Thesis Proposal

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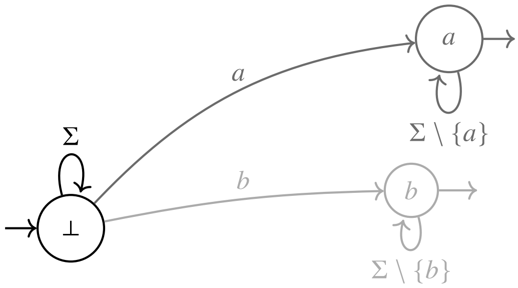
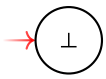
1 Motivating example

Consider an automata over the input alphabet Σ with the state space $\{\perp\} \cup \Sigma$. Define its transition relation by:

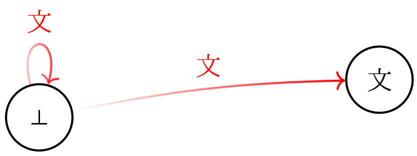


Even though only $\perp$ and 2 other states are shown, it is clear how we should fill in the blanks when given another letter $c \in \Sigma$ — as well as how we can extend Σ with a foreign letter c.

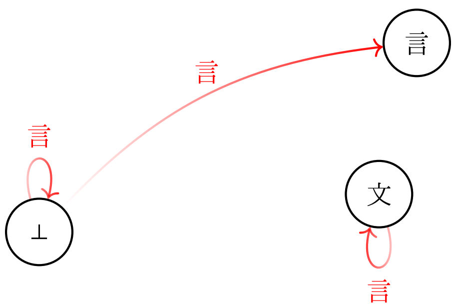
An example execution. Let us process the word 文言文. Unsurprisingly, we start at the initial state:



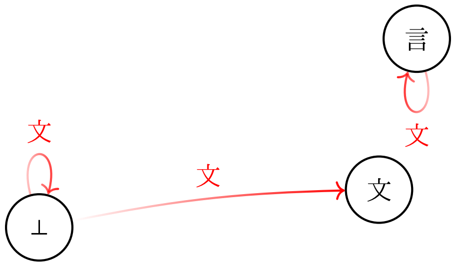
Then we read in the first letter, 文, from the left. There are two possible transitions, and we follow both:



So we are in two states at once. We continue by processing the next letter 言, again following all possible transitions:



— and this time, we have three active states. Now we read in a second 文:



Note that transitions are fired from only two states. There are no more letters to process, so we follow the final transitions:

giving 2 accepting runs in the end.

The formalism. Let Q denote the state space $\{\perp\} \cup \Sigma$. By $\operatorname{Lin} Q$ we mean the formal linear combinations of states in Q, say with rational coefficients. The initial states — here there is only one, $\perp$ — define a vector

$$\iota = \perp \in \operatorname{Lin} Q;$$

the transitions labelled with $a \in \Sigma$ define a linear map

$$\begin{aligned} \delta_a : \operatorname{Lin} Q &\to \operatorname{Lin} Q \\ \perp &\mapsto \perp + a \\ a &\mapsto 0 \\ b &\mapsto b \qquad \text{for } b \neq a \end{aligned}$$

on the basis Q; and the final states define a covector

$$\begin{aligned} \phi : \operatorname{Lin} Q &\to \mathbb{Q} \\ \perp &\mapsto 0 \\ a &\mapsto 1. \end{aligned}$$

The execution above can then be compactly written as

$$\iota = \perp \quad \overset{\delta_{文}}{\longmapsto} \quad \perp + 文 \quad \overset{\delta_{言}}{\longmapsto} \quad \perp + 言 + 文 \quad \overset{\delta_{文}}{\longmapsto} \quad \perp + 文 + 言 \quad \overset{\phi}{\mapsto} \quad 0 + 1 + 1 = 2.$$

It is not hard to see that in general, the output is the number of distinct letters in the input word.

The reachability problem. Processing the word $a_1 a_2 \ldots a_n \in \Sigma^*$ takes the automaton to the configuration

$$(\delta_{a_n} \circ \cdots \circ \delta_{a_2} \circ \delta_{a_1})(\iota) \in \operatorname{Lin} Q.$$

The linear span of the configurations reachable after processing at most n letters is thus given by

$$V_0 = \{r \cdot \iota \mid r \in \mathbb{Q}\},$$
$$V_{n+1} = V_n + \sum_{a \in \Sigma} \delta_a(V_n).$$

Notice that $V_n \subseteq V_{n+1}$ as subspaces of Lin Q, and if $V_n = V_{n+1}$ then in fact

$$V_n = V_{n+1} = V_{n+2} = \cdots .$$

When Q is finite, such an n necessarily exists:

$$V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n$$

means that $n \leq \dim(\mathrm{Lin}\, Q) = |Q|$. Consequently, the reachability problem for weighted automata is decidable in $O(|\Sigma| \cdot |Q|^3)$ time [Kie20, Proposition 2.2] and, due to an algorithm of Schützenberger that may be extracted from [Sch61], so is the equivalence problem.

The definitions of ι, δ_a, ϕ above also make sense for an infinite alphabet Σ and hence an infinite state space Q: for example, on input $q \in Q$, the transition δ_a just checks if q is equal to the constants $\perp$ or a, and produces an output that only uses $\perp$, a, and the input q; this is a finite description of an infinite object. It is therefore still reasonable to ask whether the reachability problem is decidable, but we certainly cannot consider $\dim(\mathrm{Lin}\, Q)$ — which is now infinite — again to justify why the chain of configuration spaces must stabilise in finitely many steps.

2 From automata with atoms to orbit-finite-dimensional vector spaces

Data values from an infinite domain with a limited interface arise in many settings:

- we may want to allow an unbounded number of unique process identifiers or cryptographic nonces to be generated but only tested for equality, not arbitrary individual identity;
- in timed systems, we also want to test if a timestamp is before, the same as, or after another.

Following [BKL14, §10], we will model such data values as a countably infinite homogeneous structure $\mathbb{A}$ (or equivalently, a Fraïssé limit) over a finite relational signature. Some examples include:

Data values	Fraïssé limit $\mathbb{A}$	Aut $\mathbb{A}$
Nonces	$(\mathbb{N}, =)$	all bijections $\mathbb{N} \to \mathbb{N}$
Timestamps	$(\mathbb{Q}, =, <)$	increasing bijections $\mathbb{Q} \to \mathbb{Q}$
?	$(\mathbb{R}\mathrm{ado}, =, \sim)$	graph automorphisms of $\mathbb{R}$ado

We also refer to these as the equality atoms, the ordered atoms, and the graph atoms respectively.

As demonstrated in [BKL14, Lemma 6.1] and [Boj19, Lemma 1.1], equivariance under the automorphism group Aut $\mathbb{A}$ ensures that the interface of the atoms is respected. Hence we will use the following blend of [KF94, Definition 1] and [BKM21, Definition 7.2].

Definition 1
A *weighted d-register automaton* over $\Sigma = \mathbb{A}$ consists of

- states $Q = C \times (\{\perp\} \cup \mathbb{A})^d$ where C is a finite set of control states,
- an initial vector $\iota \in \operatorname{Lin} Q$,
- a family $\delta_{(-)} : \Sigma \to (\operatorname{Lin} Q \xrightarrow{\text{linear}} \operatorname{Lin} Q)$ of weighted transition relations,
- and a final covector $\phi : \operatorname{Lin} Q \xrightarrow{\text{linear}} \mathbb{Q}$

such that ι, $\delta_{(-)}$, and ϕ are $\operatorname{Aut}\mathbb{A}$-equivariant. It recognises the weighted language

$$a_1 \dots a_n \in \Sigma^* \mapsto (\phi \circ \iota_{a_n} \circ \dots \circ \iota_{a_1})(\iota) \in \mathbb{Q}.$$

We note that Q is orbit-finite: under the action $\pi \cdot \big(c, (a_1, \dots, \perp, \dots, a_n)\big) = \big(c, (\pi(a_1), \dots, \perp, \dots, \pi(a_n))\big)$ the set Q splits into finitely many orbits. We can therefore describe the vector space $\operatorname{Lin} Q$ as orbit-finite-dimensional since it has an orbit-finite basis Q.

With the equivariance of ι and $\delta_{(-)}$ in mind, we can say more about the configuration spaces: we have

$$\pi \cdot V_0 = \{\pi \cdot (r \cdot \iota) = r \cdot (\pi \cdot \iota) = r \cdot \iota \mid r \in \mathbb{Q}\} = V_0$$

and, by induction,

$$\begin{aligned} \pi \cdot V_{n+1} &= \pi \cdot V_n + \pi \cdot \sum_{a \in \Sigma} \delta_a(V_n) \\ &= V_n + \sum_{a \in \Sigma} \pi \cdot \delta_a(\pi^{-1} \cdot V_n) = V_n + \sum_{a \in \Sigma} \delta_{\pi \cdot a}(V_n) = V_{n+1} \end{aligned}$$

for any $\pi \in \operatorname{Aut}\mathbb{A}$. Therefore the configuration spaces are equivariant as subspaces of $\operatorname{Lin} Q$, being closed under both linear combinations and atom automorphisms.

An algebraist will prefer saying that each V_i is a submodule of $\operatorname{Lin} Q$, viewed as a module of the group algebra $\operatorname{Lin}(\operatorname{Aut}\mathbb{A})$. In their language $\operatorname{Lin} Q$ is *Noetherian* if all ascending chains of submodules eventually stabilise, and $\operatorname{Lin} Q$ has finite *length* — the generalisation of the dimension that accounts for equivariance — if all descending chains of submodules eventually stabilise as well. The Noetherian property is what we need for the reachability checking procedure to terminate.

Conjecture 2
(a) If $\mathbb{A}$ is homogeneous over a finite relational signature, then $\operatorname{Lin}\mathbb{A}^n$ has finite length for all $n \in \mathbb{N}$.
(b) If $\operatorname{Aut}\mathbb{A}$ is oligomorphic, then $\operatorname{Lin}\mathbb{A}^n$ is Noetherian for all $n \in \mathbb{N}$.

Here the underlying field $\mathbb{Q}$ implicit, but we will write $\operatorname{Lin}_{\mathbb{k}}\mathbb{A}$ if we want to work over a different field $\mathbb{k}$.

We note homogeneity over a finite relational signature is a strictly stronger assumption than oligomorphicity, and having finite length is a strictly stronger conclusion than being Noetherian. The countable-dimensional vector space $\mathbb{V}_2$ over the two-element field $\mathbb{Z}/2\mathbb{Z}$ — alternatively, for computer scientists, bit strings with XOR — is oligomorphic, but cannot be made homogeneous over any finite relational signature [Mac11, before Theorem 3.17]. The orbit-finite-dimensional $\mathbb{Z}/2\mathbb{Z}$-vector space $\operatorname{Lin}_{\mathbb{Z}/2\mathbb{Z}}\mathbb{V}_2$ has an infinite strictly descending chain of equivariant subspaces [BFKM24, Theorem 4.16].

Very recently, David Evans has pointed out to me that the example of $\mathbb{V}_2$ had already been described by model theorists back in [AZ91], and that if some $\mathrm{Lin}_{\Bbbk}\,\mathbb{A}^n$ has infinitely many equivariant subspaces over a finite field $\Bbbk$ (e.g., if (a) fails), then we can construct a negative answer to [Mac11, Question 2.2.7 4.] and a counterexample to the conjecture of Thomas in [Tho91].

3 Current results

In [BPP13, §1] Bodirsky et al. claim to "know very little about [Thomas's] conjecture, beyond the fact that it is true for some fundamental homogeneous structures." Similarly we understand very little about Conjecture 2(a) in general, with $\mathrm{Lin}_{\mathbb{Z}/2\mathbb{Z}}\,\mathbb{V}_2$ being the only known failure of finite length (and owing to a lack of counterexamples, we believe (b) is true too.) The equality atoms, the ordered atoms, and the graph atoms all have finite length as we now show.

> **Theorem 3 (generalisation of [Yan23, Theorem 3.9])**
> *Let $d \in \mathbb{N}$. Suppose there is a uniform upper bound l on the number of local orbits: namely, every $S \subseteq_{fin} \mathbb{A}$ is contained in some $\overline{S} \subseteq_{fin} \mathbb{A}$ where — writing $(\mathrm{Aut}\,\mathbb{A})_{\overline{S}} \subseteq \mathrm{Aut}\,\mathbb{A}$ for the setwise stabiliser of $\overline{S}$ — we have*
>
> $$\#\left\{(\mathrm{Aut}\,\mathbb{A})_{\overline{S}} \cdot (s_1, \ldots, s_{2d}) \;\middle|\; (s_1, \ldots, s_{2d}) \in \overline{S}^{2d}\right\} \leq l.$$
>
> *Then, assuming the field $\Bbbk$ has characteristic 0, we have*
>
> $$\mathit{length}(\mathrm{Lin}_{\Bbbk}\,\mathbb{A}^d) \leq l.$$

The proof relies on some basic representation theory of finite groups and fits in a page. We deduce that any *smoothly approximated* structure studied by [KLM89] has finite length. In particular the result applies to the equality atoms and even to $\mathbb{V}_2$, showing that the problem with $\mathrm{Lin}_{\mathbb{Z}/2\mathbb{Z}}\,\mathbb{V}_2$ is due to the positive characteristic of the field $\mathbb{Z}/2\mathbb{Z}$.

Ehud Hrushovski thinks the result applies to the graph atoms as well, by constructing the enveloping graph $\overline{S}$ above from a symplectic bilinear form as described in the beginning of the monograph [CH03]; David Evans thinks there is an elementary proof not needing the hard machinery of the monograph.

In any case, this technique of approximating by finite substructures does not work for the ordered atoms: if a monotone $\pi \in \mathrm{Aut}\,\mathbb{A}$ fixes a finite set, then it must fix every point in that set; so the local orbits are all singletons, and the number of these is certainly unbounded. Instead, we can adapt and strengthen [BFKM24, §4.1 and §4.3] to determine all the equivariant subspaces — and *a fortiori* the length — of any orbit-finite-dimensional vector space. This approach works not only for the ordered atoms but also for the equality atoms, the graph atoms, the Henson triangle-free graphs, and the ordered Rado graph, though the proof is *ad hoc* and quite unsightly. Here we only state the consequence for the equality atoms and $d = 2$ — write $\mathbb{N}^{(2)} = \{(a, b) \in \mathbb{N}^2 \mid a \neq b\}$. Let

$$v \in \mathrm{Lin}_{\Bbbk}\,\mathbb{N}^{(2)} = \sum_{a \neq b} v(a, b)$$

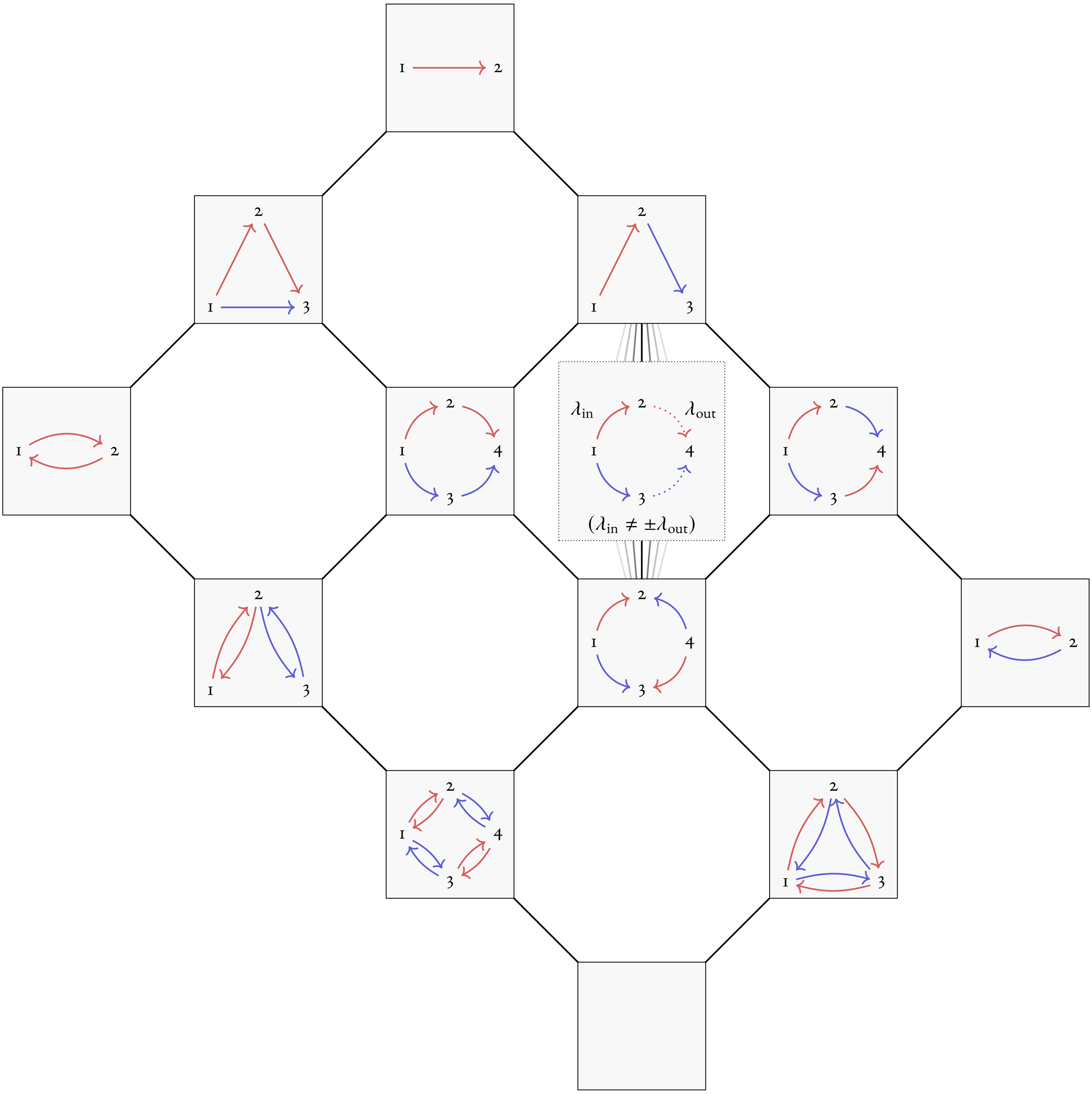
be a vector; then $v(a, b) \neq 0$ only if $a, b \in A$ for some finite $A \subseteq \mathbb{A}$. Given an arbitrary ordering $<$ on A, calculate

- the span of $\left(\sum_{-<*} v(-, *), \quad \sum_{-<*} v(*, -)\right) \in \Bbbk^2$;
- the span of $\left(\sum_{-<a} v(a, -), \quad \sum_{-<a} v(-, a), \quad \sum_{*>a} v(a, *), \quad \sum_{*>a} v(*, a)\right) \in \Bbbk^4$ for all $a \in A$;
- and the span of $\left(v(a, b), \quad v(b, a)\right) \in \Bbbk^2$ for all $a < b \in A$

to obtain a subspace $\flat v_< \subseteq \Bbbk^2 \oplus \Bbbk^4 \oplus \Bbbk^2$. We then define $\flat v = \flat v_{<_1} + \cdots + \flat v_{<_o}$ where $<_1, \ldots, <_o$ are all the total orders on A; we also define $\flat V = \sum_{v \in V} \flat v$ for an equivariant subspace $V \subseteq \mathrm{Lin}_{\Bbbk} \mathbb{N}^{(2)}$.

Theorem 4
The map $V \subseteq \mathrm{Lin}_{\Bbbk} \mathbb{N}^{(2)} \mapsto \flat V \subseteq \Bbbk^2 \oplus \Bbbk^4 \oplus \Bbbk^2$ *is order-preserving and injective.*

In particular $\Bbbk^2 \oplus \Bbbk^4 \oplus \Bbbk^2$ has finite dimension, so $\mathrm{Lin}_{\Bbbk} \mathbb{N}^{(2)}$ necessarily has finite length. We may also use this to derive the full lattice of equivariant subspaces shown below, assuming $2 \neq 0$ in $\Bbbk$. Each subspace is labelled by a possible generator v, which is in turn depicted as a finite graph with a red/blue edge $i \to j$ if and only if $v(i, j) = \pm 1$.



Bartek presented a very similar lattice at the IRIF automata seminar in 2022, but he missed the family parametrised by $[\lambda_{\text{in}} : \lambda_{\text{out}}]$ in the projective space. Regarding David Evans's comment, note also that this family is infinite when $\Bbbk = \mathbb{Q}$ but finite when $\Bbbk$ is finite — so we do not get a counterexample to Thomas's conjecture.

4 Future work

Free homogeneous structures. What specific model-theoretic properties on $\mathbb{A}$ did we use to prove Theorem 4? The equality atoms and the graph atoms are both Fraïssé limits of free amalgamation classes, whereas the ordered atoms and the ordered Rado graph are their generically ordered extension. We should therefore check if we can write a general proof for these two classes of structures; the recent works [Con17] and [PS17] may be of help.

Lower bounds on lengths. We can define a map

$$K \subseteq \Bbbk^2 \oplus \Bbbk^4 \oplus \Bbbk^2 \mapsto \sharp K \subseteq \operatorname{Lin}_{\Bbbk} \mathbb{N}^{(2)}$$

yielding a Galois connection $\flat \dashv \sharp$. However, given a strictly increasing chain $K_0 \subsetneq K_1 \subsetneq \cdots \subsetneq K_l$ of subspaces in $\Bbbk^2 \oplus \Bbbk^4 \oplus \Bbbk^2$, it may occur that $\sharp(K_i) = \sharp(K_{i+1})$ in the corresponding chain of equivariant subspaces in $\operatorname{Lin} \mathbb{N}^{(2)}$: indeed $2 + 4 + 2 > 5$. Can we detect when this collapse happens? That is, can we characterise when $K = \flat V$ for some equivariant subspace V? A first observation is that K must be invariant under the action of S_2 on the last $\Bbbk^2$ component.

I will comment on what is known about the lengths. For the ordered atoms, the bounds given in [BKM21, Corollary 4.11 and Theorem 4.7] is

$$2^d \leq \mathit{length}(\operatorname{Lin} \binom{\mathbb{Q}}{d}) \leq d!;$$

the analogue of Theorem 4 gives 2^d as a tighter upper bound, coinciding with the lower bound. For the equality atoms, the bounds given in [BKM21, Corollary 4.8] is

$$2^d \leq \mathit{length}(\operatorname{Lin} \mathbb{N}^{(d)}) \leq d!(1+d)!$$

which are again too generous: I found `https://oeis.org/A005425`(d) as an upper bound in [Yan23, Corollary 4.4] through a more careful application of Theorem 3, and there is evidence — namely, the remark [SS15, (8.7)] — that the bound is tight. Exhibiting a chain of this length in $\operatorname{Lin} \mathbb{N}^{(d)}$ via $\sharp(-)$ and Theorem 4 seems to be a low-hanging fruit.

For this last task, structure theorems of $\operatorname{Lin} \binom{\mathbb{N}}{d} \hookrightarrow \operatorname{Lin} \mathbb{N}^{(d)}$ will help us understand the properties of $\flat(-)$: it was shown in [CE91, Theorem 3.2] that over characteristic 0, there are exactly $d+1$ equivariant subspaces and these fit in a chain. Also, the case of positive characteristics was solved in [Gra97, Corollary 3.17] using what [Jam77, Corollary 9.4] calls *polytabloids*, and the same problem was again studied in [HR21, §7] using *simple hypergraphs*. It would seem what [BFKM24, Lemma 4.5] calls *cogs* — i.e., the workhorse of Theorem 4 — is at the very least similar to polytabloids and simple hypergraphs; we should therefore work out what these proofs share in common.

Back to automata. By [BFKM24, Lemma 5.2], the equivalence problem of two weighted 2-register automata as defined in Definition 1 with c control states in total can be reduced to the equivalence problem of two weighted automata over an $O(c)$ alphabet with $O(c^3)$ states; the latter can be decided in $O(c^{10})$ time. Can we do better if we follow [GHL22] and do nominal linear algebra? (On that note, it will also be interesting to develop a representation result for orbit-finitely spanned nominal vector spaces and linear maps analogous to [BKL14, Theorem 10.9]).

On the theoretical side, we can investigate if the linear Zariski machinery in [BS23] applies to weighted register automata; the study of algebraic geometry with atoms has already been initiated in [GL24]. On the practical side, we should implement the reachability checking algorithm in Nλ (https://www.mimuw.edu.pl/~szynwelski/nlambda/) or ONS-HS (https://github.com/Jaxan/ons-hs), and check if the chain of configuration spaces can really grow as long as the theoretical upper bound.

References

[AZ91] Gisela Ahlbrandt and Martin Ziegler. "What's so special about $(\mathbb{Z}/4\mathbb{Z})^\omega$?" In: *Archive for Mathematical Logic* 31.2 (Mar. 1991), pp. 115–132. ISSN: 1432-0665. DOI: 10.1007/bf01387764.

[BFKM24] Mikołaj Bojańczyk, Joanna Fijalkow, Bartek Klin, and Joshua Moerman. "Orbit-Finite-Dimensional Vector Spaces and Weighted Register Automata". In: *TheoretiCS* Volume 3 (May 2024). ISSN: 2751-4838. DOI: 10.46298/theoretics.24.13.

[BKL14] Mikołaj Bojańczyk, Bartek Klin, and Sławomir Lasota. "Automata theory in nominal sets". In: *Logical Methods in Computer Science* Volume 10, Issue 3 (Aug. 2014). DOI: 10.2168/lmcs-10(3:4)2014.

[BKM21] Mikołaj Bojańczyk, Bartek Klin, and Joshua Moerman. "Orbit-finite-dimensional vector spaces and weighted register automata". In: *2021 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. IEEE, June 2021. DOI: 10.1109/lics52264.2021.9470634.

[Boj19] Mikołaj Bojańczyk. *Slightly infinite sets*. Sept. 11, 2019. URL: https://www.mimuw.edu.pl/~bojan/upload/main-10.pdf.

[BPP13] Manuel Bodirsky, Michael Pinsker, and András Pongrácz. "The 42 reducts of the random ordered graph". In: *Proceedings of the London Mathematical Society* 111.3 (Sept. 9, 2013), pp. 591–632. ISSN: 1460-244X. DOI: 10.1112/plms/pdv037. arXiv: 1309.2165 [math.LO].

[BS23] Jason P. Bell and Daniel Smertnig. *Computing the linear hull: Deciding Deterministic? and Unambiguous? for weighted automata over fields*. 2023. arXiv: 2209.02260 [cs.FL]. URL: https://arxiv.org/abs/2209.02260.

[CE91] Alan R. Camina and David M. Evans. "Some infinite permutation modules". In: *The Quarterly Journal of Mathematics* 42.1 (1991), pp. 15–26. ISSN: 1464-3847. DOI: 10.1093/qmath/42.1.15.

[CH03] Gregory Cherlin and Ehud Hrushovski. *Finite structures with few types*. Annals of Mathematics Studies, Volume 152. Princeton University Press, 2003. ISBN: 0691113327; 9780691113326. URL: libgen.li/file.php?md5=c833b92189d127cb5927999c4c087d14.

[Con17] Gabriel Conant. "An axiomatic approach to free amalgamation". In: *The Journal of Symbolic Logic* 82.2 (June 2017), pp. 648–671. ISSN: 1943-5886. DOI: 10.1017/jsl.2016.42. arXiv: 1505.00762.

[GHL22] Arka Ghosh, Piotr Hofman, and Sławomir Lasota. "Solvability of orbit-finite systems of linear equations". In: (Jan. 2022). DOI: 10.48550/ARXIV.2201.09060. arXiv: 2201.09060 [cs.CL].

[GL24] Arka Ghosh and Sławomir Lasota. "Equivariant ideals of polynomials". In: (Feb. 2024). DOI: 10.48550/ARXIV.2402.17604. arXiv: 2402.17604 [cs.LO].

[Gra97] Darren G.D. Gray. "The structure of some permutation modules for the symmetric group of infinite degree". In: *Journal of Algebra* 193.1 (July 1997), pp. 122–143. ISSN: 0021-8693. DOI: 10.1006/jabr.1996.6963.

[HR21] Piotr Hofman and Jakub Różycki. “Linear equations for unordered data vectors in $[D]^k \to Z^d$”. In: *Logical Methods in Computer Science* Volume 18, Issue 4 (Dec. 2021). ISSN: 1860-5974. DOI: 10.46298/lmcs-18(4:11)2022. arXiv: 2109.03025 [cs.CC].

[Jam77] Gordon D. James. “A characteristic-free approach to the representation theory of $\mathfrak{S}_n$”. In: *Journal of Algebra* 46.2 (June 1977), pp. 430–450. ISSN: 0021-8693. DOI: 10.1016/0021-8693(77)90380-5.

[KF94] Michael Kaminski and Nissim Francez. “Finite-memory automata”. In: *Theoretical Computer Science* 134.2 (Nov. 1994), pp. 329–363. ISSN: 0304-3975. DOI: 10.1016/0304-3975(94)90242-9.

[Kie20] Stefan Kiefer. *Notes on Equivalence and Minimization of Weighted Automata*. 2020. arXiv: 2009.01217.

[KLM89] W. M. Kantor, Martin W. Liebeck, and H. D. Macpherson. “$\aleph_0$-Categorical Structures Smoothly Approximated by Finite Substructures”. In: *Proceedings of the London Mathematical Society* s3-59.3 (Nov. 1989), pp. 439–463. ISSN: 0024-6115. DOI: 10.1112/plms/s3-59.3.439.

[Mac11] Dugald Macpherson. “A survey of homogeneous structures”. In: *Discrete Mathematics* 311.15 (Aug. 2011), pp. 1599–1634. ISSN: 0012-365X. DOI: 10.1016/j.disc.2011.01.024.

[PS17] Gianluca Paolini and Saharon Shelah. “The Strong Small Index Property for Free Homogeneous Structures”. In: (Mar. 30, 2017). DOI: 10.48550/ARXIV.1703.10517. arXiv: 1703.10517 [math.LO].

[Sch61] Marcel-Paul Schützenberger. “On the Definition of a Family of Automata”. In: *Information and Control* 4.2 (1961), pp. 245–270. DOI: 10.1016/S0019-9958(61)80020-X.

[SS15] Steven V Sam and Andrew Snowden. “Stability patterns in representation theory”. In: *Forum of Mathematics, Sigma* 3 (June 2015). ISSN: 2050-5094. DOI: 10.1017/fms.2015.10.

[Tho91] Simon Thomas. “Reducts of the random graph”. In: *Journal of Symbolic Logic* 56.1 (Mar. 1991), pp. 176–181. ISSN: 1943-5886. DOI: 10.2307/2274912.

[Yan23] Jingjie Yang. *Equivariant subspaces of orbit-finite-dimensional vector spaces*. Master’s thesis. 2023. URL: https://www.cs.ox.ac.uk/files/14521/masters-thesis.pdf.