb{N} \cdot b.a$ — and thus describable in a finite manner. To obtain a precise and robust notion of finitely presentable sets that are amenable to algorithmic manipulation, we need a sufficiently well-behaved structure $\mathbb{A}$; to spell the model-theoretic desiderata out, we consider a countably infinite, homogeneous, oligomorphic structure over a finite relational signature that has no algebraicity and admits least finite supports. Although such terms are explained in ample detail in textbooks like [Hod93, Chapters 4–7] and [Kir19, Part V], many more have made the effort to introduce them in an approachable and self-contained manner; drawing on [Cam90, Chapter 2] [Mac11, §2–4] [Eva13, §1–2] [Boj19, Chapter 7], [Bod21, Chapters 2–4] I shall attempt to do the same.

Contents

1 The atoms structure

Fix a signature $\mathcal{R}$ of relation symbols $R_0, R_1, \ldots$ of arities $n_0, n_1, \ldots$; an $\mathcal{R}$-*structure* $(\mathbb{A}, \mathcal{R})$ is a domain $\mathbb{A}$ together with interpretations $R_i^{\mathbb{A}} \subseteq \mathbb{A}^{n_i}$, where we always assume R_0 is the binary symbol $=$ interpreted as $\{(a, a) \mid a \in \mathbb{A}\}$. We fix a countably infinite $\mathbb{A}$ throughout. Also, by $\mathbb{A}$ we will mean both the structure and the domain.

Every subset $A \subseteq \mathbb{A}$ automatically defines a new $\mathcal{R}$-structure with interpretations $R_i^A \overset{\text{def}}{=} R_i^{\mathbb{A}} \cap A^{n_i}$; by a *substructure* of $\mathbb{A}$ we simply mean a subset of $\mathbb{A}$. When A is finite, we more specifically write $A \subseteq_{\mathit{fin}} \mathbb{A}$. A function $f : A \to B$ between two $\mathcal{R}$-structures is an *embedding* if

$$\forall (a_1, \ldots, a_{n_i}) \in A^{n_i} : (a_1, \ldots, a_{n_i}) \in R_i^A \iff (f(a_1), \ldots, f(a_{n_i})) \in R_i^B$$

for every i. In particular any embedding is always injective, and if it is moreover surjective we call it an *isomorphism*. We call an isomorphism mapping $\mathbb{A}$ into $\mathbb{A}$ an *automorphism*; all the automorphisms form a group, which we denote by $\operatorname{Aut}\mathbb{A}$.

1.1 Homogeneity

Definition 1.1
$\mathbb{A}$ is *homogeneous* if every isomorphism between $A, B \subseteq_{\mathit{fin}} \mathbb{A}$ extends to an automorphism of $\mathbb{A}$. That is, given any isomorphism $f : A \to B$, there exists some $\pi \in \operatorname{Aut}\mathbb{A}$ such that $\pi|_A = f$.

Example 1.2 *The equality atoms* $(\mathbb{N}, =)$ is homogeneous: we can first extend f to a finite bijection $A \cup A' \to A \cup A'$ since $(A \cup A') \setminus A$ has the same number of elements as $(A \cup A') \setminus f(A)$, then to an automorphism of $\mathbb{N}$ by mapping $\mathbb{N} \setminus (A \cup A')$ identically. A more commonly used name is the countable pure set.

So are *the ordered atoms* $(\mathbb{Q}, =, <)$, i.e., the countable dense linear ordering without endpoints: we can extend the monotone f to a piecewise linear, monotone bijection $\mathbb{Q} \to \mathbb{Q}$.

We adopt Fraïssé's quaint terminology [Fra74] [Fra00, §10.2.1] of calling the collection $\mathcal{K}$ of finite $\mathcal{R}$-structures, each of which isomorphic to some finite substructure of $\mathbb{A}$, the *âge of* $\mathbb{A}$. Observe that

1) $\mathcal{K}$ is closed under taking substructures and isomorphic images;
2) $\mathcal{K}$ has countably many isomorphism classes, as there are only countably many finite substructures of $\mathbb{A}$;
3) $\mathcal{K}$ satisfies the *amalgamation property* — i.e., for every $A' \subseteq A \in \mathcal{K}$, $B \in \mathcal{K}$ and embedding $f : A' \to B$, there is some $C \in \mathcal{K}$ together with embeddings $g : B \to C$, $h : A \to C$ such that $g \circ f = h|_{A'}$ making

$$\begin{tikzcd}
& g(B) \cup h(A) & \\
B \arrow[ur, dashed, "g"] & & A \arrow[ul, dashed, "h"'] \\
& A' \arrow[ul, "f"] \arrow[ur, phantom, "\subsetneq"] &
\end{tikzcd}$$

commute — provided that $\mathbb{A}$ is homogeneous: extend f to some $\pi \in \operatorname{Aut}\mathbb{A}$ and put $C \overset{\text{def}}{=} B \cup \pi(A)$.

There is a converse to this. Call a collection $\mathcal{K}$ of finite $\mathcal{R}$-structures that satisfies the properties 1), 2), and 3) an *amalgamation class*.

Theorem 1.3 ([Fra54, Théorème VI])
If $\mathcal{K}$ is an amalgamation class, then

i) we can construct a countable homogeneous $\mathcal{R}$-structure $\mathrm{FLim}\,\mathcal{K}$ *whose âge is precisely $\mathcal{K}$, and*
ii) any another countable homogeneous $\mathcal{R}$-structure that also has âge $\mathcal{K}$ is isomorphic to $\mathrm{FLim}\,\mathcal{K}$.

We shall call $\mathrm{FLim}\,\mathcal{K}$ the *Fraïssé limit*. The homogeneous condition in ii) cannot be dropped: $(\mathbb{Z}, =, <)$ and $(\mathbb{Q}, =, <)$ are different order types yet share the same âge of all finite total orders, because no $\pi \in \mathrm{Aut}\,\mathbb{Z}$ extends the isomorphism $0 \mapsto 0, 1 \mapsto 2$. This is also why $\mathbb{Q}$, rather than $\mathbb{Z}$, is the domain of the ordered atoms.

Example 1.4 An amalgamation class $\mathcal{K}$ is said to be *strong* if in 3) one can moreover choose C, g, h in a way that $g(B) \cap h(A)$ coincides with $(g \circ f)(A') = h(A')$.

a) The âge of the equality atoms is a strong amalgamation class: let $C \stackrel{\text{def}}{=} \{(0, b) \mid b \in B \setminus f(A')\} \cup \{(1, a) \mid a \in A\}$ be the amalgam, and put

$$g : \begin{aligned} b \in B \setminus f(A') &\mapsto (0, b), \\ f(a') \in f(A') &\mapsto (1, a'); \end{aligned} \qquad h : a \in A \mapsto (1, a).$$

On the other hand, consider $(\mathbb{N}, =, O)$ where the unary predicate symbol O has interpretation $O^{\mathbb{N}} = \{0\}$. This is a homogeneous structure whose âge consists of all finite sets with at most one element for which O holds, which fails to be a strong amalgamation class: any amalgam of the substructures $\{0\} \supseteq \emptyset \subseteq \{0, 1\}$ must identify the two 0's, which certainly will not be in the image of $\emptyset$.

b) Consider the signature $=, \sim$ for graphs. The collection $\mathcal{K}_\sim$ of all loopless finite undirected graphs is a strong amalgamation class: with C as above, one can check that the interpretation

$$\begin{aligned} \sim^C \stackrel{\text{def}}{=} &\left\{\big(g(b), g(b')\big) \mid b, b' \in B \setminus f(A'), (b, b') \in \sim^B\right\} \\ \cup &\left\{\big(g(b), g(f(a')) = h(a')\big) \mid b \in B \setminus f(A'), a' \in A', (b, f(a')) \in \sim^B\right\} \\ \cup &\left\{\big(g(f(a')) = h(a'), g(b)\big) \mid b \in B \setminus f(A'), a' \in A', (f(a'), b) \in \sim^B\right\} \\ \cup &\left\{\big(h(a_1), h(a_2)\big) \mid a_1, a_2 \in A, (a_1, a_2) \in \sim^A\right\} \end{aligned}$$

imposes just enough relations to make g and h embeddings. Then $\mathrm{FLim}\,\mathcal{K}_\sim$ is *the Rado graph* [Rad64, Theorem 1], which we also refer to as *the graph atoms*.

Now consider the subcollection $\mathcal{K}_\sim^{K_n \not\hookrightarrow}$ of those graphs in $\mathcal{K}_\sim$ into which the complete graph K_n on $n \geq 3$ vertices does not embed. The above construction shows that $\mathcal{K}_\sim^{K_n \not\hookrightarrow}$ is a strong amalgamation class as well. To see that, note an n-clique in the amalgam $C = g(B) \cup h(A)$ cannot have vertices in both $g(B)$ and $h(A) \setminus h(A')$ due to the lack of edges; so the n-clique must be entirely contained in $h(A)$ or else in $g(B)$, which is impossible as neither A nor B contains an n-clique. The Fraïssé limit $\mathrm{FLim}\,\mathcal{K}_\sim^{K_n \not\hookrightarrow}$ is known as *the Henson K_n-free graph* [Hen71, Theorem 2.3].

Lastly, let the cardinalities $1 \leq m, n \leq \aleph_0$ be such that $m \times n = \aleph_0$ and consider the disjoint union $\biguplus_m K_n$ of complete graphs. Its âge is an amalgamation class, which is strong if and only if $n \in \{1, \aleph_0\}$. (Note that $\biguplus_{\aleph_0} K_1$ and $\biguplus_1 K_{\aleph_0}$ are graph complements; they are just the equality atoms augmented with a constant binary predicate.) In fact, every countably infinite homogeneous graph is isomorphic to $\biguplus_m K_n$, $\mathrm{FLim}\,\mathcal{K}_\sim^{K_n \not\hookrightarrow}$, $\mathrm{FLim}\,\mathcal{K}_\sim$ or their graph complements [LW80].

c) Change the signature to $=, \to$ and let $\mathcal{K}_\to$ be the collection of all loopless finite directed graphs without double edges, or digraphs for short. Then, replacing $\sim$ by $\to$ in the construction above, we see that $\mathcal{K}_\to$ is a strong amalgamation class.

Again we can forbid certain subgraphs. Call $D \in \mathcal{K}_\to$ a *tournament* if precisely one of $x \to y$ or $y \to x$ holds for any two distinct vertices $x \neq y$ of D; then the subcollection $\mathcal{K}_\to^{D\not\hookrightarrow}$ of digraphs not embedding the tournament D forms a strong amalgamation class by the same argument. More generally, if $D_1, D_2, \ldots$ are all tournaments then $\bigcap_{n\in I} \mathcal{K}_\to^{D_n\not\hookrightarrow}$ is a strong amalgamation class given any $I \subseteq \mathbb{N}$. By choosing the D_n's so that D_n does not embed into $D_{n'}$ for $n \neq n'$, we may ensure $\bigcap_{n\in I} \mathcal{K}_\to^{D_n\not\hookrightarrow} \neq \bigcap_{n'\in I'} \mathcal{K}_\to^{D_{n'}\not\hookrightarrow}$ whenever $I, I' \subseteq \mathbb{N}$ are distinct. Thus

$$\{\mathrm{FLim}(\bigcap_{n\in I} \mathcal{K}_\to^{D_n\not\hookrightarrow}) \mid I \subseteq \mathbb{N}\}$$

is an uncountable family of non-isomorphic homogeneous directed graphs, known as the *Henson digraphs* [Hen72, Theorem 2.4].

Actually the tournaments themselves form a strong amalgamation class $\mathcal{K}_\to^{\mathrm{tourn}}$: when constructing the amalgam $C = g(B) \cup h(A)$ we have to additionally include edges from each vertex of $g(B \setminus f(A'))$ to each vertex of $h(A \setminus A')$. The Fraïssé limit $\mathrm{FLim}\,\mathcal{K}_\to^{\mathrm{tourn}}$ is called *the random tournament*.

d) Finally, consider the signature $=, <$. The collection $\mathcal{K}_<^{\mathrm{total}}$ of finite total strict orders is a strong amalgamation class: on the set $C = g(B) \cup h(A)$ impose

$$\begin{aligned}
<^C \stackrel{\mathrm{def}}{=} &\left\{\big(g(b), g(b')\big) \mid b, b' \in B \setminus f(A'), (b, b') \in <^B\right\} \\
\cup &\left\{\big(g(b), g(f(a')) = h(a')\big) \mid b \in B \setminus f(A'), a' \in A', (b, f(a')) \in <^B\right\} \\
\cup &\left\{\big(g(b), h(a)\big) \mid b \in B \setminus f(A'), a \in A \setminus A', \nexists a' \in A' : (a, a') \in <^A \wedge (f(a'), b) \in <^B\right\} \\
\cup &\left\{\big(h(a') = g(f(a')), g(b)\big) \mid a' \in A', b \in B \setminus f(A'), (f(a'), b) \in <^B\right\} \\
\cup &\left\{\big(h(a), g(b)\big) \mid a \in A \setminus A', b \in B \setminus f(A'), \exists a' \in A' : (a, a') \in <^A \wedge (f(a'), b) \in <^B\right\} \\
\cup &\left\{\big(h(a_1), h(a_2)\big) \mid a_1, a_2 \in A, (a_1, a_2) \in <^A\right\}.
\end{aligned}$$

Essentially we first make g and h order embeddings before making the rather arbitrary decision that, where possible, anything in the image of g should be smaller than anything in the image of h. One can check that $<^C$ is total, asymmetric, and transitive so long as $g(B) \cap h(A) = g(f(A')) = h(A')$. Of course $\mathrm{FLim}\,\mathcal{K}_<^{\mathrm{total}}$ is the ordered atoms up to isomorphism.

We have actually established the following: let $\mathcal{K}$ be a strong amalgamation class over a signature $\mathcal{R}$ which does not already contain the symbol $<$, and consider the collection $\mathcal{K}_<$ of finite $\mathcal{R}, <$-structures obtained by adding all possible total orders to all $\mathcal{R}$-structures in $\mathcal{K}$; then $\mathcal{K}_<$ is a strong amalgamation class — first form the amalgam over $\mathcal{R}$, and prescribe the total order $<$ as above. We call $\mathrm{FLim}\,\mathcal{K}_<$ the *generically ordered* version of $\mathrm{FLim}\,\mathcal{K}$.

1.2 Oligomorphicity

We will give four definitions — the first three model-theoretic, the last group-theoretic — and prove that they are all equivalent.

1.2.1 Theories and models

In our first-order relational framework, the $\mathcal{R}$-*formulae* are given by the grammar

$$\phi, \psi, \cdots ::= \top \mid R_i(x_1, \ldots, x_{n_i}) \mid \neg\phi \mid \phi \wedge \psi \mid \exists x \phi$$

where $=, R_1, R_2, \ldots$ are the relation symbols of $\mathcal{R}$. We define shorthands $\perp, x_1 \; R_i \; x_2, \phi \vee \psi, \phi \to \psi, \phi \leftrightarrow \psi$, and $\forall\phi$ in the standard way. By writing $\phi(x_1, \ldots, x_n)$ we mean the free variables of ϕ are amongst $x_1, \ldots, x_n$; if $n = 0$ we call ϕ an $\mathcal{R}$-*sentence*. Given an $\mathcal{R}$-structure $\mathbb{B}$ and elements $b_1, \ldots, b_n \in \mathbb{B}$, the formula ϕ substituted with b_i for x_i either holds or does not; we write $\mathbb{B} \models \phi(b_1, \ldots, b_n)$ or $\mathbb{B} \not\models \phi(b_1, \ldots, b_n)$ respectively. In particular, an $\mathcal{R}$-sentence either holds in $\mathbb{B}$ or does not. The collection of the sentences that do hold is called the *theory of* $\mathbb{B}$ and written $Th(\mathbb{B})$. Conversely, given a collection T of $\mathcal{R}$-sentences, we can ask for an $\mathcal{R}$-structure $\mathbb{B}$ in which every $\phi \in T$ holds; such a structure satisfying $T \subseteq Th(\mathbb{B})$ is called a *model of* T. When $T = Th(\mathbb{A})$, the model $\mathbb{B}$ moreover satisfies $Th(\mathbb{A}) = Th(\mathbb{B})$.

Obviously the countably infinite $\mathbb{A}$ that we fixed is itself a model of $Th(\mathbb{A})$. However, a known limitation of first-order theories is that we cannot control the size of other infinite models: by the upward Löwenheim–Skolem Theorem, there exist models of $Th(\mathbb{A})$ of arbitrarily large cardinalities. We declare ourselves as countablists and announce that uncountable models are meaningless anyway: we want to effectively represent elements in the domain and compute which relations hold between them. Restricting our attention to countably infinite models, we may demand the following.

Definition 1.5
$Th(\mathbb{A})$ is $\aleph_0$-*categorical* (or ω-*categorical*) if any countably infinite model $\mathbb{A}'$ of $Th(\mathbb{A})$ is isomorphic to $\mathbb{A}$.

1.2.2 Definable subsets

Definition 1.6
Let $B \subseteq \mathbb{A}$ be a subset. A B-*definable subset* of $\mathbb{A}^n$ is one of the form

$$\{(a_1, \ldots, a_n) \in \mathbb{A}^n : \mathbb{A} \models \phi(a_1, \ldots, a_n, b_1, \ldots, b_k)\}$$

where $\phi(x_1, \ldots, x_n, y_1, \ldots, y_k)$ is an $\mathcal{R}$-formula and $b_1, \ldots, b_k$ are constants from B; implicitly assuming an enumeration of the free variables, we denote this set by $\phi[\mathbb{A}^n, b_1, \ldots, b_k]$. We also define

- the *model-theoretic algebraic closure* $ACL(B)$ as the union of finite B-definable subsets of $\mathbb{A}$, as well as
- the *model-theoretic definable closure* $DCL(B)$ as the union of singleton B-definable subsets of $\mathbb{A}$.

Assume $B = \emptyset$ and fix variables $x_1, x_2, \cdots$. Given two formulae $\phi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$, let us write $\phi \equiv_{Th(\mathbb{A})} \psi$ if

$$\mathbb{A} \models \forall x_1 \cdots \forall x_n \big(\phi(x_1, \ldots, x_n) \to \psi(x_1, \ldots, x_n)\big);$$

then we have $\phi[\mathbb{A}^n] = \psi[\mathbb{A}^n]$ precisely when $\phi \equiv_{Th(\mathbb{A})} \psi$. Moreover, notice that

$$\begin{aligned} \phi[\mathbb{A}^n] \cup \psi[\mathbb{A}^n] &= (\phi \vee \psi)[\mathbb{A}^n], \qquad \mathbb{A}^n \setminus \phi[\mathbb{A}^n] = (\neg\phi)[\mathbb{A}^n], \\ \phi[\mathbb{A}^n] \cap \psi[\mathbb{A}^n] &= (\phi \wedge \psi)[\mathbb{A}^n]. \end{aligned}$$

Hence the $\equiv_{Th(\mathbb{A})}$-equivalence classes of $\mathcal{R}$-formulae in the free variables $x_1, \ldots, x_n$ together with $\vee, \wedge, \neg$ form a Boolean algebra, the *nth Lindenbaum algebra*; so do the $\emptyset$-definable subsets of $\mathbb{A}^n$, and the two are isomorphic.

1.2.3 Types

In the other direction, given constants $a_1, \ldots, a_n \in \mathbb{A}$ we can consider the $\mathcal{R}$-formulae that they satisfy:

$$tp_{\mathbb{A}}(a_1, \ldots, a_n) \stackrel{\text{def}}{=} \{\phi(x_1, \ldots, x_n) : \mathbb{A} \models \phi(a_1, \ldots, a_n)\}$$

is what model theorists call the *n-type of* $a_1, \ldots, a_n$ (whereas the types for type theorists are what model theorists call *sorts*). More generally, an *n-type of* $Th(\mathbb{A})$ is a collection of $\mathcal{R}$-formulae equal to some $tp_{\mathbb{C}}(c_1, \ldots, c_n)$ where $\iota : \mathbb{A} \to \mathbb{C}$ is an *elementary* embedding, i.e., an embedding satisfying

$$\mathbb{A} \models \psi(b_1, \ldots, b_k) \iff \mathbb{C} \models \psi(\iota(b_1), \ldots, \iota(b_k))$$

for any $\mathcal{R}$-formula $\psi(x_1, \ldots, x_k)$ and elements $b_1, \ldots, b_k \in \mathbb{A}$; in that case the first-order sentences cannot tell $\mathbb{A}$ and $\mathbb{C}$ apart: we have $Th(\mathbb{A}) = Th(\mathbb{C})$. Now if

$$tp_{\mathbb{C}}(c_1, \ldots, c_n) = tp_{\mathbb{A}}(a_1, \ldots, a_n)$$

for some $a_i \in \mathbb{A}$, we say the a_i's *realise* the n-type in $\mathbb{A}$; otherwise $\mathbb{A}$ *omits* it. Certainly $tp_{\mathbb{C}}(\iota(a_1), \ldots, \iota(a_n))$ is realised by the a_i's in $\mathbb{A}$.

Definition 1.7
An n-type $tp_{\mathbb{C}}(c_1, \ldots, c_n)$ is *principal* or *isolated* if it contains some $\chi(x_1, \ldots, x_n)$ such that

$$\mathbb{C} \models \forall x_1 \cdots \forall x_n \big(\chi(x_1, \ldots, x_n) \to \phi(x_1, \ldots, x_n)\big)$$

for every $\phi \in tp_{\mathbb{C}}(c_1, \ldots, c_n)$. We also call χ a *principal formula* for the n-type.

A principal n-type is always realised in $\mathbb{A}$: the sentence $\exists c_1 \cdots \exists c_n \chi(c_1, \ldots, c_n)$ belongs to $Th(\mathbb{C}) = Th(\mathbb{A})$, so $\mathbb{A} \models \chi(a_1, \ldots, a_n)$ for some $a_i \in \mathbb{A}$ and thus $tp_{\mathbb{A}}(a_1, \ldots, a_n)$ contains $tp_{\mathbb{C}}(c_1, \ldots, c_n)$. But evidently

$$tp_{\mathbb{A}}(\overline{a}) \supseteq tp_{\mathbb{C}}(\overline{c}) \implies tp_{\mathbb{A}}(\overline{a}) = tp_{\mathbb{C}}(\overline{c})$$

because $\mathbb{A} \models \phi(\overline{a})$ means $\mathbb{A} \not\models \neg\phi(\overline{a})$, so $\mathbb{C} \not\models \neg\phi(\overline{c})$ which means $\mathbb{C} \models \phi(\overline{c})$.

1.2.4 Orbits

We cover one last definition, this time without mentioning model theory at all.

Definition 1.8
Let $n \in \mathbb{N}$. The group $\operatorname{Aut}\mathbb{A}$ acts on the set $\mathbb{A}^n$ via

$$\pi \cdot (a_1, \ldots, a_n) = (\pi(a_1), \ldots, \pi(a_n));$$

we have $\mathrm{id} \cdot \overline{a} = \overline{a}$ and $(\tau \circ \pi) \cdot \overline{a} = \tau \cdot (\pi \cdot \overline{a})$ for all $\overline{a} \in \mathbb{A}^n$ and $\tau, \pi \in \operatorname{Aut}\mathbb{A}$. The $\operatorname{Aut}\mathbb{A}$*-orbit of* $\overline{a} \in \mathbb{A}^n$ is

$$\operatorname{Aut}\mathbb{A} \cdot \overline{a} \stackrel{\text{def}}{=} \{\pi \cdot \overline{a} \mid \pi \in \operatorname{Aut}\mathbb{A}\};$$

the distinct $\operatorname{Aut}\mathbb{A}$-orbits form a (possibly infinite) partition of $\mathbb{A}^n$. We call $\mathbb{A}$ *oligomorphic* under $\operatorname{Aut}\mathbb{A}$ if $\mathbb{A}^n$ contains finitely only many distinct $\operatorname{Aut}\mathbb{A}$-orbits for every $n \in \mathbb{N}$.

Given $B \subseteq \mathbb{A}$, let $\mathrm{Aut}\,\mathbb{A}/B$ denote the subgroup $\{\pi \in \mathrm{Aut}\,\mathbb{A} \mid \forall b \in B : \pi(b) = b\}$ which by restriction acts on $\mathbb{A}^n$. À la Galois, we also define

- the *group-theoretic algebraic closure* $acl(B)$ as the union of finite $\mathrm{Aut}\,\mathbb{A}/B$-orbits in $\mathbb{A}$, as well as
- the *model-theoretic definable closure* $dcl(B)$ as the union of singleton $\mathrm{Aut}\,\mathbb{A}/B$-orbits in $\mathbb{A}$.

Notice that $(\mathbb{Z}, =, <)$ fails to be oligomorphic: the pairs $(0,0), (0,1), (0,2), \dots \in \mathbb{Z}^2$ all lie in different orbits under $\mathrm{Aut}\,\mathbb{Z} = \{x \mapsto x + d \mid d \in \mathbb{Z}\}$.

Now let us draw some connections to model theory.

Remark 1.9 Given any $\pi \in \mathrm{Aut}\,\mathbb{A}$, for any $\mathcal{R}$-formula ϕ and elements $\overline{a} \in \mathbb{A}^n$ we have

$$\mathbb{A} \models \phi(a_1, \dots, a_n) \iff \mathbb{A} \models \phi(\pi(a_1), \dots, \pi(a_n));$$

indeed every automorphism of $\mathbb{A}$ is an elementary embedding. Consequently

a) $tp_{\mathbb{A}}(\overline{a}) = tp_{\mathbb{A}}(\pi \cdot \overline{a})$, so every element in the same $\mathrm{Aut}\,\mathbb{A}$-orbit of $\mathbb{A}^n$ shares the same n-type;

a') in particular, if $\overline{b} \in \mathrm{Aut}\,\mathbb{A} \cdot \overline{a}$ then $a_1 \mapsto b_1, \dots, a_n \mapsto b_n$ is an isomorphism.

b) every B-definable subset $X \subseteq \mathbb{A}^n$ is invariant under the action of $\mathrm{Aut}\,\mathbb{A}/B$, i.e., X is a union of $\mathrm{Aut}\,\mathbb{A}/B$-orbits;

c) $ACL(B) \supseteq acl(B)$ and $DCL(B) \supseteq dcl(B)$.

The point is that oligomorphicity provides converses to a), b), and c), as we shall soon see, whereas homogeneity provides the converse to a').

1.2.5 The four-way equivalence

We are now ready to prove the equivalence of all four notions above, at the heart of which is the following technique often attributed to Cantor.

Lemma 1.10 (the back-and-forth method) *Suppose that for every $m \in \mathbb{N}$, every m-type of $Th(\mathbb{A})$ is principal. Let $\mathbb{B}$ be a countable model of $Th(\mathbb{A})$ where $tp_{\mathbb{A}}(a_1, \dots, a_n) = tp_{\mathbb{B}}(b_1, \dots, b_n)$ for some $a_i \in \mathbb{A}$, $b_j \in \mathbb{B}$. Then there is an isomorphism $f : \mathbb{A} \to \mathbb{B}$ mapping $a_1 \mapsto b_1, \dots, a_n \mapsto b_n$.*

Proof. We inductively construct two sequences $a_1, \dots, a_n, a_{n+1}, \dots$ and $b_1, \dots, b_n, b_{n+1}, \dots$ enumerating $\mathbb{A}$ and $\mathbb{B}$ such that

$$tp_{\mathbb{A}}(a_1, \dots, a_{n+i}) = tp_{\mathbb{B}}(b_1, \dots, b_{n+i})$$

for all $i \geq 0$. If i+1 is odd, choose $b_{n+i+1} \in \mathbb{B} \backslash \{b_1, \dots, b_{n+i}\}$ and consider the $(n$+i+1$)$-type of $b_1, \dots, b_{n+i}, b_{n+i+1}$. By assumption it contains a principal formula χ, and $\exists x_{n+i+1} \chi(x_1, \dots, x_{n+i}, x_{n+i+1})$ belongs to $tp_{\mathbb{B}}(b_1, \dots, b_n)$; by the inductive hypothesis, $\mathbb{A} \models \exists x_{n+i+1} \chi(a_1, \dots, a_{n+i}, x_{n+i+1})$ so $a_1, \dots, a_{n+i}, a_{n+i+1}$ satisfies χ and hence realises $tp_{\mathbb{B}}(b_1, \dots, b_{n+i}, b_{n+i+1})$ for some $a_{n+i+1} \in \mathbb{A}$. Symmetrically, going forth if $i + 1$ is even, choose a_{n+i+1} in $\mathbb{A} \setminus \{a_1, \dots, a_{n+i}\}$ and find $b_{n+i+1} \in \mathbb{B}$ that helps realise the same $(n + i + 1)$-type.

Now $f : a_j \mapsto b_j$ is a well-defined function: the formula $x_j = x_{j'}$ is in $tp_{\mathbb{A}}(a_1, \dots, a_{n+i})$ precisely when $a_j = a_{j'}$, and symmetrically $x_j = x_{j'}$ is in $tp_{\mathbb{B}}(b_1, \dots, b_{n+i})$ precisely when $b_j = b_{j'}$; but the two types are equal, which furthermore ensures that f preserves and reflects all relations. Hence we have an embedding which by construction is surjective; that is, we have an isomorphism. ∎

The characterisations below are widely ascribed to Ryll-Nardzewski, though parts were given independently by him at Warsaw, by Engeler at Zurich, and by Svenonius at Uppsala all in 1959; in particular the group-theoretic condition is due to Svenonius [Hod93, History and bibliography for §7.3].

> **Theorem 1.11 (Ryll-Nardzewski, Svenonius, Engeler)**
> *Assume the signature $\mathcal{R}$ is finite or countably infinite. Then the following are equivalent:*
>
> i) *$Th(\mathbb{A})$ is $\aleph_0$-categorical;*
> ii) *for every $n \in \mathbb{N}$, there are only finitely many $\emptyset$-definable subsets of $\mathbb{A}^n$;*
> iii) *for every $n \in \mathbb{N}$, any n-type of $Th(\mathbb{A})$ is principal;*
> iv) *$\mathbb{A}$ is oligomorphic.*

Proof. We will prove i) $\iff$ iii) $\implies$ iv) $\implies$ ii) $\implies$ iii).

i) $\Rightarrow$ iii) If $tp_{\mathbb{C}}(c_1, \ldots, c_n)$ is non-principal, then by Vaught's Omitting Types Theorem there is a countable model $\mathbb{B}$ of $Th(\mathbb{A})$ that omits it whilst — by the downward Löwenheim–Skolem Theorem — a countably infinite model $\mathbb{C}'$ of $Th(\mathbb{A})$ realises it; particularly $\mathbb{B}$ cannot be isomorphic to $\mathbb{C}'$. Other than our countablist considerations, this is the only place where we require the cardinality assumption on the signature $\mathcal{R}$.

i) $\Leftarrow$ iii) This is a direct consequence of Lemma 1.10: start with the empty tuple.

iii) $\Rightarrow$ iv) Combining Remark 1.9a) with Lemma 1.10, we see that n-types correspond bijectively to Aut $\mathbb{A}$-orbits of $\mathbb{A}^n$. Now suppose towards a contradiction that the collection of all distinct n-types $\{tp_{\mathbb{C}_i}(\overline{c_i}) \mid i \in I\}$ is infinite; pick a principal formula $\chi_i(\overline{x})$ for each i. We claim that $\{\neg\chi_i \mid i \in I\}$ is finitely satisfiable: given $\neg\chi_{i_1}, \ldots, \neg\chi_{i_k}$ we can find $i' \in I \setminus \{i_1, \ldots, i_k\}$; since the n-types are distinct, we necessarily have

$$\mathbb{C}_{i'} \models \neg\chi_{i_1}(\overline{c_{i'}}) \wedge \cdots \wedge \neg\chi_{i_k}(\overline{c_{i'}}).$$

By compactness, there is an n-type $tp_{\mathbb{C}}(\overline{c})$ of $Th(\mathbb{A})$ that contains $\{\neg\chi_i \mid i \in I\}$; yet $tp_{\mathbb{C}}(\overline{c}) \neq tp_{\mathbb{C}_i}(\overline{c_i})$ for any i as the latter contains χ_i, which is the desired contradiction. We conclude that $\mathbb{A}$ must be oligomorphic.

iv) $\Rightarrow$ ii) Suppose there are k distinct Aut $\mathbb{A}$-orbits of $\mathbb{A}^n$. As we noted in Remark 1.9b), any $\emptyset$-definable subset of $\mathbb{A}^n$ is a union of Aut $\mathbb{A}$-orbits; there are only 2^k of those.

ii) $\Rightarrow$ iii) Suppose that $\phi_1, \ldots, \phi_p$ are representatives of the $\equiv_{Th(\mathbb{A})}$-equivalence classes of the $\mathcal{R}$-formulae in the free variables $x_1, \ldots, x_n$. Say only $\phi_{i_1}, \ldots, \phi_{i_k}$ appear in the n-type $tp_{\mathbb{C}}(c_1, \ldots, c_n)$; then their conjunction $\phi_{i_1} \wedge \cdots \wedge \phi_{i_k}$ is a principal formula. ■

> **Remark 1.12** Let $B = \{b_1, \ldots, b_k\} \subseteq \mathbb{A}$ be finite. Expand the signature to $\mathcal{R}_B$ by adding unary predicates for each $b \in B$, and interpret $\mathbb{A}$ as an $\mathcal{R}_B$-structure $\mathbb{A}_B$ in the obvious way. Then Aut $\mathbb{A}_B$ = Aut $\mathbb{A}/B$, and $(a_1, \ldots, a_n)$ is in the same Aut $\mathbb{A}/B$-orbit as $(a'_1, \ldots, a'_n)$ precisely when $(a_1, \ldots, a_n, b_1, \ldots, b_k)$ is in the same Aut $\mathbb{A}$-orbit as $(a'_1, \ldots, a'_n, b_1, \ldots, b_k)$. Therefore there are at most as many Aut $\mathbb{A}_B$-orbits of $\mathbb{A}^n$ as Aut $\mathbb{A}$-orbits of $\mathbb{A}^{n+k}$; if $\mathbb{A}$ is oligomorphic, so is $\mathbb{A}_B$. In this case:
>
> a') Every n-type of $Th(\mathbb{A}_B)$ is principal and thus realised in $\mathbb{A}_B$, so it follows from Lemma 1.10 that
>
> $$tp_{\mathbb{A}_B}(a_1, \ldots, a_n) \mapsto \text{Aut}\,\mathbb{A}/B \cdot (a_1, \ldots, a_n)$$
>
> is a well-defined bijection between n-types with parameters from B and Aut $\mathbb{A}/B$-orbits of $\mathbb{A}^n$.

b') The $\operatorname{Aut}\mathbb{A}/B$-orbit of any $\overline{a} \in \mathbb{A}^n$ is $\emptyset$-definable by any principal $\mathcal{R}_B$-formula Φ for $tp_{\mathbb{A}_B}(\overline{a})$. Indeed $\Phi[(\mathbb{A}_B)^n]$ contains $\overline{a}$ and thus its $\operatorname{Aut}\mathbb{A}/B$-orbit; on the other hand, by Lemma 1.10 we have

$$\mathbb{A}_B \models \Phi(\overline{b}) \implies tp_{\mathbb{A}_B}(\overline{b}) = tp_{\mathbb{A}_B}(\overline{a}) \implies \overline{b} \in \operatorname{Aut}\mathbb{A}/B \cdot \overline{a}.$$

Furthermore, the $\operatorname{Aut}\mathbb{A}/B$-orbit of $\overline{a}$ is B-definable by an $\mathcal{R}$-formula — in $\Phi(x_1, \ldots, x_n)$ replace every subformula $b_i(y)$ by $z_i = y$ with z_i fresh to obtain $\phi(x_1, \ldots, x_n, z_1, \ldots, z_k)$, and consider $\phi[\mathbb{A}^n, b_1, \ldots, b_k]$ — so in $\mathbb{A}^n$, the $\operatorname{Aut}\mathbb{A}/B$-invariant subsets are precisely the B-definable ones.

c') $ACL(\{b_1, \ldots, b_k\}) = acl(\{b_1, \ldots, b_k\})$, $DCL(\{b_1, \ldots, b_k\}) = dcl(\{b_1, \ldots, b_k\})$ and are both finite.

1.3 Computability

The term "atom" is overloaded.

0) In the introductory historical notes, atoms mean urelements are synonyms in Fraenkel–Mostowski permutation models of sets with atoms.
1) In Example 1.4 we referred to the elements of various Fraïssé limits as atoms.
2) An atom in a partially ordered set is a minimal element that is not the bottom element. We note that principal formulae for n-types (which are ultrafilters) are precisely the atoms in the nth Lindenbaum algebra. Accordingly, as an oligomorphic structure $\mathbb{A}$ realises only principal (and thus all) types, it is called an atomic (and saturated) model of $Th(\mathbb{A})$.
3) $R_i(x_1, \ldots, x_{n_i})$ is also known as an atomic formula.

0) is the reason why we chose the terminology in 1): we will build sets with equality atoms, ordered atoms, or graph atoms in the next section. But our homogeneous atoms have a more than coincidental connection with 2) and 3). To see this, recall that the $\mathcal{R}$-structure $\mathbb{A}$ admits quantifier elimination if for any $\mathcal{R}$-formula $\phi(\overline{x})$, there is a quantifier-free $\mathcal{R}$-formula $\Phi(\overline{x})$ — i.e., a formula built up from $\top$, the atomic formulae, and logical connectives — with $\phi[\mathbb{A}^n] = \Phi[\mathbb{A}^n]$. We can now state two model-theoretic generalities about the countable model $\mathbb{A}$:

(†) Assume $\mathbb{A}$ is oligomorphic. Then it is homogeneous if and only if it eliminates quantifiers.
(‡) Assume $\mathcal{R} = R_0, R_1, \ldots, R_r$ is finite, so that the quantifier-free formulae in n free variables can only define finitely many distinct subsets of $\mathbb{A}^n$. Then $\mathbb{A}$ is oligomorphic whenever it is homogeneous or eliminates quantifiers.

The upshot is that every homogeneous structure of atoms in Example 1.4 is oligomorphic by (‡) and eliminates quantifiers by (†). We will comment more on these structures below, but first we give two examples that fail to be homogeneous over a finite relational signature.

1.3.1 Bad: infinite signatures

Example 1.13 (the canonical relational structure) Let $\mathcal{R}_\dagger$ be the expansion of $\mathcal{R}$ by an n-ary predicate for each $\operatorname{Aut}\mathbb{A}$-orbit of $\mathbb{A}^n$ for every $n \in \mathbb{N}$. From $\mathbb{A}$ we can naturally define an $\mathcal{R}_\dagger$-structure $\mathbb{A}_\dagger$, where an isomorphism $a_1 \mapsto b_1, \ldots, a_n \mapsto b_n$ means that — tautologically — $\pi \cdot \overline{a} = \overline{b}$ for some $\pi \in \operatorname{Aut}\mathbb{A} = \operatorname{Aut}\mathbb{A}_\dagger$. In other words, any structure can be hardcoded to be homogeneous (and to eliminate quantifiers if $\mathbb{A}$ is oligomorphic) in an infinite language.

So, in general, we cannot hope for an effective quantifier elimination procedure when the signature of $\mathbb{A}$ is infinite. We can still be more lenient by demanding the existence of a finite relational signature $\mathcal{R}'$ and a homogenous $\mathcal{R}'$-structure $\mathbb{A}'$ with the same domain as $\mathbb{A}$ and with $\operatorname{Aut}\mathbb{A} = \operatorname{Aut}\mathbb{A}'$. In the terminology of [Cov90], we demand that $\mathbb{A}$ be *homogenisable*. Here is an example where we pay the price for insisting on a relational signature.

Example 1.14 (bit strings with XOR) Consider the vector space $\mathbb{V}_2$ with basis $\mathbb{N}$ over the 2-element field $\mathbb{Z}_2$, which can be regarded as a structure with a unary relation $(-) = 0$ and a ternary relation $(-)+(-) = (-)$. Each $v \in \mathbb{V}_2$ is a sequence of 0 and 1's with only finitely many 1's, so by discarding all the infinitely many trailing 0's we can identify v with a bit string; in this way we can identify vector addition with bitwise XOR.

Note that $\operatorname{Aut}\mathbb{V}_2 = \mathrm{GL}(\aleph_0, \mathbb{Z}_2)$, and the $\operatorname{Aut}\mathbb{V}_2$-orbit of $(v_1, \ldots, v_n)$ is determined by the linear relations that hold between them. It follows that there are at most $(2^n)^{(2^n)}$ orbits and therefore $\mathbb{V}_2$ is oligomorphic. However $\mathbb{V}_2$ is not homogeneous: given linearly independent vectors e_1, e_2, e_3, e_4 the map

$$(e_1, e_2, e_3, e_4) \mapsto (e_1, e_2, e_3, e_1 + e_2 + e_3)$$

is an isomorphism — indeed $\exists v(x_1 + x_2 = v \wedge v + x_2 = x_3)$ cannot be expressed without quantifiers in this language — that clearly cannot extend to an automorphism. Even worse, $\mathbb{V}_2$ is not even homogenisable: the same argument works with $n + 1$ in place of 3 with n being the largest arity in the finite signature.

1.3.2 Good: finite signatures

We are able to say something more concrete in a structure $\mathbb{A}$ that is homogeneous over a finite relational signature. Since $\mathbb{A}$ is necessarily oligomorphic, every n-type corresponds to a unique $\operatorname{Aut}\mathbb{A}$-orbit of $\mathbb{A}^n$; by homogeneity, every $\operatorname{Aut}\mathbb{A}$-orbit of $\mathbb{A}^n$ corresponds to a unique labelled isomorphism class (also called *isomorphism type* or, when $\mathbb{A}$ is $(\mathbb{Q}, =, <)$, *order type*) of substructures with at most n elements. Indeed, let $\overline{a} = (a_1, \ldots, a_{n-1}, a_n) \in \mathbb{A}^n$. Given $j_1, \ldots, j_{n_i} \in \{1, \ldots, n-1, n\}^{n_i}$ where n_i is the arity of the symbol R_i, put

$$\rho_{i,(j_1,\ldots,j_{n_i})} \overset{\text{def}}{=} \begin{cases} R_i(x_{j_1}, \ldots, x_{j_{n_i}}) & \text{if } \mathbb{A} \models R_i(a_{j_1}, \ldots, a_{j_{n_i}}), \\ \neg R_i(x_{j_1}, \ldots, x_{j_{n_i}}) & \text{if } \mathbb{A} \not\models R_i(a_{j_1}, \ldots, a_{j_{n_i}}); \end{cases}$$

$$\Phi_{\overline{a}}(x_1, \ldots, x_{n-1}, x_n) \overset{\text{def}}{=} \bigwedge_{i=1}^{r} \bigwedge_{(j_1,\ldots,j_{n_i}) \in A^{n_i}} \rho_{i,(j_1,\ldots,j_{n_i})}$$

so that $\mathbb{A} \models \Phi_{\overline{a}}(\overline{b})$ precisely when $a_1 \mapsto b_1, \ldots, a_{n-1} \mapsto b_{n-1}, a_n \mapsto b_n$ is an isomorphism. Then $\Phi_{\overline{a}}$ turns out to be a quantifier-free principal formula for $tp_{\mathbb{A}}(\overline{a})$ and $\Phi_{(a_1,\ldots,a_{n-1},a_n)}[\mathbb{A}^n] = \operatorname{Aut}\mathbb{A} \cdot (a_1, \ldots, a_{n-1}, a_n)$.

But now we know how to eliminate quantifiers explicitly. Observe first that

$$\Phi_{(a_1,\ldots,a_{n-1})}[\mathbb{A}^{n-1}] = \operatorname{Aut}\mathbb{A} \cdot (a_1, \ldots, a_{n-1}) = (\exists x_n \Phi_{(a_1,\ldots,a_{n-1},a_n)})[\mathbb{A}^n],$$

and $\Phi_{(a_1,\ldots,a_{n-1})}$ can be obtained from $\Phi_{(a_1,\ldots,a_{n-1},a_n)}$ by removing from the conjunction any atomic formula or its negation $\rho_{i,(j_1,\ldots,j_{n_i})}$ where some $k = 1, \ldots, n_i$ is such that $j_k = n$. Now if $\psi(x_1, \ldots, x_{n-1}, x_n)$ is a quantifier-free formula and we have a finite decomposition $\mathbb{A}^n = \bigcup_t \operatorname{Aut}\mathbb{A} \cdot (a_1^{(t)}, \ldots, a_{n-1}^{(t)}, a_n^{(t)})$, we can work out a decomposition

$$\psi[\mathbb{A}^n] = \Big(\bigvee_{t:\mathbb{A} \models \psi(a_1^{(t)},\ldots,a_{n-1}^{(t)},a_n^{(t)})} \Phi_{(a_1^{(t)},\ldots,a_{n-1}^{(t)},a_n^{(t)})}\Big)[\mathbb{A}^n]$$

which means that

$$\begin{aligned}(\exists x_n \psi)[\mathbb{A}^{n-1}] &= \Big(\bigvee_{t:\mathbb{A}\models\psi(a_1^{(t)},\ldots,a_{n-1}^{(t)},a_n^{(t)})} \exists x_n \Phi_{(a_1^{(t)},\ldots,a_{n-1}^{(t)},a_n^{(t)})}\Big)[\mathbb{A}^{n-1}] \\ &= \Big(\bigvee_{t:\mathbb{A}\models\psi(a_1^{(t)},\ldots,a_{n-1}^{(t)},a_n^{(t)})} \Phi_{a_1^{(t)},\ldots,a_{n-1}^{(t)}}\Big)[\mathbb{A}^{n-1}].\end{aligned}$$

Finally, by removing quantifiers from the inside out we can eliminate the quantifiers from any $\mathcal{R}$-formula. In fact, the following is true:

> **Theorem 1.15 ([Boj19, Theorem 7.20])**
> *Let $\mathcal{R}$ be a finite relational signature. Suppose $\mathcal{K}$ is an amalgamation class of finite $\mathcal{R}$-structures such that we can compute, given any $n \in \mathbb{N}$, representatives for isomorphism classes of n-element structures and that there are only finitely many of them. Then*
>
> *i)* $\operatorname{FLim}\mathcal{K}$ *is homogeneous and oligomorphic;*
> *ii) its elements can be represented in a finite way;*
> *iii) whether* $\operatorname{FLim}\mathcal{K} \models \phi(a_1,\ldots,a_n)$ *for an* $\mathcal{R}$*-formula* $\phi(\overline{x})$ *and parameters* $\overline{a} \in (\operatorname{FLim}\mathcal{K})^n$ *is decidable.*

2 Nominal sets

Now that we have settled on the model-theoretic assumptions on the atoms structure $\mathbb{A}$ — that it is a countably infinite, homogeneous, and oligomorphic structure over a finite relational signature which we assume henceforth — we can start building sets with atoms.

2.1 As hereditarily finitely supported sets with atoms

> ... out of nothing I have created a strange new universe.
>
> János Bolyai in an 1823 letter to his father, referring to his non-Euclidean geometry

One can follow the instruction to the letter by constructing, given a set A, a cumulative hierarchy

$$\begin{aligned} V_0(A) &\stackrel{\text{def}}{=} A, \\ V_{\alpha+1}(A) &\stackrel{\text{def}}{=} V_\alpha(A) \cup \wp(V_\alpha(A)), \\ V_{\lambda=\bigcup_{\alpha<\lambda}\alpha\neq 0}(A) &\stackrel{\text{def}}{=} \bigcup_{\alpha<\lambda} V_\alpha(A) \end{aligned}$$

by transfinite recursion like Zermelo did in [Zer30, p.36] so that

$$V_0(A) \subseteq V_1(A) \subseteq \cdots \subseteq V_\omega \subseteq \cdots$$

and $\bigcup_\alpha V_\alpha(\emptyset)$ is the classical, so-called von Neumann universe of sets sitting inside the bigger universe $\bigcup_\alpha V_\alpha(\mathbb{A})$

of atoms — every $a \in \mathbb{A}$ is empty but not a set — and sets with atoms. Then, because any descending chain of $\in$'s must be finite, one can inductively define a group action of $\mathrm{Aut}\,\mathbb{A}$ on $\bigcup_\alpha V_\alpha(\mathbb{A})$ by

$$\pi \cdot a \overset{\text{def}}{=} \pi(a),$$
$$\pi \cdot X \overset{\text{def}}{=} \{\pi \cdot x \mid x \in X\}$$

for $\pi \in \mathrm{Aut}\,\mathbb{A}$. Notice that $\pi \cdot x \in \pi \cdot X \iff x \in X$ and that $\pi \cdot X = Y \iff X = \pi^{-1} \cdot Y$.

2.1.1 Supports

Here comes the central notion: we say $S \subseteq \mathbb{A}$ is a *support* of $X \in \bigcup_\alpha V_\alpha(\mathbb{A})$ if every $\pi \in \mathrm{Aut}\,\mathbb{A}/S$ satisfies $\pi \cdot X = X$. Observe that S supports X if and only if

$$\forall \tau, \pi \in \mathrm{Aut}\,\mathbb{A} : \tau|_S = \pi|_S \implies \tau \cdot X = \pi \cdot X.$$

If S can be taken to be finite, we say X is *finitely supported*; if S can be taken to be empty, we say X is *equivariant*. As basic examples, every atom $a \in \mathbb{A} = V_0(\mathbb{A})$ is supported by $\{a\}$, and so are the sets $\{a\}$ and $\mathbb{A} \setminus \{a\}$ in $V_1(\mathbb{A})$; actually, since $\mathbb{A}$ is oligomorphic, a subset of the atoms is finitely supported if and only if it is S-definable for some $S \subseteq_{\mathit{fin}} \mathbb{A}$. We only want to consider sets that are finitely supported from the ground up. To make this rigorous, given a property $\mathfrak{P}$ of objects in $\bigcup_\alpha V_\alpha(\mathbb{A})$, we recursively define the collection $H_\mathfrak{P}$ of *hereditarily* $\mathfrak{P}$ *sets (with atoms)* by:

$$X \text{ is a } \mathfrak{P} \text{ set } \frac{\forall x \in X : x \text{ is a } \mathfrak{P} \text{ atom or } x \in H_\mathfrak{P}}{X \in H_\mathfrak{P}}$$

Abbreviating 'finitely supported' as *f.s.*, we arrive at the Fraenkel–Mostowski universe $H_{\mathit{f.s.}}$ which encompasses the classical universe $\bigcup_\alpha V_\alpha(\emptyset)$ of sets without atoms; of course the equivariant set $\mathbb{A}$ is in $H_{\mathit{f.s.}}$ as well. We shall refer to the inhabitants of $H_{\mathit{f.s.}}$ as *nominal sets (with atoms)*.

Like with normal sets, there are many ways to build new nominal sets with atoms from old. What is new is that if $X \in \bigcup_\alpha V_\alpha(\mathbb{A})$ is supported by $S_X \subseteq_{\mathit{fin}} \mathbb{A}$, then $\pi \cdot X$ is supported by $\pi \cdot S_X \subseteq_{\mathit{fin}} \mathbb{A}$ and any superset; so if $X \in H_{\mathit{f.s.}}$ then $\pi \cdot X \in H_{\mathit{f.s.}}$ too.

Remark 2.1 (folklore) Let $X, Y \in H_{\mathit{f.s.}}$. Then the following sets also live in $H_{\mathit{f.s.}}$:

a) the collections

$$\{X, Y\}, \quad \{\pi \cdot X \mid \pi \in \mathrm{Aut}\,\mathbb{A}/S\}$$

where $S \subseteq_{\mathit{fin}} \mathbb{A}$ (and we can even allow X and Y to be atoms);

b) the usual set operations

$$X \cup Y, \quad X \cap Y, \quad X \setminus Y, \quad X \times Y, \quad \bigcup X;$$

c) the finitely supported elements of the power set and of the function set

$$\wp_{\mathit{f.s.}}(X) \overset{\text{def}}{=} \{Z \subseteq X \mid Z \text{ is } \mathit{f.s.}\}, \quad (X \to_{\mathit{f.s.}} Y) \overset{\text{def}}{=} \{f \mid f \text{ is a } \mathit{f.s.} \text{function } X \to Y\};$$

d) the image of a finitely supported function $f : X \to Y$,

$$f(X) = \{f(x) \mid x \in X\};$$

e) the equivalence class and the quotient set

$$[x]_\sim \stackrel{\text{def}}{=} \{x' \in X \mid x' \sim x\}, \quad X/\sim \stackrel{\text{def}}{=} \{[x]_\sim \mid x \in X\}$$

of X under a finitely supported equivalence relation $\sim$.

As is customary in set theory, here $(x, y) \in X \times Y$ is defined as the Kuratowski pair

$$\{\{x\}, \{x, y\}\},$$

and a binary relation R on $X \times Y$ is defined as the subset of pairs

$$\{(x, y) \in X \times Y \mid x\ R\ y\}.$$

As an equivalence relation $\sim$ on X and a function $f : X \to Y$ are just binary relations satisfying additional properties, the action of $\operatorname{Aut}\mathbb{A}$ on $H_{f.s.}$ prescribes what $\pi \cdot \sim$, $\pi \cdot f$, and their supports are:

1) the equivalence relation $\sim$ is supported by $S \subseteq \mathbb{A}$ precisely if for all $\pi \in \operatorname{Aut}\mathbb{A}/S$ we have

$$\begin{aligned}\{(x_1, x_2) \in X \times X \mid x_1 \sim x_2\} &= \pi \cdot \{(x_1, x_2) \in X \times X \mid x_1 \sim x_2\} \\ &= \{(\pi \cdot x_1, \pi \cdot x_2) \mid (x_1, x_2) \in X \times X, x_1 \sim x_2\},\end{aligned}$$

i.e., if we have $\pi \cdot X = X$ and $x_1 \sim x_2 \iff \pi \cdot x_1 \sim \pi \cdot x_2$ for all $x_1, x_2 \in X$;

2) the function f is supported by $S \subseteq \mathbb{A}$ precisely if for all $\pi \in \operatorname{Aut}\mathbb{A}/S$ we have

$$\begin{aligned}\{(x, f(x)) \mid x \in X\} &= \pi \cdot \{(x, f(x)) \mid x \in X\} \\ &= \{(\pi \cdot x, \pi \cdot f(x)) \mid x \in X\},\end{aligned}$$

i.e., if we have $\pi \cdot X = X$, $\pi \cdot f(X) = f(X)$, and $f(x) = \pi \cdot f(\pi^{-1} \cdot x)$ for all $x \in X$.

The seminal Cambridge paper [GP02, §4] and the Warsaw book [Boj19, §3.1] both take this element-oriented, set-theoretic approach to foundations.

2.1.2 Orbit-finiteness

Let $X \in H_{f.s.}$ be a nominal set with atoms. Say X is supported by $S \subseteq_{fin} \mathbb{A}$; then there is a group action of $\operatorname{Aut}\mathbb{A}/S$ on X, under which X is equal to a union of $\operatorname{Aut}\mathbb{A}/S$-orbits. Even though the orbit

$$\operatorname{Aut}\mathbb{A}/S \cdot x = \{\pi \cdot x \mid \pi \in \operatorname{Aut}\mathbb{A}/S\} \subseteq X$$

of $x \in X$ can in general contain infinitely many elements, to describe it we only needed to specify x and S. So by counting the number of orbits rather than of elements, we obtain a more useful and general notion of finiteness for sets with atoms. Note that we need not worry about the choice of the support S.

Definition 2.2 ([Boj13, Lemma 2.2])
Because $\mathbb{A}$ is oligomorphic, the two conditions below are equivalent for $X \in H_{f.s.}$:

i) X is a finite disjoint union of $\operatorname{Aut}\mathbb{A}/S$-orbits for some support $S \subseteq_{fin} \mathbb{A}$ of X;
ii) X is a finite disjoint union of $\operatorname{Aut}\mathbb{A}/S'$-orbits for any support $S' \subseteq_{fin} \mathbb{A}$ of X.

If either condition holds, we call X *orbit-finite*.[a]

[a] Why not 'finite-orbit' like in 'finite-state machines'?

Proof. By definition $X \in H_{f.s.}$ admits some finite support S; that ii) implies i) is obvious. Now assume i) to write

$$X = \bigcup_i X_i, \qquad X_i \stackrel{\text{def}}{=} \operatorname{Aut}\mathbb{A}/S \cdot x_i,$$

and take any other support $S' \subseteq_{fin} \mathbb{A}$ of X. Since any $x_i \in X$ admits some finite support $\{a_1, \ldots, a_{n_i}\}$, observe that

$$\left\{(\pi \cdot (a_1, \ldots, a_{n_i}), \quad \pi \cdot x_i) \;\middle|\; \pi \in \operatorname{Aut}\mathbb{A}/S\right\}$$

is the graph of a well-defined surjective function

$$f : \operatorname{Aut}\mathbb{A}/S \cdot (a_1, \ldots, a_{n_i}) \to X_i$$

that is supported by S and hence also by $S \cup S'$. But we saw in Remark 1.12 that $\mathbb{A}^{n_i}$ is a finite union of distinct $\operatorname{Aut}\mathbb{A}/(S \cup S')$-orbits; so $\operatorname{Aut}\mathbb{A}/S \cdot (a_1, \ldots, a_{n_i}) \subseteq \mathbb{A}^{n_i}$, which is invariant under $\operatorname{Aut}\mathbb{A}/(S \cup S') \subseteq \operatorname{Aut}\mathbb{A}/S$, is a sub-union of these finitely many orbits — say $\operatorname{Aut}\mathbb{A}/S \cdot (a_1, \ldots, n_i) = \bigcup_j \operatorname{Aut}\mathbb{A}/(S \cup S') \cdot \overline{b_j}$. Now

$$\begin{aligned} X_i &= f\Big(\operatorname{Aut}\mathbb{A}/S \cdot (a_1, \ldots, a_{n_i})\Big) \\ &= f\Big(\bigcup_j \operatorname{Aut}\mathbb{A}/(S \cup S') \cdot \overline{b_j}\Big) = \bigcup_j \operatorname{Aut}\mathbb{A}/(S \cup S') \cdot f(\overline{b_j}), \end{aligned}$$

which means that $X = \bigcup_i X_i = \bigcup_i \bigcup_j \operatorname{Aut}\mathbb{A}/(S \cup S') \cdot f(\overline{b_j})$ is a finite union of $\operatorname{Aut}\mathbb{A}/(S \cup S')$-orbits. This establishes ii):

$$\operatorname{Aut}\mathbb{A}/(S \cup S') \cdot x \mapsto \operatorname{Aut}\mathbb{A}/S' \cdot x, \qquad x \in X$$

is a well-defined surjection from the $\operatorname{Aut}\mathbb{A}/(S \cup S')$-orbits in X onto the $\operatorname{Aut}\mathbb{A}/S'$-orbits in X. ∎

An orbit in X is a subset of X, so it makes little sense to speak about orbits in an atom; regardless, we also declare every $a \in \mathbb{A}$ to be orbit-finite. Abbreviating 'orbit-finite' as *o-f*, we obtain the universe $H_{o\text{-}f}$ of hereditarily orbit-finite sets inside $H_{f.s.}$. Again we give two basic examples:

- as the orbit of x in a set without atoms is just the singleton $\{x\}$, a classical set in $H_{o\text{-}f} \cap \bigcup_\alpha V_\alpha(\emptyset)$ is just a hereditarily finite set;
- with atoms we have $\mathbb{A} \in H_{o\text{-}f}$, and $\pi \cdot X \in H_{o\text{-}f}$ if $X \in H_{o\text{-}f}$.

Compared to Remark 2.1, there are almost as many ways to build new hereditarily orbit-finite sets from old.

Remark 2.3 ([Boj19, Lemma 3.24]) If $X, Y \in H_{o\text{-}f} \subseteq H_{f.s.}$, then

$$\begin{gathered} \{X, Y\}, \quad \{\pi \cdot X \mid \pi \in \operatorname{Aut}\mathbb{A}/S\}, \\ X \cup Y, \quad X \cap Y, \quad X \setminus Y, \quad X \times Y, \quad \bigcup X, \\ Z \in \wp_{f.s.}(X), \quad f(X), \quad [x]_\sim, \quad X/\sim \end{gathered}$$

also live in $H_{o\text{-}f}$, where f and $\sim$ are finitely supported (and hence orbit-finite).

However $\wp_{f.s.}(X)$ and $(X \to_{f.s.} Y)$ are not in general orbit-finite: consider $X = \mathbb{A}$ and $Y = \{\emptyset, \{\emptyset\}\}$; finite subsets of $\mathbb{A}$ with different numbers of atoms lie in different orbits.

Like how Ackermann encoded the hereditarily finite sets as natural numbers [Ack37, c) on p.308], the hereditarily orbit-finite sets can be encoded as *set-builder expressions* [Boj19, Chapter 4] generalising how $\phi[\mathbb{A}^n] \subseteq \mathbb{A}^n$ is represented by the formula ϕ. The technical specifics are spelt out in [KS16, §2–4] and [MSSKS17, §6] to develop the Haskell module Nλ (`https://www.mimuw.edu.pl/~szynwelski/nlambda/`), and in [KT17, §4] to develop the C++ library LOIS (Looping Over Infinite Sets, `https://www.mimuw.edu.pl/~erykk/lois/`).

2.2 As continuous actions of the automorphism group

> Groups, as men, will be known by their actions.
>
> as per `https://mathoverflow.net/a/7759/126582`,
> Guillermo Moreno in a differential geometry class

Previously we built up a big universe $H_{f.s.}$ with an action of $\operatorname{Aut}\mathbb{A}$ and let every set with atom inherit this action; the atoms $a \in \mathbb{A}$ and the set brackets were primitives, whilst the actions and supports were derived notions. It is possible to take a more structuralist perspective which instead places the emphasis on $\operatorname{Aut}\mathbb{A}$ and its actions, as is done in the Cambridge book [Pit13, Chapter 2] and the seminal Warsaw paper [BKL14, §4].

2.2.1 The Polish group topology

As $\mathbb{A}$ is countable, enumerate its elements as $a_1, a_2, \ldots$ and consider the following distance function on $\operatorname{Aut}\mathbb{A}$:

$$d(\pi_1, \pi_2) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } \pi_1 = \pi_2, \\ 2^{-n} & \text{otherwise, where } n = \min\{i \mid \pi_1(a_i) \neq \pi_2(a_i) \text{ or } \pi_1^{-1}(a_i) \neq \pi_2^{-1}(a_i)\}. \end{cases}$$

Then $\operatorname{Aut}\mathbb{A}$ becomes a complete metric space: if a sequence $\pi_1, \pi_2, \pi_3, \ldots$ is Cauchy then it has a limit $\pi \in \operatorname{Aut}\mathbb{A}$. We note this metric topology is the same as the subspace topology inherited from the product topology on $\mathbb{A}^{\mathbb{A}}$, where $\mathbb{A}$ is endowed with the discrete topology; the open sets are arbitrary unions of the cosets

$$\tau \circ \operatorname{Aut}\mathbb{A}/S = \{\pi \in \operatorname{Aut}\mathbb{A} : \pi|_S = \tau|_S\}$$

with $\tau \in \operatorname{Aut}\mathbb{A}$ and $S \subseteq_{fin} \mathbb{A}$. But since there are only countably many different $\tau|_S$'s, $\operatorname{Aut}\mathbb{A}$ is a so named *Polish* topological space — actually, a Polish *topological group*: the operations

$$\begin{aligned} \operatorname{Aut}\mathbb{A} &\to \operatorname{Aut}\mathbb{A} \\ \pi &\mapsto \pi^{-1}, \end{aligned} \qquad\qquad \begin{aligned} \operatorname{Aut}\mathbb{A} \times \operatorname{Aut}\mathbb{A} &\to \operatorname{Aut}\mathbb{A} \\ (\pi, \tau) &\mapsto \pi \circ \tau \end{aligned}$$

are continuous under the above topology; see, e.g., [Hod93, Lemma 4.1.5(a)].

We recall two standard definitions.

0) An $\operatorname{Aut}\mathbb{A}$-*set* is nothing but an ordinary set X equipped with a group action, that is, a function

$$\begin{aligned} \operatorname{Aut}\mathbb{A} \times X &\to X \\ (\pi, x) &\mapsto \pi \cdot x \end{aligned}$$

such that $\mathrm{id} \cdot x = x$ and $(\pi \circ \tau) \cdot x = \pi \cdot (\tau \cdot x)$ for all $x \in X$ and $\pi, \tau \in \operatorname{Aut}\mathbb{A}$.

1) An *equivariant function* between two $\operatorname{Aut}\mathbb{A}$-sets X, Y is a function $f : X \to Y$ such that

$$\begin{tikzcd} \operatorname{Aut}\mathbb{A} \times X \arrow[r] \arrow[d, "\mathrm{id}\times f"'] & X \arrow[d, "f"] \\ \operatorname{Aut}\mathbb{A} \times Y \arrow[r] & Y \end{tikzcd}$$

commutes, i.e., such that $f(\pi \cdot x) = \pi \cdot f(x)$ for all $\pi \in \operatorname{Aut}\mathbb{A}$ and $x \in X$.

But observe that the group action is continuous if and only if

$$(\operatorname{Aut}\mathbb{A})_x \overset{\text{def}}{=} \{\pi \in \operatorname{Aut}\mathbb{A} \mid \pi \cdot x = x\}$$

is open for every $x \in X$, i.e., if $(\operatorname{Aut}\mathbb{A})_x$ contains $\operatorname{Aut}\mathbb{A}/S_x$ for some $S_x \subseteq_{fin} \mathbb{A}$ [Hod93, Lemma 4.1.5(b)]. Particularly consider a nominal set with atoms $X \in H_{f.s.}$ that is equivariant. Then X is certainly an $\operatorname{Aut}\mathbb{A}$-set, where the action is moreover continuous: given any $x \in X$, saying x is supported by $S_x \subseteq_{fin} \mathbb{A}$ amounts to saying $(\operatorname{Aut}\mathbb{A})_x \supseteq \operatorname{Aut}\mathbb{A}/S_x$ — we simply ignore the inner set structure of x. This motivates the following definition.

Definition 2.4
A *nominal* $\operatorname{Aut}\mathbb{A}$*-set* is an $\operatorname{Aut}\mathbb{A}$-set X whose group action is continuous. The nominal $\operatorname{Aut}\mathbb{A}$-sets together with the equivariant functions form a category $\mathbf{NomSet}_{\operatorname{Aut}\mathbb{A}}$.

There is an analogue to Remark 2.1: $\mathbf{NomSet}_{\operatorname{Aut}\mathbb{A}}$ has products, coproducts, power objects, exponentials, and quotient objects. In fact, $\mathbf{NomSet}_{\operatorname{Aut}\mathbb{A}}$ is a Cartesian closed category and a Boolean topos [Pit13, Theorems 2.19 and 2.23].

Also, we can reuse the notion of *orbit-finiteness* from Definition 2.2; where the constructions make sense, Remark 2.3 still applies. For instance, the following are orbit-finite nominal $\operatorname{Aut}\mathbb{A}$-sets:

- any finite set X, viewed as an $\operatorname{Aut}\mathbb{A}$-set with the trivial group action $\pi \cdot x = x$ which is trivially continuous;
- any $\emptyset$-definable subset of $\mathbb{A}^n$, with the usual oligomorphic action of $\operatorname{Aut}\mathbb{A}$ from Definition 1.8;
- given an open subgroup $G \subseteq \operatorname{Aut}\mathbb{A}$, the cosets $\{\tau \circ G \mid \tau \in \operatorname{Aut}\mathbb{A}\}$ with $\pi \cdot (\tau \circ G) \overset{\text{def}}{=} (\pi \circ \tau) \circ G$ — here $\tau \circ G = \tau' \circ G$ if and only if $\tau^{-1} \circ \tau' \in G$, and $(\operatorname{Aut}\mathbb{A})_{\tau\circ G} = \tau \circ G \circ \tau^{-1}$.

2.2.2 Interpretations and reducts

In the spirit of the Erlangen program, we can study structures other than $\mathbb{A}$ through their automorphism groups.

Definition 2.5 ([Eva13, Definition 2.4])
Let $\mathcal{R}'$ be another relational signature and $\mathbb{A}'$ an $\mathcal{R}'$-structure. We say $\mathbb{A}'$ is *interpretable in* $\mathbb{A}$ if there exist

- an $\emptyset$-definable subset $D \subseteq \mathbb{A}^n$,
- an $\emptyset$-definable subset $E \subseteq \mathbb{A}^{2n}$ that is an equivalence relation on D,
- and a bijection $f : D/E \to \mathbb{A}'$

such that for every k-ary relation symbol R' in $\mathcal{R}'$, the subset

$$\{(\overline{a_1},\dots,\overline{a_k}) \in D^k : \mathbb{A}' \models R'\big(f([\overline{a_1}]_E),\dots,f([\overline{a_k}]_E)\big)\} \subseteq \mathbb{A}^{nk}$$

is $\emptyset$-definable. Moreover we say $\mathbb{A}'$ is a *reduct of* $\mathbb{A}$ if $D = \mathbb{A}$ and $E = \{(\overline{a},\overline{a}) \mid \overline{a} \in D\}$.

Suppose $\mathbb{A}'$ is interpretable in $\mathbb{A}$ and let D, E, f be as above. Then the quotient D/E is naturally an orbit-finite nominal $\operatorname{Aut}\mathbb{A}$-set, where the equivalence classes are called *imaginary elements* of $\mathbb{A}$. More importantly

$$\begin{aligned}\phi : \operatorname{Aut}\mathbb{A} &\to \operatorname{Aut}\mathbb{A}'\\ \pi &\mapsto f \circ (\pi \cdot -) \circ f^{-1}\end{aligned}$$

is a continuous group homomorphism satisfying $\phi(\pi) \cdot f([\overline{d}]_E) = f([\pi \cdot \overline{d}]_E)$, so that two k-tuples of D/E lie in the same $\operatorname{Aut}\mathbb{A}$-orbit if and only if their entrywise images under f lie in the same $\phi(\operatorname{Aut}\mathbb{A})$-orbit and *a fortiori* also lie in the same orbit under the bigger group $\operatorname{Aut}\mathbb{A}'$. But D/E has finitely many orbits under $\operatorname{Aut}\mathbb{A}$, so $(f(D/E))^k = (\mathbb{A}')^k$ has finitely many orbits under $\operatorname{Aut}\mathbb{A}'$ given any $k \in \mathbb{N}$; that is, $\mathbb{A}'$ is oligomorphic under $\phi(\operatorname{Aut}\mathbb{A})$ and $\operatorname{Aut}\mathbb{A}'$. When $\mathbb{A}'$ is a reduct of $\mathbb{A}$, the homomorphism ϕ is simply the inclusion $\operatorname{Aut}\mathbb{A} \subseteq \operatorname{Aut}\mathbb{A}'$. The converses also hold thanks to our assumptions on $\mathbb{A}$.

Proposition 2.6 ([AZ86, Theorems 1.1 and 1.2]) *As $\mathbb{A}$ is countable and oligomorphic over a finite signature,*

i) $\mathbb{A}'$ is a reduct of $\mathbb{A}$ if and only if $\operatorname{Aut}\mathbb{A} \subseteq \operatorname{Aut}\mathbb{A}'$;
ii) $\mathbb{A}'$ is interpretable in $\mathbb{A}$ if and only if there is a continuous group homomorphism $\phi : \operatorname{Aut}\mathbb{A} \to \operatorname{Aut}\mathbb{A}'$ such that $\mathbb{A}'$ is oligomorphic under $\phi(\operatorname{Aut}\mathbb{A})$.

On the topological group-theoretic side, we have the translation below:

Remark 2.7 ([Hod93, Theorem 4.1.4]) Let $\mathbb{A}_=$ be the reduct of $\mathbb{A}$ where any symbol except = is forgotten; then $\operatorname{Aut}\mathbb{A}_=$ consists of all bijections $\mathbb{A} \to \mathbb{A}$.

a) A subgroup $G \subseteq \operatorname{Aut}\mathbb{A}_=$ is closed if and only if $G = \operatorname{Aut}\mathbb{A}'$ for some $\mathcal{R}'$-structure with domain $\mathbb{A}'$; in the 'only if' direction, we can take $\mathcal{R}'$ to be the infinite canonical signature from Example 1.13 with a relation symbol for each orbit.

b) On the structures with domain $\mathbb{A}$ over arbitrary signatures, the relation of being a reduct defines a preorder. We say $\mathbb{A}_1$ and $\mathbb{A}_2$ are *first-order interdefinable* if each is a reduct of the other. It follows from the results above that reducts of the oligomorphic $\mathcal{R}$-structure $\mathbb{A}$ are, up to first-order interdefinability, in an order-reversing bijection with closed subgroups of $\operatorname{Aut}\mathbb{A}_=$ containing $\operatorname{Aut}\mathbb{A}$.

Conjecture 2.8 ([Tho91, p.177])
As a countable homogeneous structure over a finite relational signature, $\mathbb{A}$ only has finitely many reducts up to first-order interdefinability; i.e., there are only finitely many closed subgroups $G \subseteq \operatorname{Aut}\mathbb{A}_=$ with $\operatorname{Aut}\mathbb{A} \subseteq G$.

The conjecture of Thomas is obviously true for the equality atoms, where $\operatorname{Aut}\mathbb{A} \subseteq G \subseteq \operatorname{Aut}\mathbb{A}_= = \operatorname{Aut}\mathbb{A}$ does not leave space for any non-trivial reducts. It has also been confirmed for, amongst a few others:

1) the ordered atoms, with 3 non-trivial reducts up to first-order interdefinability [Cam76, Theorem 6.1];
2) the Rado graph, also with 3 non-trivial reducts up to interdefinability, and the Henson K_n-free graphs, where all reducts are trivial up to interdefinability [Tho91, Theorems 1 and 2];

3) the generically ordered Rado graph, with 42 non-trivial reducts up to interdefinability [BPP13, Theorem 1] subsuming the reducts of the ordered atoms and of the graph atoms; a nice catalogue is supplied in §2.

Apart from its validity in these individual structures, the conjecture very much remains open; the Ramsey-theoretic technique in 3) employed by Bodirsky and Pinsker et al. is perhaps the most general approach known. Regardless, these reducts turn out to be homogeneous over a finite relational signature and make a good stock of examples together the ones from Example 1.2 and Example 1.4.

2.2.3 The (strong) small index property

Sometimes the Polish topology on $\operatorname{Aut}\mathbb{A}$ can be recovered from the abstract group structure alone.

> **Definition 2.9**
> The *index* of a subgroup $G \subseteq \operatorname{Aut}\mathbb{A}$ is the size of $\{\pi \circ G \mid \pi \in \operatorname{Aut}\mathbb{A}\}$. We say G is a subgroup of *small index* if this size is at most countable, i.e., if there are finitely or countably infinitely many cosets of G in $\operatorname{Aut}\mathbb{A}$. Now
>
> i) $\mathbb{A}$ has the *small index property (SIP)* if every subgroup $G \subseteq \operatorname{Aut}\mathbb{A}$ of small index is open, i.e., satisfies
> $$\operatorname{Aut}\mathbb{A}/S \subseteq G$$
> for some $S \subseteq_{fin} \mathbb{A}$;
>
> ii) $\mathbb{A}$ has the *strong small index property (SSIP)* if every subgroup $G \subseteq \operatorname{Aut}\mathbb{A}$ of small index satisfies
> $$\operatorname{Aut}\mathbb{A}/S \subseteq G \subseteq (\operatorname{Aut}\mathbb{A})_S$$
> for some $S \subseteq_{fin} \mathbb{A}$, where $(\operatorname{Aut}\mathbb{A})_S$ is the stabiliser of S in the $\operatorname{Aut}\mathbb{A}$-set $\wp_{fin}(\mathbb{A}) \stackrel{\text{def}}{=} \{T \mid T \subseteq_{fin} \mathbb{A}\}$.

Several remarks are in order.

1) Let $S = \{a_1, \ldots, a_n\} \subseteq_{fin} \mathbb{A}$. Write $\overline{s} = (a_1, \ldots, a_n) \in \mathbb{A}^n$; then
$$\operatorname{Aut}\mathbb{A}/S = \{\pi \in \operatorname{Aut}\mathbb{A} \mid \forall s \in S : \pi(s) = s\} = (\operatorname{Aut}\mathbb{A})_{\overline{s}}$$
is the *pointwise stabiliser* of S, and the Orbit-Stabiliser Theorem says
$$\pi \circ (\operatorname{Aut}\mathbb{A})_{\overline{s}} \mapsto \pi \cdot \overline{s}, \qquad \pi \in \operatorname{Aut}\mathbb{A}$$
is an isomorphism between the cosets of $\operatorname{Aut}\mathbb{A}_{\overline{s}}$ and the orbit $\operatorname{Aut}\mathbb{A} \cdot \overline{s} \subseteq \mathbb{A}^n$, which is certainly countable. Therefore every $\operatorname{Aut}\mathbb{A}/S$ is a subgroup of small index; hence so is every open subgroup of $\operatorname{Aut}\mathbb{A}$.

2) Let S and $\overline{s}$ be as above. Compared to $\operatorname{Aut}\mathbb{A}/S = (\operatorname{Aut}\mathbb{A})_{\overline{s}}$, the subgroup
$$(\operatorname{Aut}\mathbb{A})_S = \{\pi \in \operatorname{Aut}\mathbb{A} \mid \forall s \in S : \pi(s) \in S\}$$
is the *setwise stabiliser* of S. One can check that $(\operatorname{Aut}\mathbb{A})_{\overline{s}}$ is a normal subgroup of $(\operatorname{Aut}\mathbb{A})_S$, and that
$$\begin{aligned} (\operatorname{Aut}\mathbb{A})_S/(\operatorname{Aut}\mathbb{A})_{\overline{s}} &\to \operatorname{Aut} S \\ \pi \circ (\operatorname{Aut}\mathbb{A})_{\overline{s}} &\mapsto \pi|_S \end{aligned}$$
is a well-defined injective homomorphism into the finite group $\operatorname{Aut} S$.

3) Assume that $\mathbb{A}$ has the SIP. If $\mathbb{A}'$ is $\aleph_0$-categorical too and $\phi : \operatorname{Aut}\mathbb{A} \to \operatorname{Aut}\mathbb{A}'$ is a group isomorphism, then ϕ is moreover a homeomorphism [Mac11, Proposition 5.2.2]. By Proposition 2.6b) $\mathbb{A}$ is interpretable in $\mathbb{A}'$ and $\mathbb{A}'$ is interpretable in $\mathbb{A}$, which sets up a sort of homotopy equivalence called *bi-interpretability* between $\mathbb{A}$ and $\mathbb{A}'$ — see [AZ86, Corollary 1.4(ii) and definitions above]. If furthermore $\mathbb{A}$ has the SSIP and $acl(\{a\}) = \{a\}, acl(\{a'\}) = \{a'\}$ for $a \in \mathbb{A}, a' \in \mathbb{A}'$, then any group isomorphism $\phi : \operatorname{Aut}\mathbb{A} \to \operatorname{Aut}\mathbb{A}'$ is induced by a bijection $f : \mathbb{A} \to \mathbb{A}'$ [PS17a, Corollary 2]. In other words, with the (S)SIP one may reconstruct the information lost in the passage

$$\text{the permutation group } (\operatorname{Aut}\mathbb{A}, \mathbb{A}) \mapsto \text{the topological group } \operatorname{Aut}\mathbb{A} \mapsto \text{the abstract group } \operatorname{Aut}\mathbb{A}.$$

4) Structures with the SSIP include the equality atoms [DNT86, Theorem 1], the ordered atoms [Tru89, Theorem 2.12], the Rado graph, the Henson K_n-free graphs, and their directed counterparts [PS17b, Corollary 3]. Such a property is not easy to prove generally: whether even the SIP holds for the random tournament [Mac11, Question 5.2.7i)] is still a standing question.

We are mainly interested in the 'strong' part of the SSIP, which curiously is known better to model theorists as *weak elimination of imaginary elements* introduced in [Poi83, §2 except the modern definition uses the algebraic closure in place of « la clôture rationnelle », i.e., the definable closure]; see also [Hod93, §4.4]. Fortunately, more intrinsic characterisations exist. Recall first the algebraic closures from Definition 1.8; we note that $acl(-)$ is indeed a closure operator on finite subsets of the oligomorphic structure $\mathbb{A}$: for $B \subseteq C \subseteq_{fin} \mathbb{A}$, we have

$$B \subseteq acl(B) \subseteq acl(C) = acl(acl(C)) \subseteq_{fin} \mathbb{A}.$$

We say B is *algebraically closed* if $acl(B) = B$.

Definition 2.10
We say $\mathbb{A}$ *admits least finite, algebraically closed supports* if any of the following equivalent [EH93, Lemma 1.3] conditions holds.

i) $\mathbb{A}$ has weak elimination of imaginaries.
ii) For $S, T \subseteq_{fin} \mathbb{A}$ algebraically closed, the subgroup of $\operatorname{Aut}\mathbb{A}$ generated by

$$\operatorname{Aut}\mathbb{A}/S \cup \operatorname{Aut}\mathbb{A}/T$$

coincides with $\operatorname{Aut}\mathbb{A}/(S \cap T)$, where $S \cap T \subseteq_{fin} \mathbb{A}$ is also algebraically closed.
iii) If $G \subseteq \operatorname{Aut}\mathbb{A}$ is an open subgroup, then there is an algebraically closed $\operatorname{SUPP}(G) \subseteq_{fin} \mathbb{A}$ satisfying

$$\operatorname{Aut}\mathbb{A}/S \subseteq G \iff S \supseteq \operatorname{SUPP}(G)$$

for all algebraically closed $S \subseteq_{fin} \mathbb{A}$ — so $\operatorname{SUPP}(G)$ is necessarily unique.
iv) If X is a nominal $\operatorname{Aut}\mathbb{A}$-set and $x \in X$, then

$$\operatorname{Aut}\mathbb{A}/\operatorname{SUPP} x \subseteq (\operatorname{Aut}\mathbb{A})_x \subseteq (\operatorname{Aut}\mathbb{A})_{\operatorname{SUPP} x}$$

for some algebraically closed $\operatorname{SUPP} x \subseteq_{fin} \mathbb{A}$ — here $\operatorname{SUPP} x$ also must be unique.

We note that

$$\operatorname{SUPP} : X \to \wp_{fin}(\mathbb{A})$$

is an equivariant function: because $(\operatorname{Aut}\mathbb{A})_{\pi \cdot x} = \pi \circ (\operatorname{Aut}\mathbb{A})_x \circ \pi^{-1}$ and $\pi^{-1} \circ \operatorname{Aut}\mathbb{A}/S \circ \pi = \operatorname{Aut}\mathbb{A}/(\pi^{-1} \cdot S)$, from iii) we see that $\operatorname{SUPP}(\pi \cdot x) = \pi \cdot \operatorname{SUPP} x$.

Recall that we already have a sort of equivalence between the open subgroups of $\mathrm{Aut}\,\mathbb{A}$ and single-orbit nominal $\mathrm{Aut}\,\mathbb{A}$-sets: any $x \in X \in \mathbf{NomSet}_{\mathrm{Aut}\,\mathbb{A}}$ defines an open subgroup $(\mathrm{Aut}\,\mathbb{A})_x \subseteq \mathrm{Aut}\,\mathbb{A}$; conversely an open $G \subseteq \mathrm{Aut}\,\mathbb{A}$ gives rise to $G \in \{\tau \circ G \mid \tau \in \mathrm{Aut}\,\mathbb{A}\} \in \mathbf{NomSet}_{\mathrm{Aut}\,\mathbb{A}}$ with $(\mathrm{Aut}\,\mathbb{A})_G = G$. The point is that a least support assumption affords an even more concrete way to represent orbit-finite nominal $\mathrm{Aut}\,\mathbb{A}$-sets.

> **Theorem 2.11 (algebraically closed adaptation of [BKL14, Theorems 9.17 and 10.9])**
> *Suppose $\mathbb{A}$ admits least finite algebraically closed supports. (Strictly speaking we do not use the weak elimination of imaginaries characterisation, so we may drop the oligomorphicity assumption.) Then the full subcategory of $\mathbf{NomSet}_{\mathrm{Aut}\,\mathbb{A}}$ on single-orbit nominal $\mathrm{Aut}\,\mathbb{A}$-sets is equivalent to the category with*
>
> - *as objects, pairs (S, H) where $S \subseteq_{fin} \mathbb{A}$ is algebraically closed and $H \subseteq \mathrm{Aut}\,S$ is a subgroup;*
> - *as morphisms from (S_1, H_1) to (S_2, H_2), sets of embeddings $S_2 \to \mathbb{A}$ of the form $i \circ H_2 = (i \circ h_2) \circ H_2$, where $i \circ h_2 : S_2 \to S_1$ is an embedding such that $H_1 \subseteq (i \circ h_2) \circ H_2 \circ (i \circ h_2)^{-1}$ for some/any $h_2 \in H_2$*
>
> *where*
>
> - *the composition of $i \circ H_2$ followed by $j \circ H_3$ is given by*
> $$i \circ h_2 \circ j \circ h_3 \circ H_3 = i \circ j \circ H_3$$
> - *and the identity on (S, H) is given by $\mathrm{id} \circ H$ with $\mathrm{id} : S \to S$ obviously satisfying $H \subseteq \mathrm{id} \circ H \circ \mathrm{id}^{-1}$.*

Proof. We exhibit a fully faithful functor $[\![-]\!]$ from our synthetic category directly to $\mathbf{NomSet}_{\mathrm{Aut}\,\mathbb{A}}$.

- On objects, we put
$$[\![S, H]\!] \stackrel{\text{def}}{=} \{s \circ H \mid s : S \to \mathbb{A} \text{ is an embedding}\}$$
with the action $\pi \cdot (s \circ H) \stackrel{\text{def}}{=} (\pi \circ s) \circ H$ for $\pi \in \mathrm{Aut}\,\mathbb{A}$. Evidently the inclusion map $\iota_S : S \to \mathbb{A}$ is in $[\![S, H]\!]$; by homogeneity any embedding $s : S \to \mathbb{A}$ can be extended to some $\pi_S \in \mathrm{Aut}\,\mathbb{A}$, which means that $\pi_S \circ \iota_S = s$ — hence $[\![S, H]\!] = \mathrm{Aut}\,\mathbb{A} \cdot (\iota_S \circ H)$. It is instructive to check that
$$\mathrm{Aut}\,\mathbb{A}/S \subseteq (\mathrm{Aut}\,\mathbb{A})_{\iota_S \circ H} \subseteq (\mathrm{Aut}\,\mathbb{A})_S,$$
from which one deduces that $\pi \cdot S$ is a finite support (and the least such) for $\pi \circ \iota_S \circ H \in [\![S, H]\!]$. Particularly, $[\![S, H]\!]$ is a single-orbit nominal $\mathrm{Aut}\,\mathbb{A}$-set.

- On morphisms, we put
$$\begin{aligned} [\![i \circ H_2]\!] : [\![S_1, H_1]\!] &\to [\![S_2, H_2]\!] \\ \pi \circ \iota_{S_1} \circ H_1 &\mapsto \pi \circ \iota_{S_1} \circ i \circ H_2 \end{aligned}$$
which is visibly equivariant; it is well-defined as a function because if $\pi \circ \iota_{S_1} \circ H_1 = \pi' \circ \iota_{S_1} \circ H_1$ then $\pi \circ \iota_{S_1} = \pi' \circ \iota_{S_1} \circ h_1$ for some $h_1 \in H_1 \subseteq i \circ H_2 \circ i^{-1}$, so
$$\begin{aligned} \pi \circ \iota_{S_1} \circ i &= \pi' \circ \iota_{S_1} \circ h_1 \circ i \\ &= \pi' \circ \iota_{S_1} \circ i \circ h_2 \end{aligned}$$
for some $h_2 \in H_2$ which shows $[\![i \circ H_2]\!](\pi \circ \iota_{S_1} \circ H_1) = [\![i \circ H_2]\!](\pi' \circ \iota_{S_1} \circ H_1)$.

- For functoriality, let $j : S_3 \to S_2$ satisfy $H_2 \subseteq j \circ H_3 \circ j^{-1}$ and consider $[\![j \circ H_3]\!] : [\![S_2, H_2]\!] \to [\![S_3, H_3]\!]$. On the one hand, by homogeneity some $\tau_i \in \mathrm{Aut}\,\mathbb{A}$ extends $i : S_2 \to S_1$ and therefore satisfies $\tau_i \circ \iota_{S_2} =$

$\iota_{S_1} \circ i$, so we get

$$\llbracket j \circ H_3 \rrbracket \circ \llbracket i \circ H_2 \rrbracket : \pi \circ \iota_{S_1} \circ H_1 \mapsto \pi \circ \tau_i \circ \iota_{S_2} \circ j \circ H_3$$
$$= \pi \circ \iota_{S_1} \circ i \circ j \circ H_3.$$

On the other hand $H_1 \subseteq i \circ H_2 \circ i^{-1} \subseteq (i \circ j) \circ H_3 \circ (i \circ j)^{-1}$, so $\llbracket (i \circ j) \circ H_3 \rrbracket$ is indeed a morphism and also sends $\pi \circ \iota_{S_1} \circ H_1$ to $\pi \circ \iota_{S_1} \circ i \circ j \circ H_3$. The other requirement that $\llbracket \mathrm{id} \rrbracket : \llbracket S, H \rrbracket \to \llbracket S, H \rrbracket$ is the identity function is apparent.

- For full faithfulness, observe that an equivariant function $f : \llbracket S_1, H_1 \rrbracket = \operatorname{Aut} \mathbb{A} \cdot (\iota_{S_1} \circ H_1) \to \llbracket S_2, H_2 \rrbracket$ in $\mathbf{NomSet}_{\operatorname{Aut} \mathbb{A}}$ must be of the form

$$\mathcal{O}_{j \circ H_2} : \pi \circ \iota_{S_1} \circ H_1 \mapsto \pi \circ j \circ H_2$$

where $j : S_2 \to \mathbb{A}$ is an embedding — here $j \circ H_2 = f(\iota_{S_1} \circ H_1)$. Now $\mathcal{O}_{j \circ H_2}$ is well-defined only if

1. $S_1 = \mathrm{SUPP}(\iota_{S_1} \circ H_1) \supseteq \mathrm{SUPP}(j \circ H_2) = j(S_2)$; and
2. for all $h_1 \in H_1$ and $\pi_{h_1} \in \operatorname{Aut} \mathbb{A}$ extending h_1,

$$\pi_{h_1} \circ j \circ H_2 = \mathcal{O}_{j \circ H_2}(\iota_{S_1} \circ h_1 \circ H_1) = \mathcal{O}_{j \circ H_2}(\iota_{S_1} \circ H_1) = j \circ H_2,$$

i.e., $h_1 \circ j = j \circ h_2$ for some $h_2 \in H$.

So $\llbracket - \rrbracket$ is full. But any embedding $j : S_2 \to \mathbb{A}$ satisfying $j(S_2) \subseteq S_1$ and $H_1 \circ j \subseteq j \circ H_2$ gives rise to an equivariant function $\mathcal{O}_{j \circ H_1} : \llbracket S_1, H_1 \rrbracket \to \llbracket S_2, H_2 \rrbracket$, and clearly $\mathcal{O}_{j \circ H_2} = \mathcal{O}_{j' \circ H_2}$ implies $j \circ H_2 = j' \circ H_2$ by evaluating at $\iota_{S_1} \circ H_1$; so $\llbracket - \rrbracket$ is faithful.

Finally, take any $\operatorname{Aut} \mathbb{A} \cdot x \in \mathbf{NomSet}_{\operatorname{Aut} \mathbb{A}}$. By our assumption on $\mathbb{A}$, there is a least finite algebraically closed support $\mathrm{SUPP}\, x \subseteq_{fin} \mathbb{A}$ of x. Note that $\operatorname{Aut} \mathbb{A} / \mathrm{SUPP}\, x$ is also normal in $(\operatorname{Aut} \mathbb{A})_x \subseteq (\operatorname{Aut} \mathbb{A})_{\mathrm{SUPP}\, x}$; let us write $H \subseteq \operatorname{Aut}(\mathrm{SUPP}\, x)$ for the image of the group homomorphism

$$(\operatorname{Aut} \mathbb{A})_x / (\operatorname{Aut} \mathbb{A} / \mathrm{SUPP}\, x) \to \operatorname{Aut}(\mathrm{SUPP}\, x)$$
$$\tau \circ (\operatorname{Aut} \mathbb{A} / \mathrm{SUPP}\, x) \mapsto \tau|_{\mathrm{SUPP}\, x}.$$

Now consider the map

$$\mathcal{O}_x : \llbracket \mathrm{SUPP}\, x, H \rrbracket \to \operatorname{Aut} \mathbb{A} \cdot x$$
$$\pi \circ \iota_{\mathrm{SUPP}\, x} \circ H \mapsto \pi \cdot x$$

which is certainly equivariant and surjective; but before that, it is well-defined and injective since

$$\begin{aligned}
& \pi \circ \iota_{\mathrm{SUPP}\, x} \circ H = \pi' \circ \iota_{\mathrm{SUPP}\, x} \circ H \\
\iff\ & \exists \tau \in (\operatorname{Aut} \mathbb{A})_x : \pi \circ \iota_{\mathrm{SUPP}\, x} = \pi' \circ \iota_{\mathrm{SUPP}\, x} \circ \tau|_{\mathrm{SUPP}\, x} \\
\iff\ & \exists \tau \in (\operatorname{Aut} \mathbb{A})_x, \forall s \in \mathrm{SUPP}\, x : \pi(s) = (\pi' \circ \tau)(s) \\
\iff\ & (\pi')^{-1} \circ \pi \in (\operatorname{Aut} \mathbb{A})_x \\
\iff\ & \pi \cdot x = \pi' \cdot x.
\end{aligned}$$

We conclude that $\mathcal{O}_x$ is an isomorphism in $\mathbf{NomSet}_{\operatorname{Aut} \mathbb{A}}$ and that an arbitrary single-orbit nominal $\operatorname{Aut} \mathbb{A}$-set $\operatorname{Aut} \mathbb{A} \cdot x$ is in the essential image of $\llbracket - \rrbracket$ (whose free coproduct completion is $\mathbf{NomSet}_{\operatorname{Aut} \mathbb{A}}$ at any rate). ∎

Such an orbit-by-orbit representation has been adopted in [BBKL12, §10] for a prototype of Nλ, and more recently in [VMR22, §5.4] to implement a C++ library Ons (Ordered Nominal Sets, `https://github.com/davidv1992/ONS`) as well as a Haskell library Ons-Hs (`https://github.com/Jaxan/ons-hs`). The main disadvantage compared to the set-builder notations based on first-order formulae is that to represent the straightforwardly defined set $\top[\mathbb{A}^n] = \mathbb{A}^n$, we need to represent each of the

$$n! + (n-1)! + \cdots + 2! + 1!$$

orbits for the ordered atoms and still as many orbits as the nth Bell number for the equality atoms. Indeed, [KLOT14, §5] reports that "only very rudimentary programs could be evaluated in reasonable time" with the orbit-based predecessor to Nλ. Nonetheless, when working with orbits in $\biguplus_I \mathbb{A}^n$ with n very small ($n \le 3$), even if the orbit count is big Ons(-Hs) can significantly outperform LOIS and Nλ; see [VMR22, Table 2] for a comparison of the running times.

2.3 As sheaves on an âge

> (Die Mathematiker sind eine Art Franzosen ...)
> Mathematicians are like Frenchmen: whatever you say to them they translate into their own language, and forthwith it is something entirely different.
>
> Johann Goethe [Goe88, 1279. on p.247]

Finally, it is possible too to take supports as the primitive notion and derive the action of $\mathrm{Aut}\,\mathbb{A}$. I learnt this approach from the notes [Kli11a; Kli11b] which are in turn inspired by [Joh02, Example 2.1.11(h)]; I mainly paraphrase what is written there while likely including too many proofs and details.

To begin with, we turn the âge of $\mathbb{A}$ into a category $\widehat{\mathbb{A}}$:

0) the objects are the finite substructures (i.e., finite subsets) of $\mathbb{A}$, and
1) the morphisms $\widehat{\mathbb{A}}(C, D)$ between $C, D \subseteq_{fin} \mathbb{A}$ are embeddings of C into D.

The homogeneity assumption means every $i \in \widehat{\mathbb{A}}(C, D)$ can be extended to some $\widehat{i} \in \mathrm{Aut}\,\mathbb{A}$.

Definition 2.12
A *presheaf* S is just a functor $\widehat{\mathbb{A}} \to \mathbf{Set}$. We say S is a *sheaf* if the *sheaf condition*

$$\begin{tikzcd} C \arrow[d, "i"'] \\ D \arrow[d, shift right, "j"'] \arrow[d, shift left, "k"] \\ E \end{tikzcd} \qquad \begin{tikzcd} SC \arrow[d, "Si"'] & 1 \arrow[l, dashed, "x"'] \arrow[dl, "y"] \\ SD \arrow[d, shift right, "Sk"'] \arrow[d, shift left, "Sj"] & \\ SE & \end{tikzcd}$$

$$\forall i, \forall y : \Big(\forall E, j, k : j \circ i = k \circ i \implies Sj(y) = Sk(y)\Big) \implies \exists! x : Si(x) = y.$$

holds. We write $\mathbf{PSh}(\mathbb{A})$ and $\mathbf{Sh}(\mathbb{A})$ for the categories of presheaves and sheaves with natural transformations.

Note our sheaves are precisely category theorists' sheaves for the atomic topology on $(\widehat{\mathbb{A}})^{\mathrm{op}}$ [MM94, Lemma 2 on p.126 of §III.4], where the Ore condition is satisfied [MM94, Example 2(f) on p.115 of §III.4] precisely because the amalgamation property is satisfied by the âge of the homogeneous structure $\mathbb{A}$.

2.3.1 Between nominal sets and presheaves

Definition 2.13
We describe a functor $\sharp : \mathbf{NomSet}_{\mathrm{Aut}\,\mathbb{A}} \to \mathbf{Sh}(\mathbb{A})$.

0) Given a nominal $\mathrm{Aut}\,\mathbb{A}$-set X, define $X^\sharp : \widehat{\mathbb{A}} \to \mathbf{Set}$ by:

$$\begin{array}{ccc} C & & X^\sharp C \overset{\text{def}}{=} \{x \in X \mid C \text{ supports } x\} \\ \Big\downarrow i & \mapsto & \Big\downarrow X^\sharp i : x \mapsto \widehat{i} \cdot x \\ D & & X^\sharp D \overset{\text{def}}{=} \{x \in X \mid D \text{ supports } x\} \end{array}$$

One checks that $X^\sharp i$ does not depend on the choice of $\bar{i} \in \mathrm{Aut}\,\mathbb{A}$ and indeed maps into $X^\sharp D$, that $X^\sharp$ is functorial, and that $X^\sharp$ satisfies the sheaf condition — y is supported by $i(C)$, so take $x = \widehat{i}^{-1} \cdot y$.

1) Given an equivariant function

$$X \xrightarrow{f} Y$$

between nominal $\mathrm{Aut}\,\mathbb{A}$-sets, one easily checks that

$$\begin{tikzcd} X^\sharp C \arrow[r, "f^\sharp_C : x \mapsto f(x)"] \arrow[d, "X^\sharp i"'] & Y^\sharp C \arrow[d, "Y^\sharp i"] \\ X^\sharp D \arrow[r, "f^\sharp_D : x \mapsto f(x)"'] & Y^\sharp D \end{tikzcd}$$

commutes and the horizontal functions are well-defined for any $C, D \subsetneq_{fin} \mathbb{A}$. So the family $f^\sharp_\bullet$ is a natural transformation $X^\sharp \Rightarrow Y^\sharp$, and the functoriality of $\sharp$ is immediate.

Definition 2.14
Now we describe a functor $\flat : \mathbf{PSh}(\mathbb{A}) \to \mathbf{NomSet}_{\mathrm{Aut}\,\mathbb{A}}$.

0) Let $S : \widehat{\mathbb{A}} \to \mathbf{Set}$ be a functor. Consider the wide subcategory $\widehat{\mathbb{A}}_\subseteq$ of $\widehat{\mathbb{A}}$ with only the inclusions $\iota_{D \supseteq C} : C \to D$ as morphisms; note $\widehat{\mathbb{A}}_\subseteq$ is thin and filtered. Let $I_\subseteq : \widehat{\mathbb{A}}_\subseteq \hookrightarrow \widehat{\mathbb{A}}$ denote the inclusion functor, and define the quotient set

$$\operatorname{colim} SI_\subseteq \cong S^\flat \overset{\text{def}}{=} \Big(\coprod_{C \subsetneq_{fin} \mathbb{A}} SC \Big) / \sim$$

where $(C \subsetneq_{fin} \mathbb{A}, x \in SC) \sim (D \subsetneq_{fin} \mathbb{A}, y \in SD)$ if and only if $S\iota_{E \supseteq C}(x) = S\iota_{E \supseteq D}(y)$ in SE for some $E \subsetneq_{fin} \mathbb{A}$ that contains $C \cup D$.

Now given $\pi \in \mathrm{Aut}\,\mathbb{A}$, put

$$\pi \cdot [C, x]_\sim \overset{\text{def}}{=} [\pi \cdot C, S\pi|_C(x)]_\sim.$$

- This is well-defined because if $E \supseteq C$, then $\pi|_C : C \to \pi \cdot C$ and $\pi|_E : E \to \pi \cdot E$ satisfy $\iota_{\pi \cdot E \supseteq \pi \cdot C} \circ \pi|_C \iota = \pi_E \circ \iota_{E \supseteq C}$. (Here we also co-restrict a restriction to its image.)

- It defines a group action because $(\tau \circ \pi)|_C = \tau|_{\pi \cdot C} \circ \pi|_C$.
- The resulting $\mathrm{Aut}\,\mathbb{A}$-set $S^\flat$ is nominal: each $[C, x]_\sim$ is supported by C.

1) Let $\alpha : S \Rightarrow T$ be a natural transformation. We define

$$\begin{aligned} \alpha^\flat &: S^\flat \to T^\flat \\ [C, x]_\sim &\mapsto [C, \alpha_C(x)]_\sim; \end{aligned}$$

then

- $\alpha^\flat$ is well-defined: this follows from the naturality of α;
- $\alpha^\flat$ is equivariant: we check that

$$\begin{aligned} \alpha^\flat(\pi \cdot [C, x]_\sim) &= \alpha^\flat([\pi \cdot C, S\pi|_C(x)]_\sim) \\ &= [\pi \cdot C, (\alpha_{\pi \cdot C} \circ S\pi|_C)(x)]_\sim \\ &= [\pi \cdot C, (T\pi|_C \circ \alpha_C)(x)])_\sim \\ &= \pi \cdot [C, \alpha_C(x)]_\sim = \pi \cdot \alpha^\flat([C, x]_\sim). \end{aligned}$$

So $\alpha^\flat$ is a morphism in $\mathbf{NomSet}_{\mathrm{Aut}\,\mathbb{A}}$. Again, the functoriality of $\flat$ is immediate.

Let us be pedantic and write $\underline{\flat}$ for the composite $\mathbf{Sh}(\mathbb{A}) \hookrightarrow \mathbf{PSh}(\mathbb{A}) \xrightarrow{\flat} \mathbf{NomSet}_{\mathrm{Aut}\,\mathbb{A}}$.

Lemma 2.15 *Given any nominal* $\mathrm{Aut}\,\mathbb{A}$*-set X,*

$$\begin{aligned} \eta_X &: X \to (X^\sharp)^{\underline{\flat}} \\ x &\mapsto [C_x, x]_\sim, \text{ where } C_x \text{ is any support of } x \end{aligned}$$

defines an equivariant bijection. Furthermore this isomorphism is natural in X.

Proof. The nominal $\mathrm{Aut}\,\mathbb{A}$-set $(X^\sharp)^{\underline{\flat}}$ has

$$\{(C, x) \mid C \subseteq_{fin} \mathbb{A} \text{ supports } x \in X\}/\sim$$

as its underlying set, where $(C, x) \sim (D, y)$ if and only if $x = y$, and is equipped with the action

$$\pi \cdot [C, x]_\sim = [\pi \cdot C, \pi \cdot x]_\sim.$$

It follows that η_X is well-defined, equivariant, and bijective.

For the naturality of $\eta_\bullet$, let $f : X \to Y$ be an equivariant function between nominal $\mathrm{Aut}\,\mathbb{A}$-sets; we see that

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & (X^\sharp)^{\underline{\flat}} \arrow[d, "(f^\sharp)^{\underline{\flat}}:[C,x]_\sim \mapsto [C, f(x)]_\sim"] \\ Y \arrow[r, "\eta_Y"'] & (Y^\sharp)^{\underline{\flat}} \end{tikzcd}$$

commutes as desired. ∎

Lemma 2.16 *Given any sheaf $S \in$* $\mathbf{Sh}(\mathbb{A})$,

$$\varepsilon_S \quad : (S^{\underline{\flat}})^\sharp \quad \Rightarrow S$$

$$
\begin{aligned}
\varepsilon_{S,C} : (S^{\underline{\flat}})^{\sharp} C &\to SC \\
[C, x]_{\sim} &\mapsto x
\end{aligned}
$$

defines a natural (in C) isomorphism between sheaves. Furthermore this isomorphism is natural in S.

Proof. The sheaf $(S^{\underline{\flat}})^{\sharp}$ assigns

- to each $C \subseteq_{fin} \mathbb{A}$, the set $\{[D \subseteq_{fin} \mathbb{A}, y \in SD]_{\sim} \mid \forall \pi \in \operatorname{Aut} \mathbb{A}/C : [D, y]_{\sim} = [\pi \cdot D, S\pi|_D(y)]_{\sim}\}$, and
- to each embedding $l : C \to F$, the function $[D, y]_{\sim} \mapsto [\widehat{l} \cdot D, S\widehat{l}|_D(y)]_{\sim}$.

But we can say more about an element $[D, y]_{\sim}$ supported by C now that S is a sheaf. Firstly, we have $(D, y) \sim (C \cup D, S\iota_{C\cup D \supseteq D}(y)) \stackrel{\text{def}}{=} (D', y')$. Now let $j, k : D' \to E$ be embeddings satisfying $j \circ \iota_{D' \supseteq C} = k \circ \iota_{D' \supseteq C}$; we wish to show $Sj(y') = Sk(y')$. Well, because $\widehat{j}, \widehat{k} \in \operatorname{Aut} \mathbb{A}$ satisfy $\widehat{j}|_C = \widehat{k}|_C$, the support assumption gives

$$
[\widehat{j} \cdot D', S\widehat{j}|_{D'}(y')]_{\sim} = [\widehat{k} \cdot D', S\widehat{k}|_{D'}(y')]_{\sim}.
$$

Since $E \supseteq j(D') = \widehat{j} \cdot D'$ and $j = \iota_{E \supseteq j(D')} \circ \widehat{j}|_{D'}$, we also know $(\widehat{j} \cdot D', S\widehat{j}|_{D'}(y')) \sim (E, Sj(y'))$ by the definition of $\sim$; similarly $(\widehat{k} \cdot D', S\widehat{k}|_{D'}(y')) \sim (E, Sk(y'))$, so by transitivity we obtain

$$
(E, Sj(y')) \sim (E, Sk(y'))
$$

in $S^{\underline{\flat}}$. We are done: by the definition of $\sim$, there is some $F \supseteq E$ such that $S\iota_{F \supseteq E}(Sj(y')) = S\iota_{F \supseteq E}(Sk(y'))$; the uniqueness in the sheaf condition forces $Sj(y') = Sk(y')$. We conclude by applying the sheaf condition again, this time using the existence: $y' = S\iota_{D' \supseteq C}(x)$ for some $x \in SC$, and therefore

$$
[D, y]_{\sim} = [D', y']_{\sim} = [C, x]_{\sim}.
$$

Conversely, for any $x \in SC$ the element $[C, x]_{\sim}$ in $S^{\underline{\flat}}$ is supported by C, and we have $(C, x) \sim (C, x')$ if and only if $x = x'$ by the sheaf condition yet again. All in all, $\varepsilon_{S,C}$ is a well-defined bijection of sets.

Next we check the naturality of $\varepsilon_{S,\bullet}$: given an embedding $l : C \to F$, we want

$$
\begin{tikzcd}
(S^{\underline{\flat}})^{\sharp} C \arrow[r, "\varepsilon_{S,C}"] \arrow[d, "(S^{\underline{\flat}})^{\sharp} l : [C,x]_{\sim} \mapsto [\widehat{l} \cdot C, S\widehat{l}|_C(x)]_{\sim}"'] & SC \arrow[d, "Sl"] \\
(S^{\underline{\flat}})^{\sharp} F \arrow[r, "\varepsilon_{S,F}"] & SF
\end{tikzcd}
$$

to commute. Our wish is granted: $Sl(x) = (S\iota_{F \supseteq l(C)} \circ S\widehat{l}|_C)(x)$ indeed.

Finally, we verify the naturality of $\varepsilon_{\bullet}$. To this end, let $\alpha : S \Rightarrow T$ be a natural transformation and let $C \subseteq_{fin} \mathbb{A}$. Then the component of $(\alpha^{\underline{\flat}})^{\sharp}$ at C is

$$
[C, x]_{\sim} \in (S^{\underline{\flat}})^{\sharp} C \qquad \mapsto \qquad \alpha^{\underline{\flat}}([C, x]_{\sim}) = [C, \alpha_C(x)]_{\sim} \in (T^{\underline{\flat}})^{\sharp} C.
$$

We conclude that $(\alpha_C \circ \varepsilon_{S,C})([C, x]_{\sim}) = \alpha_C(x) = (\varepsilon_{T,C} \circ (\alpha^{\underline{\flat}})^{\sharp}_C)([C, x]_{\sim})$ and that $\alpha \circ \varepsilon_S = \varepsilon_T \circ (\alpha^{\underline{\flat}})^{\sharp}$. ■

Theorem 2.17
$\mathbf{Sh}(\mathbb{A})$ *and* $\mathbf{NomSet}_{\operatorname{Aut} \mathbb{A}}$ *are equivalent as categories.*

Proof. $(\sharp, \underline{\flat}, \eta_{\bullet}, \varepsilon_{\bullet})$ is an adjoint equivalence. ■

2.3.2 The sheaf condition versus preserving pullbacks

When $\mathbb{A}$ is the equality atoms $\mathbf{Sh}(\mathbb{A})$ is the *Schanuel topos*, whose objects are perhaps more easily known as the pullback-preserving functors $\widehat{\mathbb{A}} \to \mathbf{Set}$ [Joh02, Example 2.1.11(h)]. For a general $\mathbb{A}$, the âge category $\widehat{\mathbb{A}}$ also has pullbacks: given embeddings $i : C \to E$ and $j : D \to E$,

$$\begin{tikzcd}
 & & C \arrow[dr, "i"] & \\
B \arrow[urr, bend left, "k"] \arrow[drr, bend right, "l"'] \arrow[r, dashed] & P \overset{\text{def}}{=} i(C) \cap j(D) \arrow[ur, "p\mapsto i^{-1}(p)"] \arrow[dr, "p\mapsto j^{-1}(p)"'] & & E \\
 & & D \arrow[ur, "j"'] &
\end{tikzcd} \tag{2.18}$$

is a pullback square — if $i \circ k = j \circ l$ then

$$\begin{aligned}
(i \circ k)(B) &\subseteq i(C), \\
(i \circ k)(B) &= (j \circ l)(B) \subseteq j(D)
\end{aligned}$$

so $b \mapsto (i \circ k)(b)$ is a mediating morphism $B \to P$ which is easily seen to be unique. In general, however, no containment relation holds between the pullback-preserving functors in $\mathbf{PSh}(\mathbb{A})$ and the sheaves in $\mathbf{Sh}(\mathbb{A})$.

Theorem 2.19
The following conditions are equivalent in the countable homogeneous structure $\mathbb{A}$.

i) The âge of $\mathbb{A}$ is a strong amalgamation class — see Example 1.4.
ii) $\mathbb{A}$ has no algebraicity, i.e.,

$$\forall C \subseteq_{fin} \mathbb{A} : acl(C) = C$$

— see Definition 1.8 and Remark 1.12c') for why we want to assume $\aleph_0$-categoricity.
iii) $\mathbb{A}$ has trivial definable closure (or is "fungible" as in [BKL14, Definition 9.6]), i.e.,

$$\forall C \subseteq_{fin} \mathbb{A} : dcl(C) = C.$$

iv) Every pullback-preserving functor $S : \widehat{\mathbb{A}} \to \mathbf{Set}$ is a sheaf.

Proof. i) ⟺ ii) is explained well in [Cam90, §2.7]. Also, ii) ⟹ iii) is obvious since

$$C \subseteq dcl(C) \subseteq acl(C).$$

For ii) ⟸ iii) let $d \in acl(C)$ so that $\operatorname{Aut} \mathbb{A}/C \cdot d$ is finite, say with the elements $d_1, \ldots, d_{n-1}, d_n = d$. Then

$$\operatorname{Aut} \mathbb{A}/(C \cup \{d_1, \ldots, d_{n-1}\}) \cdot d = \{d\}$$

which entails that $d \in dcl(C \cup \{d_1, \ldots, d_{n-1}\}) = C \cup \{d_1, \ldots, d_{n-1}\}$ and hence that $d \in C$.

Now we will show iii) ⟸ iv), so assume iv). Notice that the inclusion functor $I : \widehat{\mathbb{A}} \hookrightarrow \mathbf{Set}$ preserves pullbacks, so I is a sheaf. Let $C \subseteq_{fin} \mathbb{A}$ and take $d \in \mathbb{A} \setminus C$; the sheaf condition for $S = I$ and $i = \iota_{C \cup \{d\} \supseteq C}$ says that

$$\forall y \in C \cup \{d\} : \left(\forall E, \forall C \cup \{d\} \xrightarrow{j,k} E : j|_C = k|_C \implies j(y) = k(y) \right) \implies y \in C.$$

When $y = d \notin C$, we see that there are $j, k : C \cup \{d\} \to E$ such that $j|_C = k|_C$ but $j(d) \neq k(d)$. It follows that $(\widehat{j}^{-1} \circ \widehat{k}) \cdot d$ and d are distinct elements of $\operatorname{Aut} \mathbb{A}/C \cdot d$, so $d \notin dcl(C)$ and thus $dcl(C) \subseteq C$.

Finally, we tackle iii) $\Rightarrow$ iv). Suppose that $S \in \mathbf{PSh}(\mathbb{A})$ preserves pullbacks. To check the sheaf condition, let $i : C \to D$ and let $y \in SD$ be such that

$$\forall E, \forall D \xrightarrow{j,k} E : j \circ i = k \circ i \implies Sj(y) = Sk(y);$$

in the end we wish to establish $Si(x) = y$ for a unique $x \in SC$. Enumerate the elements of $D \setminus i(C)$ as $d_1, \ldots, d_n$; using the assumption on the decidable closure, we will cook up $d'_1, \ldots, d'_n$ and k so that

$$\begin{tikzcd}
C \arrow[r, "i"] \arrow[d, "i"'] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d, "\iota_{D \cup \{d'_1, \ldots, d'_n\} \supseteq D}"] \\
D \arrow[r, "k"'] & D \cup \{d'_1, \ldots, d'_n\}
\end{tikzcd}$$

is a pullback square. We do so inductively: having picked distinct $d'_1, \ldots, d'_\alpha, \ldots, d'_m \in \mathbb{A} \setminus D$ such that

$$d_\alpha \neq d'_\alpha = \pi_\alpha \cdot d_\alpha \in \operatorname{Aut} \mathbb{A}/(D \cup \{d'_1, \ldots, d'_{\alpha-1}\} \setminus \{d_\alpha\}) \cdot d_\alpha,$$

we can continue pick $d'_{m+1} \notin D \cup \{d'_1, \ldots, d'_m\}$ because $d_{m+1} \notin \mathit{dcl}(D \cup \{d'_1, \ldots, d'_m\} \setminus \{d_{m+1}\})$. Once we are done, put $k = (\pi_n \circ \cdots \circ \pi_2 \circ \pi_1)|_D$ so that

$$\begin{aligned}
e \in i(C) &\mapsto (\pi_n \circ \cdots \circ \pi_2)(e) = \cdots = e \\
d_\alpha &\mapsto (\pi_n \circ \cdots \circ \pi_\alpha)(d_\alpha) = \cdots = d'_\alpha
\end{aligned}$$

making the square commute; to see that C is a pullback, notice the mediating morphism $C \to i(C)$ from (2.18) is an isomorphism. Now S preserves pullbacks, and $P = \{(x_1, x_2) \in SD \times SD \mid Sk(x_1) = S\iota_{D \cup \{d'_1, \ldots, d'_n\} \supseteq D}(x_2)\}$ with the two projections is a pullback in **Set**: so

$$\begin{tikzcd}
P \arrow[dr, dashed, "f"] \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\
& SC \arrow[r, "Si"] \arrow[d, "Si"'] \arrow[dr, phantom, "\lrcorner", very near start] & SD \arrow[d, "S\iota_{D \cup \{d'_1, \ldots, d'_n\} \supseteq D}"] \\
& SD \arrow[r, "Sk"'] & S(D \cup \{d'_1, \ldots, d'_n\})
\end{tikzcd}$$

where the mediating morphism $f : P \to SC$ is a bijection. But by our assumption on $y \in SD$ we know that $(y, y) \in P$, so $f(y, y)$ is the unique $x \in SC$ satisfying $Si(x) = y$ that we hoped for. ∎

Example 2.20 (structures with and without algebraicity) It is unsurprising that an algebraic structure like (the canonical homogeneous expansion of) Example 1.14 "has algebraicity": we have

$$\mathit{acl}(\{e_1, e_2\}) = \{0, e_1, e_2, e_1 + e_2\}$$

where $e_1, e_2 \in \mathbb{V}_2$ are linearly independent. More trivially, the expansion by constants $\mathbb{A}_B$ in Remark 1.12 also "has algebraicity": we always have

$$b \in \mathit{dcl}(C)$$

whenever $b \in B$. A less evident example is the disjoint union of complete graphs $\mathbb{A} = \biguplus_m K_n$ when $n > 1$ is finite. There, given two distinct vertices $x, y \in K_n$, making $\pi \in \operatorname{Aut} \mathbb{A}$ fix the vertex $(i, x) \in \mathbb{A}$ also forces $\pi(i, y)$ to be from the finite vertex set

$$\{(i, y') \mid y' \in K_n, y' \neq x\}$$

— so the singleton $\{(i, x)\} \subseteq \mathbb{A}$ already fails to be algebraically closed.

Nonetheless any other relational structure from Example 1.4 has no algebraicity, because their âge is a strong amalgamation class.

It remains to study when $S \in \mathbf{Sh}(\mathbb{A})$ preserves pullbacks.

Theorem 2.21

i) If $S \in \mathbf{Sh}(\mathbb{A})$ preserves pullbacks, then $S^{\underline{\flat}} \in \mathbf{NomSet}_{\mathrm{Aut}\,\mathbb{A}}$ has least finite supports with respect to set inclusion.

ii) If $X \in \mathbf{NomSet}_{\mathrm{Aut}\,\mathbb{A}}$ has least finite supports with respect to inclusion, then $X^\sharp$ preserves pullbacks.

Now $\underline{\flat} : \mathbf{Sh}(\mathbb{A}) \rightleftarrows \mathbf{NomSet}_{\mathrm{Aut}\,\mathbb{A}} : \sharp$ is an equivalence by Theorem 2.17; therefore the following conditions are equivalent:

I) every sheaf preserves pullbacks;

II) $\mathbb{A}$ admits least finite supports.

Proof. Assume $S \in \mathbf{Sh}(\mathbb{A})$ preserves pullbacks. Let $[B, w]_\sim \in S^{\underline{\flat}}$. Suppose that $C, D \subseteq_{fin} \mathbb{A}$ both support $[B, w]_\sim$; recall from the proof of Lemma 2.16 that we then have

$$(C, x) \sim (B, w) \sim (D, y),$$

which means that some $E \supseteq C \cup D$ satisfies $S\iota_{E\supseteq C}(x) = S\iota_{E\supseteq D}(y)$. Form the pullback square of inclusions in $\widehat{\mathbb{A}}$, map it to **Set** under S, and form the pullback $P = \{(x_c, y_d) \in SC \times SD \mid S\iota_{E\supseteq c}(x_c) = S\iota_{E\supseteq c}(y_d)\} \ni (x, y)$:

$$\begin{tikzcd}
C \cap D \arrow[r, "\iota_{C\supseteq C\cap D}"] \arrow[d, "\iota_{C\supseteq C\cap D}"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "\iota_{E\supseteq C}"] \\
D \arrow[r, "\iota_{E\supseteq D}"'] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[dr, dashed, "f"] \arrow[drr, bend left, "p_1"] \arrow[ddr, bend right, "p_2"'] & & \\
& S(C \cap D) \arrow[r, "S\iota_{C\supseteq C\cap D}"] \arrow[d, "S\iota_{C\supseteq C\cap D}"] \arrow[dr, phantom, "\lrcorner", very near start] & SC \arrow[d, "S\iota_{E\supseteq C}"] \\
& SD \arrow[r, "S\iota_{E\supseteq D}"'] & SE
\end{tikzcd}$$

Since S preserves pullbacks, there is a bijection f such that $z \stackrel{\text{def}}{=} f(x, y) \in S(C \cap D)$ satisfies $S\iota_{C\supseteq C\cap D}(z) = x$; therefore $(C \cap D, z) \sim (C, x) \sim (B, w)$, showing that $[B, w]_\sim$ is supported by $C \cap D$. At last,

$$\operatorname{supp}[B, w]_\sim \stackrel{\text{def}}{=} \bigcap\{F \subseteq_{fin} \mathbb{A} \mid F \text{ supports } [B, w]_\sim\} = \bigcap\{F \subseteq B \subseteq_{fin} \mathbb{A} \mid F \text{ supports } [B, w]_\sim\}$$

is a finite, non-empty intersection of finite supports for $[B, w]_\sim$; as we argued above, $\operatorname{supp}[B, w]_\sim$ is a genuine finite support of $[B, w]_\sim$ and is by construction the least such.

Now assume that $X \in \mathbf{NomSet}_{\mathrm{Aut}\,\mathbb{A}}$ admits least finite supports. Then $X^\sharp$ maps a pullback square in $\widehat{\mathbb{A}}$ like so:

$$\begin{tikzcd}
& C \arrow[dr, "i"] & \\
P \arrow[ur, "p\mapsto i^{-1}(p)"] \arrow[dr, "p\mapsto j^{-1}(p)"'] \arrow[rr, phantom, "\rangle", near start] & & E \\
& D \arrow[ur, "j"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& X^\sharp C \arrow[dr, "X^\sharp i = \widehat{i}\cdot(-)"] & \\
X^\sharp P \arrow[ur, "\widehat{i}^{-1}\cdot(-)"] \arrow[dr, "\widehat{j}^{-1}\cdot(-)"'] & & X^\sharp E \\
& X^\sharp D \arrow[ur, "X^\sharp j = \widehat{j}\cdot(-)"'] &
\end{tikzcd}$$

where $P = i(C) \cap j(D)$ and

$$\begin{aligned} X^\sharp P &= \{z \in X \mid i(C) \cap j(D) \text{ supports } z\} \\ &\subseteq \{z \in X \mid i(C) \text{ supports } z, j(D) \text{ supports } z\} \\ &= \{z \in X \mid C \text{ supports } \widehat{i}^{-1} \cdot z, D \text{ supports } \widehat{j}^{-1} \cdot z\} \\ &= \{z \in X \mid \widehat{i}^{-1} \cdot z \in X^\sharp C, \widehat{j}^{-1} \cdot z \in X^\sharp D\} \end{aligned}$$

where the $\subseteq$ on the second line is in fact an equality, as finite supports are closed under intersection by assumption. Since

$$\{(x, y) \mid x \in X^\sharp C, y \in X^\sharp D, \widehat{i} \cdot x = \widehat{j} \cdot y\}$$

with the two projections defines a pullback in **Set**, it is clear that $(x, y) \mapsto \widehat{i} \cdot x = \widehat{j} \cdot y \in X^\sharp P$ is an isomorphism of pullback cones. ∎

Note we did not specify that the supports are algebraically closed. This is intentional.

Definition 2.22
We say $\mathbb{A}$ *admits least finite supports* if any of the following equivalent conditions [BKL14, Theorem 9.3] holds. By $\langle X \rangle \subseteq \operatorname{Aut} \mathbb{A}$ we mean the subgroup generated by an arbitrary subset $X \subseteq \operatorname{Aut} \mathbb{A}$.

i) For all $E \subseteq_{fin} \mathbb{A}$ and for all distinct $c, d \in \mathbb{A} \setminus E$,

$$G_E \cdot c \subseteq \langle G_{E \cup \{c\}} \cup G_{E \cup \{d\}} \rangle \cdot c.$$

ii) For $S, T \subseteq_{fin} \mathbb{A}$, the subgroup

$$\langle \operatorname{Aut} \mathbb{A}/S \cup \operatorname{Aut} \mathbb{A}/T \rangle$$

coincides with $\operatorname{Aut} \mathbb{A}/(S \cap T)$.

iii) If $G \subseteq \operatorname{Aut} \mathbb{A}$ is an open subgroup, then there is a unique $\operatorname{supp}(G) \subseteq_{fin} \mathbb{A}$ satisfying

$$\operatorname{Aut} \mathbb{A}/S \subseteq G \iff S \supseteq \operatorname{supp}(G)$$

for all $S \subseteq_{fin} \mathbb{A}$.

iv) If X is a nominal $\operatorname{Aut} \mathbb{A}$-set and $x \in X$, then there is a unique $\operatorname{supp} x \subseteq_{fin} \mathbb{A}$ satisfying

$$\operatorname{Aut} \mathbb{A}/S \subseteq (\operatorname{Aut} \mathbb{A})_x \iff S \supseteq \operatorname{supp} x$$

for all $S \subseteq_{fin} \mathbb{A}$.

This is a stronger assumption on $\mathbb{A}$ compared to Definition 2.10 — compare ii) or iii) here with the ones there.

Now assume that $\mathbb{A}$ admits least finite supports, and let $x \in X \in \mathbf{NomSet}_{\operatorname{Aut} \mathbb{A}}$. Notice that $\operatorname{supp} x \subseteq_{fin} \mathbb{A}$ satisfies

$$\operatorname{Aut} \mathbb{A}/\operatorname{supp} x \subseteq (\operatorname{Aut} \mathbb{A})_x \subseteq (\operatorname{Aut} \mathbb{A})_{\operatorname{supp} x}.$$

To see the second containment, observe that $\operatorname{supp} : X \to \wp_{fin}(\mathbb{A})$ is still an equivariant function; so $\pi \in (\operatorname{Aut} \mathbb{A})_x$ implies $\pi \cdot \operatorname{supp} x = \operatorname{supp}(\pi \cdot x) = \operatorname{supp} x$, i.e., π fixes $\operatorname{supp} x$ setwise. One can then go back to Theorem 2.11 and prove another representation theorem for nominal orbits, where instead of requiring $S \subseteq_{fin} \mathbb{A}$ to be algebraically closed we impose the rather fiddly property that

$$\forall a \in S : a \notin dcl(S \setminus \{a\}),$$

as was originally done in [BKL14, see Lemma 9.8(1) and Theorem 9.17].

Let us continue by considering $acl(\operatorname{supp} x) \supseteq \operatorname{supp} x$. As passing to stabiliser subgroups is inclusion-reversing, we obtain

$$\operatorname{Aut}\mathbb{A}/acl(\operatorname{supp} x) \subseteq \operatorname{Aut}\mathbb{A}/\operatorname{supp} x \subseteq (\operatorname{Aut}\mathbb{A})_x \subseteq (\operatorname{Aut}\mathbb{A})_{\operatorname{supp} x} = (\operatorname{Aut}\mathbb{A})_{acl(\operatorname{supp} x)}.$$

To see the last left-to-right containment this time, take $\pi \in (\operatorname{Aut}\mathbb{A})_{\operatorname{supp} x}$ and suppose $a \in acl(\operatorname{supp} x)$ — that is, suppose $\operatorname{Aut}\mathbb{A}/\operatorname{supp} x \cdot a$ is finite; then

$$\begin{aligned}
&\operatorname{Aut}\mathbb{A}/\operatorname{supp} x \cdot (\pi \cdot a) \\
&= \operatorname{Aut}\mathbb{A}/(\pi \cdot \operatorname{supp} x) \cdot (\pi \cdot a) \\
&= (\pi \circ \operatorname{Aut}\mathbb{A}/\operatorname{supp} x \circ \pi^{-1}) \cdot (\pi \cdot a) \\
&= \pi \cdot (\operatorname{Aut}\mathbb{A}/\operatorname{supp} x \cdot a)
\end{aligned}$$

is finite too, so $\pi \cdot a \in acl(\operatorname{supp} x)$. This proves that π fixes $acl(\operatorname{supp} x)$ as well — the setwise stabiliser only sees the algebraic closure. Of course $acl(\operatorname{supp} x)$ is still finite because $\mathbb{A}$ is oligomorphic; so Definition 2.10 says $\operatorname{SUPP} x$ exists and

$$\operatorname{SUPP} x = acl(\operatorname{supp} x)$$

quite conveniently.

Example 2.23 (structures with and without least supports)

a) One can directly show the equality atoms [GP02, Proposition 3.4], the ordered atoms, and the graph atoms [BKL14, Corollary 9.5 and Example 10.4] admit least finite supports and therefore also least finite, algebraically closed supports. Obviously this is not saying very much in a structure with no algebraicity, where the two notions agree. As we mentioned before Definition 2.10, the SSIP is equivalent to the SIP together with weak elimination of imaginaries; so the Henson graphs with forbidden subgraphs — directed or undirected — are also examples of structures with both kinds of least supports.

b) An innocent example where the two notions start to differ is $\mathbb{N}_5$, the equality atoms with a constant $5 \in \mathbb{N}$ fixed. Here one can still straightforwardly use the criterion i) of Definition 2.22 to show that $\operatorname{supp} x$ and thus $\operatorname{SUPP} x$ always exist, with the caveat that

$$\operatorname{SUPP} x = \operatorname{supp} x \cup \{5\} \supsetneq \operatorname{supp} x.$$

c) Now consider $\mathbb{A} = \biguplus_m K_n$ where $m, n \geq 2$ and $m \times n = \aleph_0$. Then $\operatorname{Aut}\mathbb{A}$ is the wreath product $S_n \wr_m S_m$ — that is, the set $(S_n)^m \times S_m$ with the operation

$$\left((\rho_j)_{j\in m}, \tau\right) \lll \left((\sigma_j)_{j\in m}, \pi\right) \stackrel{\text{def}}{=} \left((\rho_j \circ \sigma_{\tau^{-1}(j)})_{j\in m}, \tau \circ \pi\right)$$

acting on $\mathbb{A}$ via

$$\left((\sigma_j)_{j\in m}, \pi\right) \cdot (j', i') \stackrel{\text{def}}{=} (\pi(j'), \sigma_{\pi(j')}(i')).$$

But m can also be an $\operatorname{Aut}\mathbb{A}$-set if we simply put

$$\left((\sigma_j)_{j\in m}, \pi\right) \cdot j' \stackrel{\text{def}}{=} \pi(j').$$

Here $j' \in m$ is supported by the singletons $\{(j', 1)\}, \{(j', 2)\} \subseteq_{fin} \{j'\} \times K_n \subseteq \mathbb{A}$, but not by their intersection $\{\}$ unless $m = 1$. Therefore $\mathbb{A}$ does not admit least finite supports, though $\{(j', 1)\}$ and $\{(j', 2)\}$ are both minimal supports.

If we restrict ourselves to algebraically closed supports, when n is finite the algebraicity saves us: we have

$$acl(\{(j',1)\}) = \{j\} \times K_n = acl(\{(j',2)\}).$$

When $n = \aleph_0$ we run into trouble again: $\mathbb{A}$ has no algebraicity, so it fails to admit either kind of least supports.

We close with two questions.

Question 2.24
How much of [MM94, §III.9], [GMM06, §3], and [CKM10, Criterion 4.1 and Theorem 4.12]'s wheel did we reinvent, and how well did we do?

Question 2.25
Consider presheaves valued in **Mon** *or* $\mathbf{Vect}_{\mathbb{Q}}$ *instead of* **Set**. *What can be said about the sheaf condition and pullback preservation?*

Summary. We work in the following setting:

- $\mathcal{R}$ is a finite relational signature;
- $\mathcal{K}$ is an amalgamation class of finite $\mathcal{R}$-structures;
- $\mathbb{A}$ is the Fraïssé limit of $\mathcal{K}$.

Then:

- $\mathbb{A}$ is homogeneous;
- $\operatorname{Aut}\mathbb{A}$ acts oligomorphically on $\mathbb{A}^n$;
- $Th(\mathbb{A})$ eliminates quantifiers;
- $\mathbb{A}$ has no algebraicity if and only if $\mathcal{K}$ is a strong amalgamation class;
- $\mathbb{A}$ admits least finite supports if $\mathcal{K}$ is moreover a *free* amalgamation class [Con17, Theorem 1.1].

Note that $(\mathbb{Q}, =, <)$ admits least finite supports, even though its âge is an amalgamation that is strong but not free [Mac11, Remark 2.1.5 1.]; it is the generically ordered expansion of $(\mathbb{N}, =)$, whose âge is a free amalgamation class. We can say more: the generically ordered expansion $\mathcal{K}_<$ of a free amalgamation class $\mathcal{K}$ is *Ramsey* [EHN21, Theorem 1.3]; equivalently, $\operatorname{Aut}(\operatorname{FLim}\mathcal{K}_<)$ is extremely amenable [KPT05, Theorem 4.7]. I do not know why (or whether) $\operatorname{FLim}\mathcal{K}_<$ eliminates imaginaries in general.

There are more elementary ways to get new nice structures from old: e.g., take reducts, or expand by finitely many constants. One may need to check for homogeneity over a finite signature and the existence of least algebraically finite supports.

Having fixed a structure $\mathbb{A}$, we can build nominal sets out of $\{\}$ and the atoms $a \in \mathbb{A}$ alone; we can and should view them as continuous $\operatorname{Aut}\mathbb{A}$-sets, and also as sheaves on $\widehat{\mathbb{A}}$. So we understand orbit-finite nominal sets well through the lens of set theory, topological group theory, and category theory; we are now ready to study orbit-finite-dimensional vector spaces.

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