The Finite Length Property of the Rado Graph and Friends

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EXAMPLE. $\mathrm{Lin}_{\mathbb{R}}(\mathbb{N})$, the $\mathbb{R}$-vector space with basis $\{\underline{1}, \underline{2}, \underline{3}, \cdots\}$, allows 2 operations:

$\mathbb{R}$-linear combination
$3 \cdot (\underline{1} + \underline{2}) - \underline{9} \;=\; 3 \cdot \underline{1} + 3 \cdot \underline{2} - \underline{9}$

group action of Sym($\mathbb{N}$)
$\pi \cdot (\underline{1} + 0.5 \cdot \underline{2}) \;=\; \underline{\pi(1)} + 0.5 \cdot \underline{\pi(2)}$

What are the **equivariant subspaces**, i.e., subsets closed under both?

$\{0\}$ — $\{\sum_i r_i \cdot \underline{i} \mid \sum_i r_i \text{ is zero}\}$ — $\mathrm{Lin}_{\mathbb{R}}(\mathbb{N})$

and no other V:

If $V \ni \underline{1} + \underline{2} - 2 \cdot \underline{12}$, then $V \ni \underline{1} + \underline{2} - 2 \cdot \underline{1200}$ and hence $V \ni \underline{12} - \underline{1200}$. (or any $v \neq 0$)

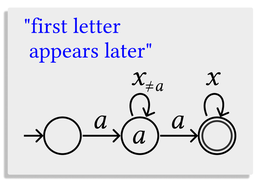
If $V \ni \underline{1} + \underline{2} + \underline{12}$, then, since $V \ni \underline{i} - \underline{i'}$, $V \ni -3 \cdot \underline{1200}$ and $V \ni \underline{1}$. (or anything non-zero-sum)

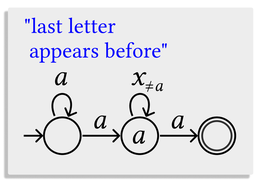
So **length**$(\mathrm{Lin}_{\mathbb{R}}(\mathbb{N}))$ = 2, even though $\dim(\mathrm{Lin}_{\mathbb{R}}(\mathbb{N})) = \infty$.

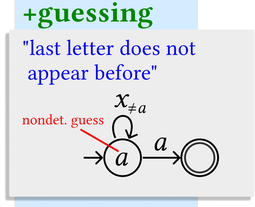
MOTIVATION. Over the alphabet $\mathbb{N}$,

Data values, only testable for = (and ≠).

Deterministic **R**egister **A**utomata $\subsetneq$ **U**nambiguous **R**egister **A**utomata $\subsetneq$ **U**nambiguous **R**egister **A**utomata **+guessing**







$\cap$ $\hookrightarrow$ $\subseteq$

Equivariant transitions

Deterministic **O**rbit **-F**inite **A**utomata $\hookrightarrow$ **W**eighted **O**rbit **-F**inite **A**utomata $\subseteq$ **O**rbit **-F**initely **-S**panned **A**utomata (With dual spaces 😳)

Still, language equivalence is decidable if:

#registers #states

$\forall d$: $k \times (1+\mathbb{N})^d$ has finitely many **orbits of** elements ✓

[Schützenberger's algorithm] $\forall d$: $\mathrm{Lin}_{\mathbb{R}}(k \times (1+\mathbb{N})^d)$ has finite **length** = "equivariant dimension" ✓

More examples: Can registers store richer data?

Finite structures	Fraïssé limit $\mathbb{A}$	$\mathrm{Lin}_{\mathbb{F}}(\mathbb{A}^d)$ has **finite length?**
pure sets	$(\mathbb{N}; =)$	✓ • for $\mathbb{C}$: [Sam–Snowden '15] • for any $\mathbb{F}$: [Evans–Rashwan '01]
total orders	$(\mathbb{Q}; =, <)$	✓ [B.–Fijalkow–K.–Moerman '24]
undirected graphs	(Rado graph; E) — One construction: • take $\mathbb{N}$ as vertices, • with an edge between $i < j$ if the i-th bit of j is 1,	✓
ordered undirected graphs	(Rado graph; E, $<$) — Generically ordered • ordered by $i\sqrt{2} - \lfloor i\sqrt{2} \rfloor$.	✓
finite-dim. $\mathbb{F}_2$-vector spaces	$(\mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \cdots;\ \square \stackrel{?}{=} 0,\ \square + \square \stackrel{?}{=} 0, \cdots)$	• for $\mathbb{C}$: (char. 0) ✓ • for $\mathbb{F}_2$: ✗ [B.–Fijalkow–K.–Moerman '24] [Evans–Gray '96] • for $\mathbb{F}_q$: ✗ ⇒ ¬Thomas's Conjecture [Evans '26]
???'s over a finite relational signature	??? that is finitely homogeneous	

No dual spaces ...

[LICS '26]

Proof sketches?

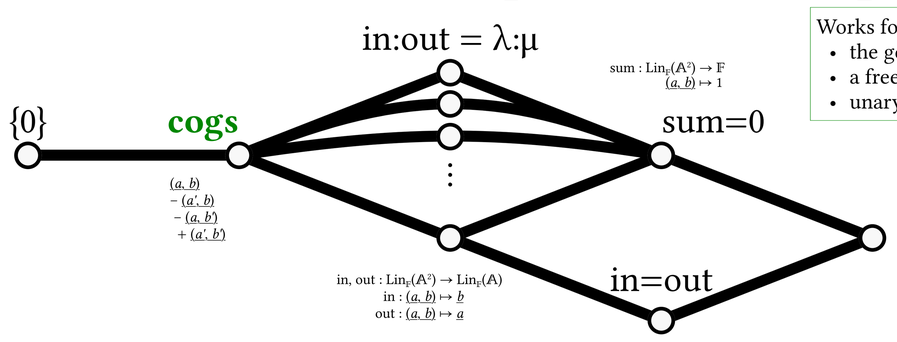
I. Approximate!

Works (best) when:
- $\mathbb{F}$ has characteristic 0
- $\mathbb{A}$ is oligomorphically approximable

length of $\mathrm{Lin}_{\mathbb{F}}(\{1, 2, \cdots, n\})$	1	2	3	4	5	6	7	8	9	⋯	∞
$\mathbb{Q}$	1	2	2	2	2	2	2	2	2	⋯	2
$\mathbb{F}_2$	1	2	2	3	2	3	2	3	2	⋯	2

II. Just find all the equivariant subspaces ...

Works for $(\mathbb{A}; \cdots, a_1, \cdots, a_k)$, where age($\mathbb{A}$) is:
- the generically ordered expansion of ⋯
- a free amalgamation class over ⋯
- unary and binary relations.

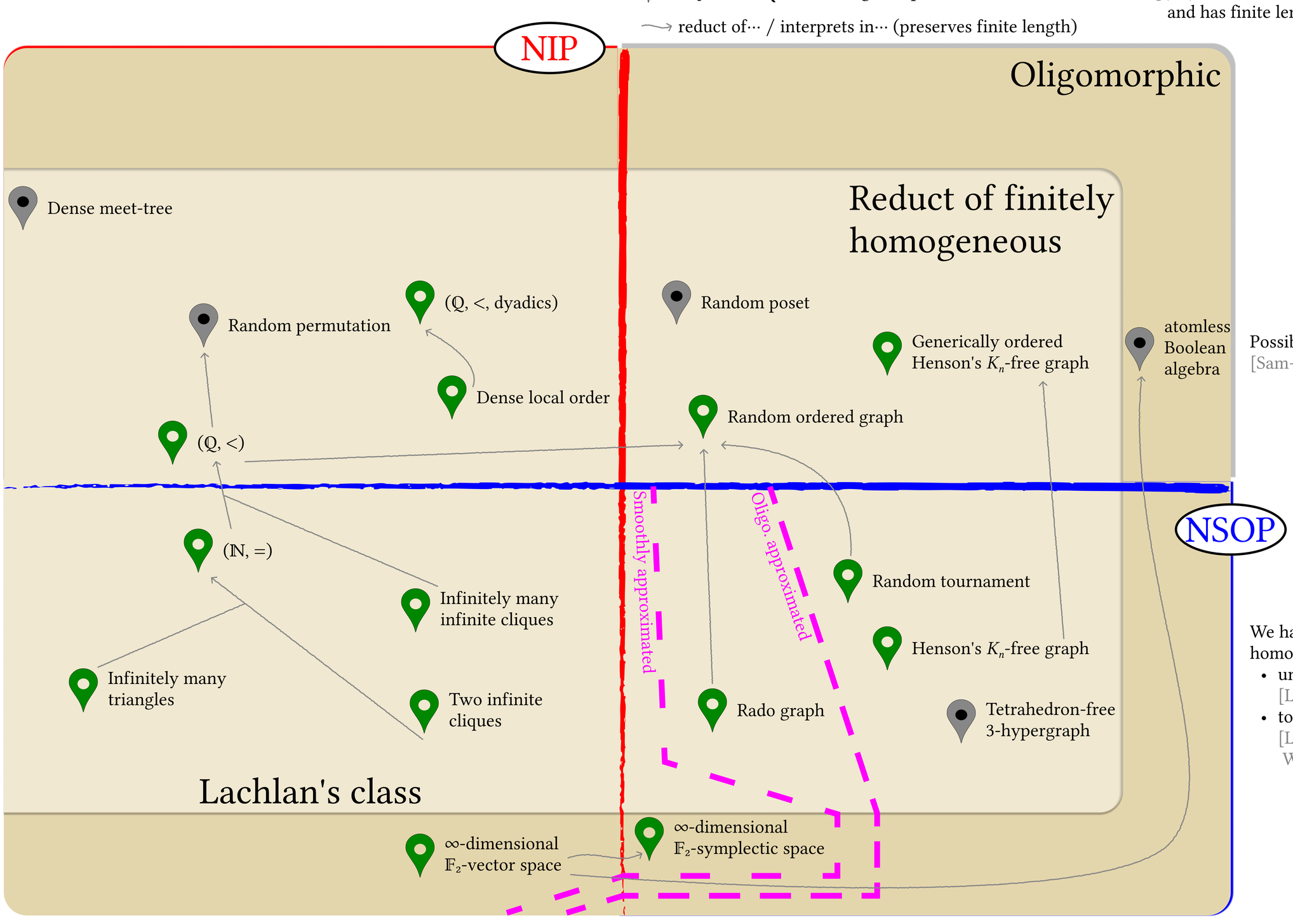


$\mathrm{Lin}_{\mathbb{F}}(\mathcal{O})$ where $\mathcal{O} \subset \mathbb{A}^2$ is an orbit of distinct atoms in $(\mathbb{Q}, <)$, the ordered Rado graph, ⋯

THE LENGTHSCAPE.

Theorem / **Conjecture**
- $\mathbb{A}$ is finitely homogeneous ⇒ $\forall d, \mathbb{F}$: $\mathrm{Lin}_{\mathbb{F}}(\mathbb{A}^d)$ has finite length
- $\mathbb{A}$ is oligomorphic ⇒ $\forall d, \mathbb{F}$: $\mathrm{Lin}_{\mathbb{F}}(\mathbb{A}^d)$ is Noetherian and has finite length if char($\mathbb{F}$)=0

⇝ reduct of⋯ / interprets in⋯ (preserves finite length)



has the Noetherian property by WQO + Hilbert's Basis Thm. [Ghosh–Lasota '24] [Laudone–Snowden '26]

Possibly helpful: [Sam–Snowden '17]

We have covered all homogeneous:
- undirected graphs [Lachlan '80]
- tournaments [Lachlan–Woodrow '84]

Lachlan's class:
- all interprets in (Q, <)
- ⊂ ω-stable, so has dual spaces 😳 [Cherlin–Harrington–Lachlan '85]

Here the Noetherian property follows from ∃AZ-enumeration + [Evans–Camina '91]