Lectures on Categorical Quantum Mechanics

Chris Heunen and Jamie Vicary

February 28, 2012

Preface

Physical systems cannot be studied in isolation, since we can only observe their behaviour with respect to other systems, such as a measurement apparatus. The central idea of this course is that the ability to group individual systems into compound systems should be taken seriously. We take the action of grouping systems together as a primitive notion, and build models of quantum mechanics from there.

The mathematical tool we use for this is category theory, one of the most beautiful parts of modern mathematics. It has become abundantly clear that it provides a deep and powerful language for describing *compositional structure* in an abstract fashion. It provides a unifying language for an incredible variety of areas, including quantum theory, quantum information, logic, topology and representation theory.

This course will tell this story right from the beginning, focusing on monoidal categories and their applications in quantum information.

Much of this relatively recent field of study is covered only fragmentarily or at the research level, see e.g. [16]. We feel there is a need for a self-contained text introducing categorical quantum mechanics at a more leisurely pace; these notes are intended to fill this space.

Prerequisites

Ideal foundations for this course are given by the Michaelmas term course "Categories, Proofs and Processes" and the Hilary term course "Quantum Computer Science". Students who have not taken these courses will need to be familiar with basic elements of both of these subjects, including categories, functors, natural transformations, vector spaces, Hilbert spaces and tensor products.

Acknowledgements

Thanks to the students who let us use them as guinea pigs for testing out this material! These notes would not exist were it not for the motivation and assistance of Bob Coecke. We are also grateful to Daniel Marsden for careful reading and useful feedback on early versions.

Contents

Chapter 1

Monoidal categories

This chapter introduces monoidal categories. These categories form the core of the course, as they provide the basic language with which the rest of the material will be developed. They have a powerful graphical calculus, which provides an intuitive, pictorial way to work with them that elegantly sidesteps much of their technical difficulty. We also introduce our main examples of monoidal categories, which will be used as running examples throughout the course.

1.1 Monoidal categories

Roughly, a monoidal category is a category equipped with extra data, allowing one to group objects and morphisms together into bigger objects and morphisms.

Scope

We will soon give the precise mathematical definition of a monoidal category. To appreciate it, it is good to realize first what sort of situation it aims to represent. In general, one can think of objects $A, B, C, \ldots$ of a category as *systems* of a certain type, and of morphisms $A \xrightarrow{f} B$ as *processes* turning a system of type A into a system of type B. This can be applied to a vast range of structures:

- physical systems, and physical processes governing them;
- data types in computer science, and algorithms manipulating them;
- algebraic or geometric structures in mathematics, and structure-preserving functions;
- logical propositions, and proofs of implications between them;
- one could even think of ingredients in stages of cooking, and recipes to process them into each other.

The extra structure of monoidal categories then simply says that we can consider processes *in parallel*, as well as in sequence. In the examples above, one could interpret this as:

- letting physical systems evolve next to each other;
- running computer algorithms in parallel;
- taking products or sums of algebraic or geometric structures;
- proving conjunctions of logical implications by proving both implications;
- chopping carrots while boiling rice.

Monoidal categories are the concept that makes all this precise.

Definition and coherence

Definition 1.1 (Monoidal category)**.** A monoidal category is a category $\mathbf{C}$ equipped with the following *data*, satisfying a property called *coherence*:

- a functor $\bigotimes \colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, called the *tensor product;*
- an object $I \in \mathbf{C}$, called the *unit object;*
- a family of natural isomorphisms $\alpha_{A,B,C} \colon (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ for all objects $A, B, C \in \mathbf{C}$, called the *associators;*
- a family of natural isomorphisms $\lambda_A \colon I \otimes A \to A$ for all objects $A \in \mathbf{C}$, called the *left unitors;*
- a family of natural isomorphisms $\rho_A \colon A \otimes I \to A$ for all objects $A \in \mathbf{C}$, called the *right unitors.*

The coherence property is that every well-formed equation built from $\circ$, $\otimes$, id, α, α^{-1}, λ, λ^{-1}, ρ and ρ^{-1} is satisfied.

Interesting examples of such equations are the *triangle* and *pentagon* equations:

$$\begin{array}{ccc} (A \otimes I) \otimes B & \xrightarrow{\alpha_{A,I,B}} & A \otimes (I \otimes B) \\ & {\scriptstyle \rho_A \otimes \mathrm{id}_B} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_A \otimes \lambda_B} & \\ & A \otimes B & \end{array} \tag{1.1}$$

$$
\begin{array}{ccccc}
 & \big(A \otimes (B \otimes C)\big) \otimes D & \xrightarrow{\alpha_{A,B\otimes C,D}} & A \otimes \big((B \otimes C) \otimes D\big) & \\
\alpha_{A,B,C} \otimes \mathrm{id}_D \nearrow & & & & \searrow \mathrm{id}_A \otimes \alpha_{B,C,D} \\
\big((A \otimes B) \otimes C\big) \otimes D & & & & A \otimes \big(B \otimes (C \otimes D)\big) \\
 & \searrow \alpha_{A\otimes B,C,D} & & \alpha_{A,B,C\otimes D} \nearrow & \\
 & & (A \otimes B) \otimes (C \otimes D) & &
\end{array}
\tag{1.2}
$$

By the coherence property, for any monoidal category, these diagrams must commute.

Surprisingly, it turns out that these equations (1.1) and (1.2) are sufficient (MacLane, Kelly).

Theorem 1.2 (Coherence for monoidal categories)**.** *Given the data for a monoidal category, α, λ and ρ are coherent iff* (1.1) *and* (1.2) *hold.*

This is a very important and beautiful theorem. It implies the following result — try to prove this yourself! (It's not easy.)

Corollary 1.3. *Equations* (1.1) *and* (1.2) *imply $\rho_I = \lambda_I$.*

Proof. See Exercise Sheet 1. □

Strictness

Some types of monoidal category seem particularly simple.

Definition 1.4 (Strict monoidal category)**.** A monoidal category is *strict* if the natural isomorphisms α, λ and ρ are identities.

In fact, every monoidal category can be 'made' into a strict one.

Theorem 1.5 (Strictification)**.** *Every monoidal category is monoidally equivalent to a strict monoidal category.*

We will not give a definition of monoidal equivalence in this course, which determines when two monoidal categories encode the same information.

This theorem means that, if you prefer, you can always 'strictify' your monoidal category to obtain an equivalent one for which α, λ and ρ are all identities. However, this often isn't very useful. For example, you often have some idea of what you want the objects of your category to be — but this might might not be compatible with a strict version of your category.

In particular, it's often useful for categories to be *skeletal*, meaning that if any pair A and B of objects are isomorphic, then they are equal. Every monoidal category is equivalent to a skeletal monoidal category, and skeletal categories are often particularly easy to work with. However, *not* every monoidal category is monoidally equivalent to a strict, skeletal category — you can't necessarily have both. If you have to choose, it usually turns out that skeletality is the more useful property to have.

The interchange law

Monoidal categories have an important property called the *interchange law.*

Theorem 1.6 (Interchange)**.** *Any morphisms* $A \xrightarrow{f} B$, $B \xrightarrow{g} C$, $D \xrightarrow{h} E$ *and* $E \xrightarrow{j} F$ *in a monoidal category satisfy the interchange law:*

$$(g \circ f) \otimes (j \circ h) = (g \otimes j) \circ (f \otimes h) \tag{1.3}$$

Proof. This holds because of properties of the category $\mathbf{C} \times \mathbf{C}$, and from the fact that $\bigotimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is a functor.

$$\begin{aligned}
(g \circ f) \otimes (j \circ h) &\equiv \textstyle\bigotimes(g \circ f, j \circ h) \\
&= \textstyle\bigotimes\big((g, j) \circ (f, h)\big) && \text{(definition of } \mathbf{C} \times \mathbf{C}) \\
&= \big(\textstyle\bigotimes(g, j)\big) \circ \big(\textstyle\bigotimes(f, h)\big) && \text{(functoriality of } \textstyle\bigotimes) \\
&\equiv (g \otimes j) \circ (f \otimes h)
\end{aligned}$$

Recall that the functoriality property for a functor F says that $F(f \circ g) = F(f) \circ F(g)$. □

1.2 Graphical calculus

We now describe a graphical way to denote the basic protagonists of monoidal categories: objects, morphisms, composition, and tensor product. This graphical calculus faithfully captures the essence of working with monoidal categories. In fact, in most cases, it makes them a lot easier to work with.

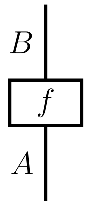
Graphical calculus for ordinary categories

First, we describe a graphical notation for *ordinary*, non-monoidal categories. We draw an object A like this:

(1.4)

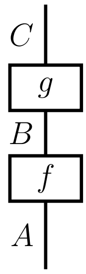
It's just a line. In fact, really, you shouldn't think of this as a picture of the object A; you should think of it as a picture of the identity morphism $\mathrm{id}_A : A \to A$. Remember: in category theory, the morphisms are more important than the objects.

We draw a general morphism $A \xrightarrow{f} B$ like this:



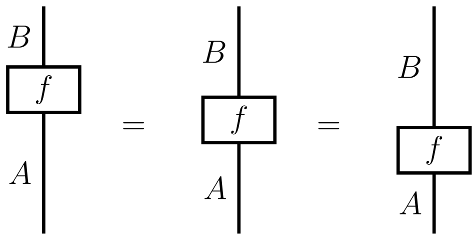
(1.5)

Composition of $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ is drawn like this:



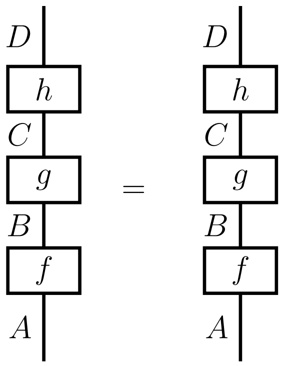
(1.6)

Let's see what the identity law $f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$ looks like:



(1.7)

It's completely trivial — we just have to remember that what is important is the *homotopy class* of the diagram. Categories also have an associativity axiom: given $C \xrightarrow{h} D$, we must have $(h \circ g) \circ f = h \circ (g \circ f)$. We can see what this looks like:



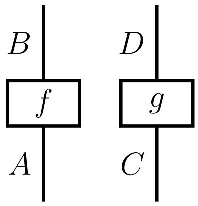
(1.8)

It's also trivial.

So even for *ordinary* categories, the graphical calculus is extremely useful: it somehow 'absorbs' our axioms, making them a consequence of the notation. This is because the axioms of a category have something to do with the *topology of space* at a fundamental level — in particular, 1-dimensional manifolds. Of course, this graphical representation isn't so unfamiliar. We usually call it *algebra*.

Graphical calculus for monoidal categories

Morphisms and composition are drawn in the same way as for ordinary categories. Given morphisms $A \xrightarrow{f} B$ and $C \xrightarrow{g} D$, we draw $f \otimes g : A \otimes C \to B \otimes D$ in the following way:



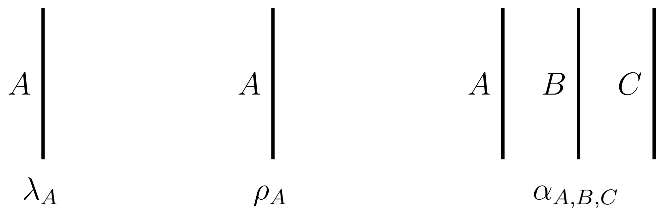
(1.9)

The idea is that f and g represent separate processes, taking place at the same time. Whereas the graphical calculus for ordinary categories was *linear*, the graphical calculus for monoidal categories is *planar*.

The monoidal unit object I is drawn as the empty diagram:

(1.10)

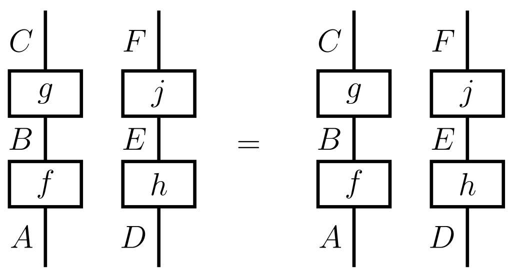
The left unitor $\lambda_A : I \otimes A \to A$, the right unitor $\rho_A : A \otimes I \to A$ and the associator $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ are drawn like this:



(1.11)

They are completely trivial. The *coherence* of α, λ and ρ is therefore important for the graphical calculus to function: since there can only be a single morphism formed from these natural isomorphisms between any two given objects, it doesn't matter that their graphical calculus encodes no information.

We now consider the graphical representation of the interchange law (1.3):



(1.12)

Once again, we see that it is completely trivial — what seemed to be a mysterious algebraic identity becomes very clear from the graphical perspective.

The point of the graphical calculus is that all of the superficially complex aspects of the algebraic definition of monoidal categories — the unit law, the associativity law, associators, left unitors, right unitors, the triangle equation, the pentagon equation, the interchange law — simply melt away, leaving one able to use the formalism much more directly. These features are still there, but they are absorbed into the geometry of the plane, of which our species has an excellent automatic understanding.

We will give a formal statement of the correctness of the graphical calculus later in the course.

1.3 Examples

It is now high time to have some examples.

Hilbert spaces

We now describe some examples of monoidal categories. Our first example is **Hilb**, which will play a central role in this course. We also define **FHilb** and $\mathbf{FHilb_{ss}}$, which are closely related.

Definition 1.7. The monoidal category **Hilb** is defined in the following way:

- **Objects** are separable complex Hilbert spaces $H, J, K, \ldots$;
- **Morphisms** are bounded linear maps $f, g, h, \ldots$;
- **Composition** is composition of linear maps;
- **Identity maps** are given by the identity linear maps;
- **Tensor product** $\bigotimes$: $\mathbf{Hilb} \times \mathbf{Hilb} \to \mathbf{Hilb}$ is tensor product of Hilbert spaces;
- **Unit object** $I \in \mathrm{Ob}(\mathbf{Hilb})$ is the 1-dimensional Hilbert space $\mathbb{C}$;
- **Associators** $\alpha_{H,J,K} : (H \otimes J) \otimes K \to H \otimes (J \otimes K)$ are the unique linear maps sending $|\phi\rangle \otimes \big(|\chi\rangle \otimes |\psi\rangle\big) \mapsto \big(|\phi\rangle \otimes |\chi\rangle\big) \otimes |\psi\rangle$ for all $|\phi\rangle \in H$, $|\chi\rangle \in J$ and $|\psi\rangle \in K$;
- **Left unitors** $\lambda_H : \mathbb{C} \otimes H \to H$ are the unique linear maps sending $1 \otimes |\phi\rangle \mapsto |\phi\rangle$ for all $|\phi\rangle \in H$;
- **Right unitors** $\rho_H : H \otimes \mathbb{C} \to H$ are the unique linear maps sending $|\phi\rangle \otimes 1 \mapsto |\phi\rangle$ for all $|\phi\rangle \in H$.

Recall that a Hilbert space is *separable* if it has finite or countable dimension, and that a linear maps between Hilbert spaces is *bounded* iff it is continuous. Such maps are defined on every vector in their domain.

You might have noticed that this definition of **Hilb** makes no mention of the *inner products* on the Hilbert spaces. This structure is crucial for quantum mechanics, so it's perhaps surprising it hasn't made an appearance here. In fact, in the development of this subject, it took a long time for people to understand the correct way to deal with it. We will encounter the inner product later on.

We also define a finite-dimensional variant of **Hilb**.

Definition 1.8. The monoidal category **FHilb** is the restriction of the monoidal category **Hilb** to finite-dimensional Hilbert spaces.

This is particularly appropriate for the purposes of quantum information, where the main results are often in finite dimensions.

Neither of the monoidal categories **Hilb** or **FHilb** are strict, and neither of them are skeletal. However, for **FHilb**, there is a *monoidally equivalent* monoidal category which *is* strict and skeletal, which we call $\mathbf{FHilb_{ss}}$.

Definition 1.9. The strict, skeletal monoidal category $\mathbf{FHilb_{ss}}$ is defined in the following way:

- **Objects** are natural numbers $0, 1, 2, \ldots$;
- **Morphisms** $n \to m$ are matrices of complex numbers with m rows and n columns;
- **Composition** is given by matrix multiplication;
- **Tensor product** $\bigotimes : \mathbf{FHilb_{ss}} \times \mathbf{FHilb_{ss}} \to \mathbf{FHilb_{ss}}$ is Kronecker product of matrices:

$$(f \otimes g) := \begin{pmatrix} \big(f_{11} g \big) & \big(f_{21} g \big) & \cdots & \big(f_{1n} g \big) \\ \big(f_{12} g \big) & \big(f_{22} g \big) & \ldots & \big(f_{2n} g \big) \\ \vdots & \vdots & \ddots & \vdots \\ \big(f_{m1} g \big) & \big(f_{m2} g \big) & \ldots & \big(f_{mn} g \big) \end{pmatrix};$$

- **Associators, left unitors and right unitors** are the identity matrices.

Objects n in $\mathbf{Hilb_{ss}}$ can be thought of as the Hilbert space $\mathbb{C}^n$, which has a privileged basis. Linear maps between such Hilbert spaces can be canonically represented as matrices. In practice, this monoidal category $\mathbf{FHilb_{ss}}$ is the most convenient place to work when doing calculations involving finite-dimensional Hilbert spaces.

We do not give a full treatment of the notion of *monoidal equivalence* in this course, but it seems intuitively possible that $\mathbf{FHilb_{ss}}$ somehow 'captures' everything that is important about **FHilb** as a monoidal category.

Question. What might go wrong if you try to include infinite-dimensional Hilbert spaces in this strict, skeletal category?

Sets and functions

While **Hilb** will be an important setting for *quantum* physics, the monoidal category **Set** is an important setting for *classical* physics.

Definition 1.10. The monoidal category **Set** is defined in the following way:

- **Objects** are sets;
- **Morphisms** are functions;
- **Composition** is function composition;
- **Identity morphisms** are given by the identity functions;
- **Tensor product** is Cartesian product of sets, written '×';
- **The unit object** is a 1-element set $\{\bullet\}$ (you choose which one!);
- **Associators** $\alpha_{A,B,C} : (A \times B) \times C \to A \times (B \times C)$ are the functions taking $\big((a,b),c\big) \mapsto \big(a,(b,c)\big)$, for $a \in A$, $b \in B$ and $c \in C$;
- **Left unitors** $\lambda_A : I \times A \to A$ are the functions taking $(\bullet, a) \mapsto a$ for $a \in A$;
- **Right unitors** $\rho_A : A \times I \to A$ are the functions taking $(a, \bullet) \mapsto a$ for $a \in A$.

Definition 1.11. The monoidal category **FSet** is the restriction of the monoidal category **Set** to finite sets.

If you have studied some category theory, you might know that the Cartesian product in **Set** is a *product*, which is a type of *limit* in category theory. This is a general phenomenon: if a category has products, then these can be used to give a monoidal structure. The same is true for coproducts.

This gives us an important difference between **Hilb** and **Set**: while the tensor product on **Set** comes from a categorical product, the tensor product on **Hilb** does not. We will discover many more differences between **Hilb** and **Set** in the coming lectures, which often tell us things about the differences between quantum and classical information.

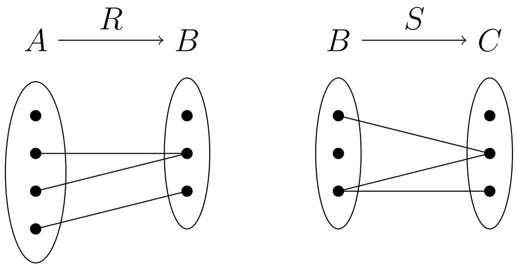
Sets and relations

While **Hilb** is a setting for quantum physics and **Set** is a setting for classical physics, **Rel**, the category of sets and relations, is somewhere in the middle. It seems like it should be a lot like **Set**, but in fact, its properties are much more like those of **Hilb**. This makes it a fascinating test-bed for investigating different aspects of quantum mechanics from a categorical perspective.

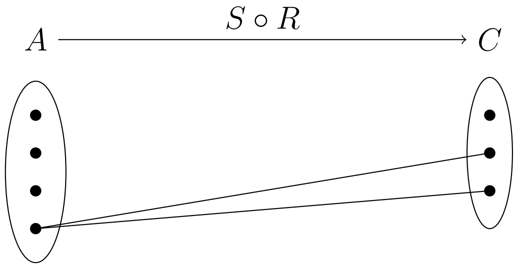
Definition 1.12. Given sets A and B, a *relation* $A \xrightarrow{R} B$ is a subset $R \subseteq A \times B$.

If elements $a \in A$ and $b \in B$ are such that $(a, b) \in R$, then we often indicate this by writing $a\,R\,b$, or even $a \sim b$ when R is clear. Since a subset can be defined by giving its elements, we can define our relations by listing the related elements, in the form $a_1\,R\,b_1$, $a_2\,R\,b_2$, $a_3\,R\,b_3$, and so on.

We can think of a relation $A \xrightarrow{R} B$ as indicating the possible ways from elements of A to elements of B, as follows.



Composition then connects up paths.



This corresponds to the following algebraic definition:

$$S \circ R := \{(a, c) \mid \exists b \in B\colon aRb \text{ and } bSc\} \subseteq A \times C \tag{1.13}$$

We can write this differently as

$$a\,(S \circ R)\,c \;\Leftrightarrow\; \bigvee_b (bSc \wedge aRb), \tag{1.14}$$

where $\vee$ represents logical OR, and $\wedge$ represents logical AND. Compare this with the definition of matrix multiplication:

$$(g \circ f)_{ij} = \sum_k g_{ik} f_{kj} \tag{1.15}$$

This gives us an interesting analogy between quantum mechanics and the theory of relations. Firstly, a relation $A \xrightarrow{R} B$ tells us, for each $a \in A$ and $b \in B$, whether it is *possible* for a to produce b. whereas a matrix $H \xrightarrow{L} J$ gives us an *amplitude* for a to evolve to b. Secondly, relational composition tells us the *possibility* of evolving via an intermediate point; where as matrix composition tells us the *amplitude* for this to happen.

We now define our monoidal category of relations.

Definition 1.13. The monoidal category **Rel** is defined in the following way:

- **Objects** are sets;
- **Morphisms** $A \xrightarrow{R} B$ are relations;
- **Composition** of two relations $A \xrightarrow{R} B$ and $B \xrightarrow{S} C$ is given as above;
- **Identity morphisms** $A \xrightarrow{\mathrm{id}_A} A$ are the relations $\{(a,a) \mid a \in A\} \subseteq A \times A$;
- **Tensor product** is Cartesian product of sets, written '×';
- **The unit object** is a 1-element set $\{\bullet\}$ (you choose which one!);
- **Associators** $\alpha_{A,B,C} : (A \times B) \times C \to A \times (B \times C)$ are the relations defined by $\big((a,b),c\big) \sim \big(a,(b,c)\big)$ for all $a \in A$, $b \in B$ and $c \in C$;
- **Left unitors** $\lambda_A : I \times A \to A$ are the relations defined by $(\bullet, a) \sim a$ for all $a \in A$;
- **Right unitors** $\rho_A : A \times I \to A$ are the relations defined by $(a, \bullet) \sim a$ for all $a \in A$.

The monoidal category **FRel** is the restriction of the monoidal category **Rel** to finite sets.

1.4 States

States of general objects

Morphisms out of the tensor unit I play a special role in a monoidal category. In many cases we can think of such morphisms as generalized 'states' or 'points', even though the objects might not be sets at all.

Definition 1.14 (State)**.** A *state* of an object A in a monoidal category is a morphism $I \to A$.

We now examine what the states are in our three example categories.

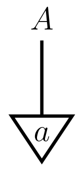
- In **Set**, points of a set A are morphisms $\{\bullet\} \to A$, which correspond to elements of A;
- In **Hilb**, points of a Hilbert space H are morphisms $\mathbb{C} \to H$, which correspond to elements of H by considering the image of $1 \in \mathbb{C}$;
- In **Rel**, points of a set A are relations $\{\bullet\} \xrightarrow{R} A$, which correspond to subsets of A.

Definition 1.15 (Well-pointed)**.** We say a monoidal category is *well-pointed* if for all parallel pairs of morphisms $A \xrightarrow{f,g} B$, we have $f = g$ iff $f \circ a = g \circ a$ for all points $I \xrightarrow{a} A$.

The idea is that in a well-pointed category, we can tell whether or not morphisms are equal just by seeing how they affect points of their source objects. The categories **Set**, **Hilb**, and **Rel** are all well-pointed.

Graphical representation

States $I \xrightarrow{a} A$ have the following graphical representation:

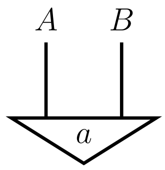


(1.16)

We can think of this dynamically as a *method of preparation* of the system A.

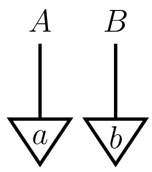
Entanglement and product states

For objects A and B of a monoidal category, a morphism $I \xrightarrow{a} A \otimes B$ is a *joint state* of A and B. We depict it graphically in the following way:



(1.17)

A joint state is a *product state*, or *separable*, if it is of the form $I \xrightarrow{\lambda_I^{-1}} I \otimes I \xrightarrow{a \otimes b} A \otimes B$, for $I \xrightarrow{a} A$ and $I \xrightarrow{b} B$:



(1.18)

An *entangled state* is a joint state which is not a product state. Entangled states represent preparations of $A \otimes B$ which cannot be decomposed as a preparation of A alongside a preparation of B. In this case, there is some essential connectivity between A and B which means that they cannot have been prepared independently.

Let's see what these look like in our example categories.

- In **Hilb**:
  - **Joint states** of H and J are elements of $H \otimes J$;
  - **Product states** are factorizable states;
  - **Entangled states** are elements of $H \otimes J$ which cannot be factorized.
- In **Set**:
  - **Joint states** of S and T are elements of $S \times T$;
  - **Product states** are elements $(s, t) \in S \times T$ coming from $s \in S$ and $t \in T$;

  - **Entangled states** don't exist!

- In **Rel**:
  - **Joint states** of A and B are subsets of $A \times B$;
  - **Product states** are subsets $P \subseteq A \times B$ such that, for some $R \subseteq A$ and $S \subseteq B$, $(a, b) \in P$ iff $a \in R$ and $b \in S$;
  - **Entangled states** are subsets of $A \times B$ which are not of this form.

This gives us an insight on why entanglement is so difficult for us to understand intuitively: classically, in the worldview encoded in the category **Set**, it simply does not occur. If we allow *probabilistic* behaviour, then a form of entanglement *is* classically meaningful — but quantum entanglement cannot be understood in this way.

1.5 Braiding and Symmetry

Braided monoidal categories

Definition 1.16. A *braided monoidal category* is a monoidal category **C** equipped with a family of natural isomorphisms

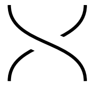
$$\sigma_{A,B} : A \otimes B \to B \otimes A \tag{1.19}$$

satisfying the *hexagon equations*:

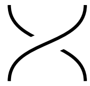
$$\begin{array}{ccccc}
 & A\otimes(B\otimes C) & \xrightarrow{\sigma_{A,B\otimes C}} & (B\otimes C)\otimes A & \\
 {}^{\alpha_{A,B,C}}\nearrow & & & & \searrow^{\alpha_{B,C,A}} \\
(A\otimes B)\otimes C & & & & B\otimes(C\otimes A) \\
 {}_{\sigma_{A,B}\otimes \mathrm{id}C}\searrow & & & & \nearrow_{\mathrm{id}B\otimes\sigma_{A,C}} \\
 & (B\otimes A)\otimes C & \xrightarrow[\alpha_{B,A,C}]{} & B\otimes(A\otimes C) &
\end{array} \tag{1.20}$$

$$\begin{array}{ccccc}
 & A\otimes(B\otimes C) & \xrightarrow{\sigma^{-1}_{B\otimes C,A}} & (B\otimes C)\otimes A & \\
 {}^{\alpha_{A,B,C}}\nearrow & & & & \searrow^{\alpha_{B,C,A}} \\
(A\otimes B)\otimes C & & & & B\otimes(C\otimes A) \\
 {}_{\sigma^{-1}_{B,A}\otimes \mathrm{id}_C}\searrow & & & & \nearrow_{\mathrm{id}_B\otimes\sigma^{-1}_{C,A}} \\
 & (B\otimes A)\otimes C & \xrightarrow[\alpha_{B,A,C}]{} & B\otimes(A\otimes C) &
\end{array} \tag{1.21}$$

We can include the braiding in our graphical notation:



$\sigma_{A,B} : A \otimes B \to B \otimes A$



$\sigma^{-1}_{A,B} : A \otimes B \to B \otimes A$ (1.22)

Invertibility has the following graphical representation:

(1.23)

This captures part of the topological behaviour of strings.

Since they cross over each other, they are not lying on the plane — they are in 3d space. So while categories have a 1d graphical notation, and monoidal categories have a 2d graphical notation, we see that braided monoidal categories have a 3d graphical notation. Because of this, braided monoidal categories have an important connection to 3d topological quantum field theories.

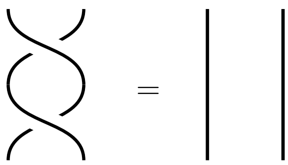
Symmetric monoidal categories

Definition 1.17. A braided monoidal category is *symmetric* if

$$\sigma_{B,A} \circ \sigma_{A,B} = \mathrm{id}_{A \otimes B} \tag{1.24}$$

for all objects A, B,

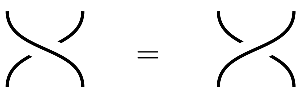
Graphically, this has the following representation:



(1.25)

Intuitively, this means our strings can pass through each other, and there can be no non-trivial linkages.

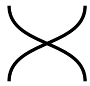
Lemma 1.18. *In a symmetric monoidal category we have $\sigma_{A,B} = \sigma^{-1}_{B,A}$, with the following graphical representation:*



(1.26)

Proof. Combine (1.23) and (1.25). □

In a symmetric monoidal category, there is therefore no distinction between over- and under-crossings, and so we simplify our graphical notation, drawing



(1.27)

for both.

Suppose we imagine our string diagrams as curves embedded *4d space*. Then we can continuously deform one crossing into the other, by making use of the extra dimension. In this sense, symmetric monoidal categories have a 4d graphical notation.

Examples

We now see what these structures look like for our example categories **Hilb**, **Set** and **Rel**, all of which can be equipped with a symmetry.

- In **Hilb**, $\sigma_{H,J} : H \otimes J \to J \otimes H$ is the unique linear map sending $|\phi\rangle \otimes |\psi\rangle \mapsto |\psi\rangle \otimes |\phi\rangle$ for all $|\phi\rangle \in H$ and $\psi \in J$.
- In **Set**, $\sigma_{S,T} : S \times T \to T \times S$ is defined by $\sigma_{S,T}(s,t) = (t,s)$ for all $s \in S$, $t \in T$.
- In **Rel**, $\sigma_{S,T} : S \times T \to T \times S$ is defined by $(s,t) \sim (t,s)$ for all $s \in S$, $t \in T$.

1.6 Exercises

Notes and further reading

(Symmetric) monoidal categories were introduced independently by Bénabou and Mac Lane in 1963 [10, 51]. Early developments centred around the problem of coherence, and were resolved by Mac Lane's Coherence Theorem 1.2. For a comprehensive treatment, see the textbooks [52, 13].

The graphical language dates back to 1971, when Penrose used it to abbreviate tensor contraction calculations [58]. It was formalized for monoidal categories by Joyal and Street in 1991 [38], who later also introduced and generalized to braided monoidal categories [40]. For a modern survey, see [68].

Our remarks about the dimensionality of the graphical calculus are a shadow of higher category theory, as first hinted at by Grothendieck [31]. For a modern overview, see [48]. Monoidal categories are 2-categories with one object, braided monoidal categories are 3-categories with one object and one 1-cell, and symmetric monoidal categories are 4-categories with one object, one 1-cell and one 2-cell — and n-categories have an n-dimensional graphical calculus; see [7].

Chapter 2

Abstract linear algebra

2.1 Scalars

Definition and examples

States of the tensor unit I play a very special role in monoidal categories. The are called the *scalars*, and generalize the idea of a 'base field' in linear algebra.

Definition 2.1 (Scalars). In a monoidal category, the *scalars* are the morphisms $I \to I$.

The scalars form a *monoid* under composition. They are very different in each of our example categories:

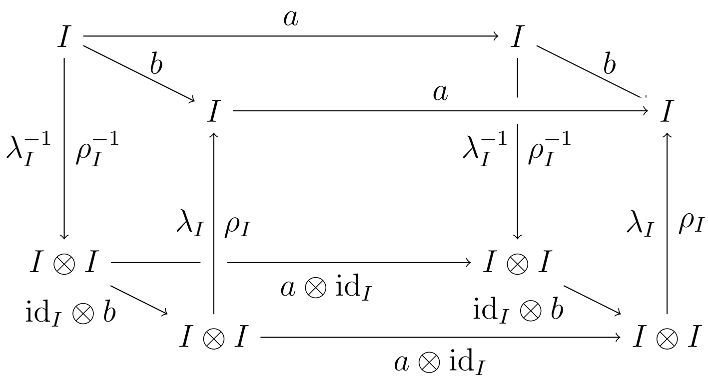
- In **Hilb**, the scalars are $\mathbb{C}$, the complex numbers, under multiplication;
- In **Set**, the scalars are 1, the trivial one-element monoid;
- In **Rel**, the scalars are {true, false} under AND.

Commutativity

In fact, for any monoidal category, this monoid commutative.

Lemma 2.2 (Scalars commute). *In a monoidal category, the scalars are commutative.*

Proof. We consider the following diagram, for any two scalars $a, b : I \to I$:



(2.1)

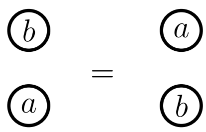
The four side cells of the cube commute by naturality of λ_I and ρ_I, and the bottom cell commutes by an application of the interchange law 1.6. Hence we have $ab = ba$. Note the importance of coherence, as we rely one the fact that $\rho_I = \lambda_I$. □

Graphical calculus

A scalar $I \xrightarrow{a} I$ takes the following form in the graphical calculus:

(2.2)

So commutativity of scalars has the following graphical representation:



(2.3)

This is another triviality.

Scalar multiplication

Objects in an arbitrary monoidal category do not have to be anything like vector spaces. Nevertheless, we replicate many of the features of the mathematics of vector spaces; in particular, we can multiply morphisms by scalars.

Definition 2.3 (Left scalar multiplication)**.** Let $I \xrightarrow{a} I$ be a scalar in a monoidal category, and $A \xrightarrow{f} B$ an arbitrary morphism. Define a new morphism $A \xrightarrow{a \bullet f} B$ as the following composite.

$$\begin{array}{ccc} A & \xrightarrow{a \bullet f} & B \\ {\scriptstyle \lambda_I^{-1}} \downarrow & & \uparrow {\scriptstyle \lambda_I} \\ I \otimes A & \xrightarrow{a \otimes f} & I \otimes B \end{array}$$

This abstract scalar multiplication satisfies many properties we know from the actual scalar multiplication of vector spaces. For example, it is associative, as in the following lemma. The proof is not hard – try it yourself.

Lemma 2.4 (Scalar multiplication). *Let $I \xrightarrow{a,b} I$ be scalars, and $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ be arbitrary morphism in a monoidal category. Then:*

1. $\mathrm{id}_I \bullet f = f$;
2. $a \bullet b = a \circ b$;
3. $a \bullet (b \bullet f) = (a \bullet b) \bullet f$;
4. $(b \bullet g) \circ (a \bullet f) = (b \circ a) \bullet (g \circ f)$.

2.2 Superposition

In quantum mechanics, given linear maps $H \xrightarrow{f,g} J$, we can form their sum $H \xrightarrow{f+g} J$, another linear map. When $H = \mathbb{C}$ this allows us to form superpositions of states, a fundamental part of quantum theory. We analyze this abstractly with the help of various categorical structures.

Zero morphisms

Definition 2.5. An object 1 (or $\top$) is a *terminal object* if for all objects A, there is a unique morphism $A \to 1$. An object 0 (or $\perp$) is an *initial object* if for all objects A, there is a unique morphism $0 \to A$. An object 0 is a *zero object* if it is both initial and terminal.

In a category with a zero object, for all objects A and B, there is a unique morphism $A \to 0 \to B$ factorizing through the zero object. We write this as $A \xrightarrow{0_{A,B}} B$, and call it the *zero morphism.*

Lemma 2.6. *A zero object is unique up to unique isomorphism.*

Of our example categories, **Hilb** and **Rel** have zero objects, whereas **Set** does not.

- In **Hilb**, the 0-dimensional vector space is a zero object, and the zero morphisms are the zero linear maps.
- In **Rel**, the empty set is a zero object, and the zero morphisms are the empty relations.

Superposition rules

We first define a *superposition rule* on a category, more formally known as an *enrichment in commutative monoids.*

Definition 2.7. An addition operation $(A \xrightarrow{f,g} B) \mapsto (A \xrightarrow{f+g} B)$ for every hom-set is an *superposition rule* if it has the following properties:

- **Commutativity:**
$$f + g = g + f \tag{2.4}$$
- **Associativity:**
$$(f + g) + h = f + (g + h) \tag{2.5}$$
- **Units:** for all A, B there is a unit morphism $A \xrightarrow{u_{A,B}} B$ such that for all $A \xrightarrow{f} B$:
$$f + u_{A,B} = f \tag{2.6}$$
- **Addition is compatible with composition:**
$$(g + g') \circ f = (g \circ f) + (g' \circ f) \tag{2.7}$$
$$g \circ (f + f') = (g \circ f) + (g \circ f') \tag{2.8}$$
- **Zeros are compatible with composition:**
$$u_{B,C} \circ u_{A,B} = u_{A,C} \tag{2.9}$$

Both **Hilb** and **Rel** have a superposition rule, while once again **Set** does not. In **Hilb**, it is given by addition of linear maps, and in **Rel** it is given by union of subsets.

Lemma 2.8. *In a category with a zero object and a superposition rule, then for all objects A, B, we have $u_{A,B} = 0_{A,B}$.*

Proof.
$$\begin{aligned} u_{A,B} &= u_{0,B} \circ u_{A,0} && \text{[by equation (2.7)]} \\ &= 0_{A,B} && \text{[by definition]} \end{aligned}$$
□

Because of this lemma, we write $0_{A,B}$ instead of $u_{A,B}$ whenever we are working in such a category. We can see this 'in action' in both **Hilb** and **Rel**: the zero linear map is the unit for addition, and the empty set is the unit for taking unions.

The existence of a zero object and a superposition rule turns our scalars into a *commutative semiring with an absorbing zero*, a set equipped with commutative multiplication and addition operations with the following properties:

$$\begin{aligned} (a + b)c &= ac + bc \\ a(b + c) &= ab + ac \\ a + b &= b + a \\ a + 0 &= 0 + a \\ a0 = 0 &= 0a \end{aligned}$$

In **Hilb**, this is the field $\mathbb{C}$. In **Rel**, this is {true, false}, with multiplication given by AND and addition given by OR.

Biproducts

In a category with a superposition rule, we can define the following structure.

Definition 2.9 (Biproducts)**.** In a category with a zero object and a superposition operation, the *biproduct* of an object A and B is an object $A \oplus B$ equipped with morphisms $A \xrightarrow{i_A} A \oplus B$, $B \xrightarrow{i_B} A \oplus B$, $A \oplus B \xrightarrow{p_A} A$ and $A \oplus B \xrightarrow{p_B} B$, satisfying the following equations:

$$\mathrm{id}_A = p_A \circ i_A \tag{2.10}$$
$$0_{B,A} = p_A \circ i_B \tag{2.11}$$
$$0_{A,B} = p_B \circ i_A \tag{2.12}$$
$$\mathrm{id}_B = p_B \circ i_B \tag{2.13}$$
$$\mathrm{id}_{A\oplus B} = (i_A \circ p_A) + (i_B \circ p_B) \tag{2.14}$$

Both **Hilb** and **Rel** have biproducts: in **Hilb**, they are the *direct sum* of Hilbert spaces, and in **Rel** they are union of sets.

Lemma 2.10. *In a category with biproducts, $(A \oplus B, p_A, p_B)$ form a product of A and B, and $(A \oplus B, i_A, i_B)$ form a coproduct of A and B.*

In fact, biproducts are automatic if a category has both products and a superposition rule.

Lemma 2.11. *If a category has products or coproducts, a zero object, and a superposition rule, then it has biproducts.*

Proof. See Exercise Sheet 1. □

Also, in the presence of biproducts, superposition rules are unique.

Lemma 2.12. *If a category has a biproducts and a zero object, then it has a unique superposition rule.*

Proof. Our category necessarily has at least one superposition rule, since it is required for the definition of biproducts. Using this, for any $f, g : A \to B$, we can define a morphism $A \xrightarrow{f \boxplus g} B$ as follows:

$$A \xrightarrow{\Delta_A} A \times A \xrightarrow{(i_1 \circ p_1)+(i_2 \circ p_2)} A + A \xrightarrow{[f,g]} B \tag{2.15}$$

Here, we make use of the fact that biproducts give rise to both products and coproducts. We can simplify this in the following way:

$$\begin{aligned} f \boxplus g &= [f,g] \circ (i_1 \circ p_1 + i_2 \circ p_2) \circ \Delta_A \\ &= [f,g] \circ i_1 \circ p_1 \circ \Delta_A + [f,g] \circ i_2 \circ p_2 \circ \Delta_A \\ &= f \circ \mathrm{id}_A + g \circ \mathrm{id}_A \\ &= f + g. \end{aligned}$$

However, from the biproduct equation (2.14), the composite $(i_1 \circ p_1) + (i_2 \circ p_2)$ used in (2.15) is the identity, and so our definition of $f \boxplus g$ is *independent* of our chosen superposition rule. Since we have shown $f \boxplus g = f + g$, our superposition rule is therefore unique. □

Matrix notation

In a category with biproducts, we can use a matrix notation for our morphisms. For example, for morphisms $A \xrightarrow{f} C$, $A \xrightarrow{g} D$, $B \xrightarrow{h} C$ and $B \xrightarrow{j} D$, we write

$$A \oplus B \xrightarrow{\begin{pmatrix} f & h \\ g & j \end{pmatrix}} C \oplus D \tag{2.16}$$

to denote the following map:

$$A \oplus B \xrightarrow{(i_C \circ f \circ p_A) + (i_D \circ g \circ p_A) + (i_C \circ h \circ p_B) + (i_D \circ j \circ p_B)} C \oplus D \tag{2.17}$$

Matrices with any finite number of rows and columns can be defined in a similar way.

Lemma 2.13. *Every morphism $A \oplus B \xrightarrow{k} C \oplus D$ has a matrix representation.*

Proof. Suppose we have a morphism $A \oplus B \xrightarrow{k} C \oplus D$. Then by introducing identities and expanding, we can rewrite it in the following way:

$$\begin{aligned} k &= \mathrm{id}_{C \oplus D} \circ k \circ \mathrm{id}_{A \oplus B} \\ &= \big((i_C \circ p_C) + (i_D \circ p_D)\big) \circ k \circ \big((i_A \circ p_A) + (i_B \circ p_B)\big) \\ &= i_C \circ (p_C \circ k \circ i_A) \circ p_A + i_C \circ (p_C \circ k \circ i_B) \circ p_B \\ &\qquad + i_D \circ (p_D \circ k \circ i_A) \circ p_A + i_D \circ (p_D \circ k \circ i_B) \circ p_B \end{aligned} \tag{2.18}$$

But this is the morphism represented by the following matrix:

$$\begin{pmatrix} p_C \circ k \circ i_A & p_C \circ k \circ i_B \\ p_C \circ k \circ i_A & p_D \circ k \circ i_B \end{pmatrix} \tag{2.19}$$

□

Matrices compose in the way we would expect, with morphism composition replacing multiplication of entries. For example, for 2 by 2 matrices:

$$\begin{pmatrix} s & p \\ q & r \end{pmatrix} \circ \begin{pmatrix} f & g \\ h & j \end{pmatrix} = \begin{pmatrix} (s \circ f) + (p \circ h) & (s \circ g) + (p \circ j) \\ (q \circ f) + (r \circ h) & (q \circ g) + (r \circ j) \end{pmatrix} \tag{2.20}$$

Identity morphisms have a familiar matrix representation:

$$\mathrm{id}_{A \oplus B} = \begin{pmatrix} \mathrm{id}_A & 0_{B,A} \\ 0_{A,B} & \mathrm{id}_B \end{pmatrix} \tag{2.21}$$

We will prove these properties in Exercise Sheet 1.

Interaction with monoidal structure

In general, linear structure interacts badly with monoidal structure. For example, it isn't true in general that $f \otimes (g+h) = (f \otimes g) + (f \otimes h)$, or that $f \otimes 0 = 0$; for counterexamples to both of these, consider the category of Hilbert spaces with direct sum as the tensor product operation. To get this sort of good interaction, we require *duals for objects*, which we will encounter in the next chapter.

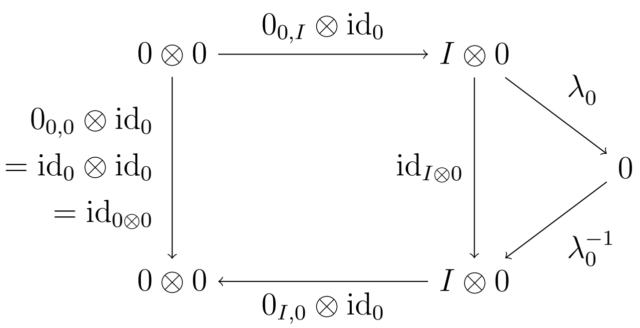
However, it is possible to prove the following result in general.

Lemma 2.14. *In a monoidal category with a zero object,* $0 \otimes 0 \simeq 0$.

Proof. We first note that $I \otimes 0$ is a zero object, since it is isomorphic to 0. We then consider the composite morphisms

$$0 \xrightarrow{\lambda_0^{-1}} I \otimes 0 \xrightarrow{0_{I,0} \otimes \mathrm{id}_0} 0 \otimes 0,$$
$$0 \otimes 0 \xrightarrow{0_{0,I} \otimes \mathrm{id}_0} I \otimes 0 \xrightarrow{\lambda_0} 0.$$

Composing them in one direction we obtain a morphism of type $0 \to 0$, which is necessarily the identity since 0 is a zero object. Composing in the other direction, we obtain the following:



Hence $0 \otimes 0$ is isomorphic to a zero object, and so is itself a zero object. □

2.3 Adjoints and the dagger-functor

Introduction

When we defined the monoidal category of Hilbert spaces in Definition 1.7 above, one peculiarity stood out: we didn't make any use of the *inner product*. As a result, only the vector space structure of the Hilbert spaces was playing a role. However, the inner products of Hilbert spaces are crucial for quantum mechanics, allowing us to compute probabilities in the theory. We capture the inner product structure with the abstract notion of a *dagger-functor*, and we explore some applications of this concept.

A Hilbert space H is equipped with an inner product, which is a *sesquilinear map*

$$\langle - \,|\, - \rangle_H \colon H \times H \to \mathbb{C}. \tag{2.22}$$

This means it is a map which is *antilinear* in the first component, *linear* in the second, *conjugate-symmetric* and *positive-definite*:

$$\langle x\phi + \chi \,|\, \psi \rangle_H = \overline{x}\langle \phi \,|\, \psi \rangle_H + \overline{x}\langle \chi \,|\, \psi \rangle_H \tag{2.23}$$

$$\langle \phi \,|\, x\chi + \psi \rangle_H = x\langle \phi \,|\, \chi \rangle_H + x\langle \phi \,|\, \psi \rangle_H \tag{2.24}$$

$$\langle \phi \,|\, \chi \rangle_H = \overline{\langle \chi \,|\, \psi \rangle_H} \tag{2.25}$$

$$\langle \phi \,|\, \phi \rangle_H \geq 0 \tag{2.26}$$

We use the inner product to define the *adjoint* to a linear map, in the following way.

Definition 2.15 (Adjoint of a bounded linear map)**.** For a bounded linear map $f \colon H \to J$, its *adjoint* $f^\dagger \colon J \to H$ is the unique linear map with the following property, for all $\phi \in H$ and $\psi \in J$:

$$\langle f\phi \,|\, \psi \rangle_J = \langle \phi \,|\, f^\dagger \psi \rangle_H. \tag{2.27}$$

This inspired the terminology for *adjoint functors* in category theory, although this is not itself an instance of a categorical adjunction.

We model this abstractly as an endofunctor on our category.

Definition 2.16. The *adjunction functor* $\dagger \colon \mathbf{Hilb} \to \mathbf{Hilb}^{\mathrm{op}}$ takes bounded linear maps to their adjoints.

This has the following property:

Lemma 2.17. *For all bounded linear maps,* $(f^\dagger)^\dagger = f$.

The dagger-functor

Definition 2.18. For a category $\mathbf{C}$, a *dagger-functor* is a functor $\dagger \colon \mathbf{C} \to \mathbf{C}^{\mathrm{op}}$ such that for all morphisms f, we have $(f^\dagger)^\dagger = f$.

Definition 2.19. A *dagger-category* is a category **C** equipped with a dagger-functor $\dagger\colon \mathbf{C} \to \mathbf{C}^{\mathrm{op}}$.

The category **Hilb** is the motivating example of a dagger-category, with the dagger-functor given by the adjunction functor. The category **Rel** can be made into a dagger-category, with the dagger-functor given by taking the relational converse: for $S \xrightarrow{R} T$, we define $T \xrightarrow{R^\dagger} S$ as the relation $t\, R^\dagger\, s$ iff $s\, R\, t$.

The category **Set** cannot be made into a dagger-category, since for finite sets A and B the hom-set $\mathrm{Hom}(A, B)$ contains $|B|^{|A|}$ elements, but the hom-set $\mathrm{Hom}(B, A)$ contains $|A|^{|B|}$ elements. A dagger-functor would give an isomorphism between these sets for all objects A and B, which is impossible.

Another important non-example is $\mathbf{Vect}_{\mathbb{C}}$: for an infinite-dimensional vector space V, the set $\mathrm{Hom}(\mathbb{C}, V)$ has a strictly smaller cardinality than the set $\mathrm{Hom}(V, \mathbb{C})$. The category $\mathbf{FVect}_{\mathbb{C}}$ *can* be given a dagger-functor: one way to do this is by assigning an inner product to every object, and then constructing the associated adjunction functor. However, it does not come with a *canonical* adjunction functor.

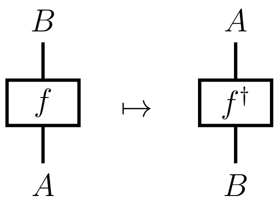
We can prove some basic lemmas about dagger-categories.

Lemma 2.20. *In a dagger-category with a zero object,* $0^\dagger_{A,B} = 0_{B,A}$.

Proof. Immediate from functoriality of the dagger-functor. □

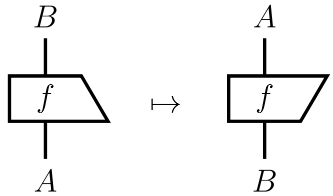
Graphical calculus

We depict the action of the dagger-functor by flipping the graphical representation about a horizontal axis:



(2.28)

To help us tell the difference, from now on we will draw our morphisms in a way that breaks their symmetry. Applying the dagger-functor then has the following representation:



(2.29)

We no longer write the † symbol within the label, as this is now indicated by the orientation of the wedge.

BRA-KET NOTATION.

Measurement

We have described how a state of an object $I \xrightarrow{f} A$ can be thought of as a *preparation* of A by the process f. Dually, a 'costate' $A \xrightarrow{f^\dagger} I$ models the *effect* of measuring A by the process $f^\dagger$. A dagger-functor gives us a correspondence between states and costates.

The Way of the Dagger

In a dagger-category, we often want to require that certain structures are compatible with the dagger-functor. This is something that we can deliberately seek to do for any particular categorical structure of interest. Often, this can give rise to interesting results. As a methodology, with tongue in cheek, we can call this the *way of the dagger*.

To help us apply this approach, we give special names to some basic properties for morphisms in a dagger-category. These are are generalizations of terms usually reserved for bounded linear maps between Hilbert spaces.

Definition 2.21. In a $\dagger$-category, given a morphism $A \xrightarrow{f} B$:

1. the *adjoint* of $A \xrightarrow{f} B$ is the morphism $B \xrightarrow{f^\dagger} A$;
2. $A \xrightarrow{f} B$ is *unitary* if $f \circ f^\dagger = \mathrm{id}_B$ and $f^\dagger \circ f = \mathrm{id}_A$;
3. $A \xrightarrow{f} B$ is an *isometry* if $f^\dagger \circ f = \mathrm{id}_A$;
4. $A \xrightarrow{f} A$ is *self-adjoint* if $A = B$ and $f = f^\dagger$;
5. $A \xrightarrow{f} A$ is *positive* when $f = g^\dagger \circ g$ for some $A \xrightarrow{g} B$.

We can combine the notion of dagger-category with the notion of monoidal category in a natural way.

Definition 2.22. A *monoidal dagger-category* is a monoidal category which is also a $\dagger$-category, such that $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ for all morphisms f and g, and such that the natural isomorphisms α, λ and ρ are unitary at every stage. A *braided monoidal dagger-category* is a monoidal dagger category equipped with a unitary braiding.

Both **Hilb** and **Rel** are monoidal dagger-categories.

Biproducts can also be compatible with a dagger-functor in an important way.

Definition 2.23. In a dagger-category with a zero object and superpositions, a *dagger-biproduct* of objects A and B is a biproduct $A \oplus B$ with injection and projection maps satisfying $i_A^\dagger = p_A$ and $i_B^\dagger = p_B$.

In **Rel**, every biproduct is a dagger-biproduct. In **Hilb**, dagger-biproducts are orthogonal direct sums.

Given two morphisms $A \xrightarrow{f,g} B$, it is interesting to ask for the largest subspace of A on which they agree. If it exists, this is given by the *equalizer* of f and g.

Definition 2.24. In a category, the equalizer of parallel morphisms $A \xrightarrow{f,g} B$ is an object E equipped with a morphism $E \xrightarrow{e} A$ such that $f \circ e = g \circ e$, and with the property that for every morphism $X \xrightarrow{x} A$ with $f \circ x = g \circ x$, then $x = e \circ \tilde{x}$ for some $X \xrightarrow{\tilde{x}} E$.

Definition 2.25. In a dagger-category, a *dagger-equalizer* of a parallel pair $A \xrightarrow{f,g} B$ is an equalizer $E \xrightarrow{e} A$ such that e is an isometry.

Intuitively, a dagger-equalizer represents the isometric embedding of the largest subspace of A on which the morphisms f and g agree. The dagger-category **Hilb** has all dagger-equalizers, whereas **Rel** does not.

The existence of dagger-equalizers gives rise to strong restrictions on the properties of a category, as indicated by the following lemma.

Theorem 2.26. *In a dagger-category with dagger-equalizers and a zero object, we have* $f \circ f^\dagger = 0 \Rightarrow f = 0$.

Proof. See Exercise Sheet 1. □

2.4 Exercises

Notes and further reading

The early uses of category theory were in algebraic topology. Therefore early developments mostly considered categories like **Vect**. The most general class of categories for which known methods worked are so-called Abelian categories, for which biproducts and what we called superposition rules are important axioms; see Freyd's book [29]. By Mitchell's embedding theorem, any Abelian category embeds into $\mathbf{Mod}_R$, the category of R-modules for some ring R, preserving all the important structure [53]. See also [13] for an overview.

Self-duality in the form of involutive endofunctors on categories has been considered as early as 1950 [49, 50]. A link between adjoint functors and adjoints in Hilbert spaces was made precise in 1974 [55]. The systematic exploitation of daggers in the way we have been using them started with Selinger in 2007 [67].

Using different terminology, Lemma 2.2 was proved in 1980 by Kelly and Laplaza [45]. The realization that endomorphisms of the tensor unit behave as scalars was made explicit by Abramsky and Coecke in 2004 [4, 2]. Heunen proved an analogue of Mitchell's embedding theorem for **Hilb** in 2009 [33]. Conditions under which the scalars embed into the complex numbers are due to Vicary [71].

Chapter 3

Duals for objects

3.1 Introduction

Definition

Definition 3.1 (Dual object)**.** An object L in a monoidal category is *left-dual* to an object R, and R is *right-dual* to L, written $L \dashv R$, when it is equipped with morphisms $\eta \colon I \to R \otimes L$ and $\varepsilon \colon L \otimes R \to I$ making the following two diagrams commute.

$$
\begin{array}{ccccc}
L & \xrightarrow{\rho_L^{-1}} & L \otimes I & \xrightarrow{\mathrm{id}_L \otimes \eta} & L \otimes (R \otimes L) \\
{\scriptstyle \mathrm{id}_L}\downarrow & & & & \downarrow{\scriptstyle \alpha_{L,R,L}^{-1}} \\
L & \xleftarrow[\lambda_L]{} & I \otimes L & \xleftarrow[\varepsilon \otimes \mathrm{id}_L]{} & (L \otimes R) \otimes L
\end{array}
\tag{3.1}
$$

$$
\begin{array}{ccccc}
R & \xrightarrow{\lambda_R^{-1}} & I \otimes R & \xrightarrow{\eta \otimes \mathrm{id}_R} & (R \otimes L) \otimes R \\
{\scriptstyle \mathrm{id}_R}\downarrow & & & & \downarrow{\scriptstyle \alpha_{R,L,R}} \\
R & \xleftarrow[\rho_R]{} & R \otimes I & \xleftarrow[\mathrm{id}_R \otimes \varepsilon]{} & R \otimes (L \otimes R)
\end{array}
\tag{3.2}
$$

The maps η and ε are called the *unit* and *counit*, respectively. When L is both left and right dual to R, we simply call L a *dual* of R.

Graphical representation

We draw an object L as a wire with an upward-pointing arrow, and its right dual R as a wire with a downward-pointing arrow.

(3.3)

We then draw $\eta : I \to R \otimes L$ and $\epsilon : L \otimes R \to I$ as bent wires:

(3.4)

The duality equations then take the following graphical form:

(3.5)

Particularly when drawn graphically, these are also called the *snake equations*, for obvious reasons.

These equations allow us to represent *oriented 1d manifolds* in our monoidal category. This is one of the simplest examples of the deep connections between topology and monoidal category theory.

Examples

Every object in **FHilb** is dual to itself. To construct the unit and counit maps for a finite-dimensional Hilbert space H, we first pick a basis $|i\rangle$ for H. We then construct $\mathbb{C} \xrightarrow{\eta} H \otimes H$ and $H \otimes H \xrightarrow{\epsilon} \mathbb{C}$ as follows:

$$\eta : 1 \mapsto \sum_i |i\rangle \otimes |i\rangle \tag{3.6}$$

$$\epsilon : |i\rangle \otimes |j\rangle \mapsto \delta_{i,j}|i\rangle \tag{3.7}$$

We will see below that duals are unique up to isomorphism, so it doesn't matter which basis we pick. However, not all bases will give rise to the same maps η, ϵ.

Question. When do two orthonormal bases for a finite-dimensional Hilbert space give rise to the same duality maps?

For any finite-dimensional Hilbert space, its dual in the sense of linear algebra also gives rise to a dual object. Since we do not make us of the inner products, this construction also makes sense for finite-dimensional vector spaces. However, it will not work for an infinite-dimensional vector space or Hilbert space, as the resulting infinite sum is not well-defined.

The category of finite-dimensional representations of a compact Lie group is a symmetric monoidal category, and also has duals. This example is so important, it leads to the following terminology:

Definition 3.2. A *compact category* is a symmetric monoidal category for which every object has a dual.

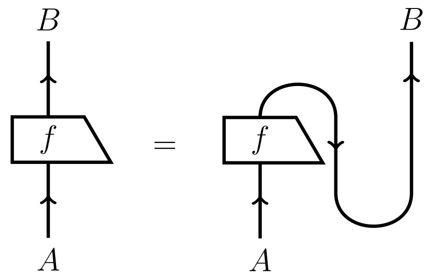
In particle physics, particles correspond to particular representations of a Lie group, which for the standard model is compact (ignoring spacetime symmetries.) If a particle corresponds to an object P, then its *antiparticle* corresponds to the dual object P^*. The unit and counit maps then correspond physically to particle-antiparticle creation and annihilation. In this context, the graphical calculus has a different name — *Feynman diagrams.*

The category **Rel** has duals for all objects, even for sets with infinite cardinality. Again, the objects are self-dual. For a set S, the duality maps $1 \xrightarrow{\eta} S \otimes S$ and $S \otimes S \xrightarrow{\epsilon} 1$ are defined in the following way:

$$\{\bullet\} \sim_\eta (s, s) \text{ for all } s \in S \tag{3.8}$$

$$(s, s) \sim_\epsilon \{\bullet\} \text{ for all } s \in S \tag{3.9}$$

The category **Set** does *not* have duals for objects. The best way to see this is to note that, in a category with duals, for all morphisms $A \xrightarrow{f} B$ we have the following equality:



(3.10)

Definition 3.3. In a monoidal category with dualities $A \dashv A^*$ and $B \dashv B^*$, given a morphism $A \xrightarrow{f} B$, we define its *name* $I \xrightarrow{\ulcorner f \urcorner} A^* \otimes B$ and *coname* $A \otimes B^* \xrightarrow{\llcorner f \lrcorner} I$ as the following morphisms:

(3.11)

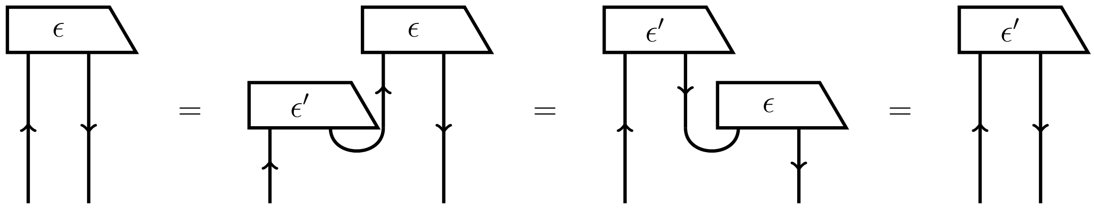
Morphisms can be recovered from their names or conames. However, in **Set**, the monoidal unit object 1 is terminal, and so all conames must be equal. If **Set** had duals, this would imply that all functions are equal, which is not true.

Basic properties

We first show that units determine counits uniquely, if they exist. Similarly, counits determine units uniquely.

Lemma 3.4. *In a monoidal category, if* (L, R, η, ϵ) *and* (L, R, η, ϵ') *both exhibit a duality, then* $\epsilon = \epsilon'$. *Similarly, if* (L, R, η, ϵ) *and* (L, R, η', ϵ) *both exhibit a duality, then* $\eta = \eta'$.

Proof. For the first case, we use the following graphical argument.



(3.12)

The second case is similar. □

Lemma 3.5. *In a monoidal category, duals have the following properties:*

1. $L \dashv R,\ L \dashv R' \Leftrightarrow R \simeq R'$, *and* $L \dashv R,\ L' \dashv R \Leftrightarrow L \simeq L'$;
2. $L \dashv R,\ L' \dashv R' \Rightarrow L \otimes R \dashv R' \otimes L'$;
3. $I \dashv I$;
4. $0 \dashv 0$ *if a zero object* 0 *exists;*
5. $L \dashv R \Rightarrow L \otimes \bot \simeq \bot \simeq \bot \otimes R$ *if an initial object* $\bot$ *exists.*
6. $L \dashv R \Rightarrow R \otimes \top \simeq \top \simeq \top \otimes L$ *if a terminal object* $\top$ *exists.*
7. $L \dashv R \Rightarrow L \otimes 0 \simeq R \otimes 0 \simeq 0 \simeq 0 \otimes L \simeq 0 \otimes R$ *if a zero object* 0 *exists.*

Proof. See Exercise Sheet 2. □

If the monoidal category has a symmetry, then a duality $L \dashv R$ gives rise to a duality $R \dashv L$, as we now investigate.

Lemma 3.6. *In a symmetric monoidal category,* $L \dashv R \Rightarrow R \dashv L$.

Proof. Suppose we have (L, R, η, ϵ) witnessing the duality $L \dashv R$. Then we construct a duality (R, L, η', ϵ') as follows, where we use the ordinary graphical calculus for the duality (L, R, η, ϵ):

(3.13)

$I \xrightarrow{\eta'} L \otimes R$ $\qquad$ $R \otimes L \xrightarrow{\epsilon'} I$

Writing out the snake equations for these new duality morphisms, we see that they are satisfied by using properties of the swap map and the snake equations for the original duality morphisms η and ϵ. □

3.2 Interaction with linear structure

We noted in Chapter 2 that linear structure does not necessarily interact well with monoidal structure. However, in the presence of duals for objects, this sort of good interaction is guaranteed. If you think about it, this is quite remarkable, since duals objects are defined independently from linear structures such as superposition rules, biproducts and zero objects. This is a reflection of the fact that, for some fundamental reason which is not yet completely understood, topological structures and linear structures are deeply related.

Lemma 3.7. *In a monoidal category, if objects A or B have a left or right dual, then for any morphism $A \xrightarrow{f} B$ and any objects C and D,*

$$\begin{aligned} f \otimes 0_{C,D} &= 0_{A\otimes C, B\otimes D}, \\ 0_{C,D} \otimes f &= 0_{C\otimes A, D\otimes B}. \end{aligned}$$

Proof. Use Lemma 3.5, part 7. □

Lemma 3.8. *In a monoidal category with biproducts, if A has a left or right dual, then for all objects B and C the following morphisms are inverse to each other:*

$$A \otimes (B \oplus C) \underset{\left(\mathrm{id}_A \otimes \begin{pmatrix} \mathrm{id}_B \\ 0_{B,C} \end{pmatrix} \;\; \mathrm{id}_A \otimes \begin{pmatrix} 0_{C,B} \\ \mathrm{id}_C \end{pmatrix}\right)}{\overset{\begin{pmatrix} \mathrm{id}_A \otimes \begin{pmatrix} \mathrm{id}_B & 0_{C,B} \end{pmatrix} \\ \mathrm{id}_A \otimes \begin{pmatrix} 0_{B,C} & \mathrm{id}_C \end{pmatrix} \end{pmatrix}}{\rightleftarrows}} (A \otimes B) \oplus (A \otimes C) \tag{3.14}$$

Proof. See Exercise Sheet 2. □

Lemma 3.9. *In a monoidal category with biproducts and a zero object, if objects A or B have a left or right dual, then for all objects C and D and all morphisms $A \xrightarrow{f} B$ and $C \xrightarrow{g,h} D$,*

$$(f \otimes g) + (f \otimes h) = f \otimes (g + h), \tag{3.15}$$

$$(g \otimes f) + (h \otimes f) = (g + h) \otimes f. \tag{3.16}$$

Proof. Composing the morphisms of Lemma 3.8 for $B = C$ to obtain an endomorphism of $A \otimes (C \oplus C)$, we can apply the interchange law to obtain

$$A \otimes (C \oplus C) \xrightarrow{\mathrm{id}_A \otimes \begin{pmatrix} \mathrm{id}_C & 0_{C,C} \\ 0_{C,C} & 0_{C,C} \end{pmatrix} + \mathrm{id}_A \otimes \begin{pmatrix} 0_{C,C} & 0_{C,C} \\ 0_{C,C} & \mathrm{id}_C \end{pmatrix}} A \otimes (C \oplus C). \tag{3.17}$$

By that lemma, this composite is equal to the identity.

By further applications of the matrix calculus and the interchange law, we see that $f \otimes (g + h)$ can be written in the following way:

$$f \otimes (g+h) = \big(\mathrm{id}_B \otimes (\mathrm{id}_D \;\; \mathrm{id}_D)\big) \circ \left(f \otimes \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}\right) \circ \left(\mathrm{id}_A \otimes \begin{pmatrix} \mathrm{id}_C \\ \mathrm{id}_C \end{pmatrix}\right) \tag{3.18}$$

Inserting the identity in the form of morphism 3.17, and using the interchange law and distributivity of composition over superposition (2.7), we obtain the following.

$$\begin{aligned} f \otimes \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} &= \left(f \otimes \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}\right) \circ \left(\mathrm{id}_A \otimes \begin{pmatrix} \mathrm{id}_C & 0_{C,C} \\ 0_{C,C} & 0_{C,C} \end{pmatrix} + \mathrm{id}_A \otimes \begin{pmatrix} 0_{C,C} & 0_{C,C} \\ 0_{C,C} & \mathrm{id}_C \end{pmatrix}\right) &(3.19) \\ &= \left(f \otimes \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix}\right) + \left(f \otimes \begin{pmatrix} 0 & 0 \\ 0 & h \end{pmatrix}\right) &(3.20) \end{aligned}$$

Inserting this into equation 3.18, this gives

$$\begin{aligned} f \otimes (g+h) &= \big(\mathrm{id}_B \otimes (\mathrm{id}_D \;\; \mathrm{id}_D)\big) \circ \left(f \otimes \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} + f \otimes \begin{pmatrix} 0 & 0 \\ 0 & h \end{pmatrix}\right) \circ \left(\mathrm{id}_A \otimes \begin{pmatrix} \mathrm{id}_C \\ \mathrm{id}_C \end{pmatrix}\right) \\ &= f \otimes \left(\big(\mathrm{id}_D \;\; \mathrm{id}_D\big) \circ \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} \mathrm{id}_C \\ \mathrm{id}_C \end{pmatrix}\right) \\ &\qquad + f \otimes \left(\big(\mathrm{id}_D \;\; \mathrm{id}_D\big) \circ \begin{pmatrix} 0 & 0 \\ 0 & h \end{pmatrix} \circ \begin{pmatrix} \mathrm{id}_C \\ \mathrm{id}_C \end{pmatrix}\right) \\ &= (f \otimes g) + (f \otimes h), \end{aligned}$$

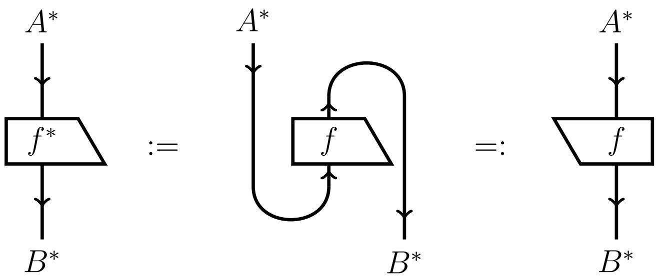
as required. The equation $(g + h) \otimes f = (g \otimes f) + (h \otimes f)$ can be proved in a similar way. □

Lemma 3.10. *In a monoidal category, $L \dashv R$, $L' \dashv R' \Rightarrow L \oplus L' \dashv R \oplus R'$ if the necessary biproducts exist.*

3.3 The duality functor

The dual of a morphism

Definition 3.11. For a morphism $A \xrightarrow{f} B$ and chosen dualities $A \dashv A^*$, $B \dashv B^*$, the *right dual* $B^* \xrightarrow{f^*} A^*$ is defined in the following way:

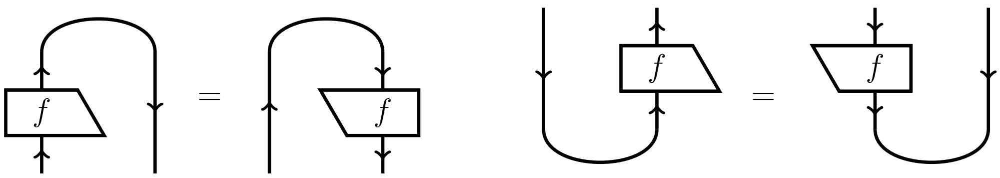


(3.21)

We represent this graphically by rotating the morphism box representing f, as shown in the third image here.

The dual can 'slide' along the cups and the caps of representing our dualities.

Lemma 3.12. *In a symmetric monoidal category, the following equations hold:*



Proof. Direct from writing out the definitions of all the components involved. □

Right duality functor

Definition 3.13. For a monoidal category **C** in which every object X has a chosen right dual X^*, we define the *right duality functor* $(-)^* : \mathbf{C} \to \mathbf{C}^{\mathrm{op}}$ as $(X)^* := X^*$ on objects, and $(f)^* := f^*$ on morphisms.

Lemma 3.14. *The right duality functor satisfies the axioms for a functor.*

Proof. See Exercise Sheet 2. □

3.4 Dagger-compact categories

If our monoidal category with duality $L \dashv R$ is also a dagger category, we introduce the following graphical representation for the adjoints of our unit and counit:

(3.22)

These adjoints provide witnesses for a duality $R \dashv L$, which gives us the following lemma.

Lemma 3.15. *In a monoidal dagger-category,* $L \dashv R \Rightarrow R \dashv L$.

Proof. Direct from the requirement the axiom $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ of a monoidal dagger-category. □

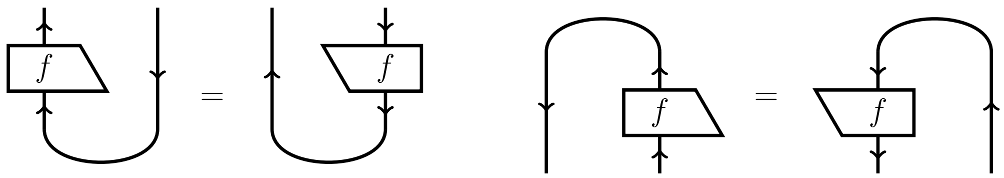
Definition 3.16. In a symmetric monoidal dagger-category, a duality $L \dashv R$ is a *dagger-duality* if $\epsilon^\dagger = \sigma_{R,L} \circ \eta$, or equivalently if $\epsilon \circ \sigma_{R,L} = \eta^\dagger$.

Graphically, these conditions have the following representation:


(3.23)

In this case we can extend the graphical rules developed in Lemma 3.12.

Lemma 3.17. *For a dagger-duality* $L \dashv R$*, the following equations hold:*



Definition 3.18. A *dagger-compact category* is a symmetric monoidal dagger-category for which every object is equipped with a dagger-dual.

When we construct the right duality functor for a dagger-compact category, we ensure that the chosen right duals for each object are dagger-duals.

Lemma 3.19. *On a dagger-compact category, the right duality functor and the dagger-functor commute.*

Proof. See Exercise Sheet 2. □

This allows us to define the *conjugation functor*.

Definition 3.20. On a dagger-compact category, the *conjugation functor* $(-)_*$ is defined as the composite of the dagger-functor and the right duality functor.

Since these functors commute, it does not matter in which order we compose them. Also, since both the dagger-functor and the right duality functor are contravariant, reversing the direction of morphisms, the conjugation functor will be covariant, mapping $A \xrightarrow{f} B$ to $A^* \xrightarrow{f_*} B^*$.

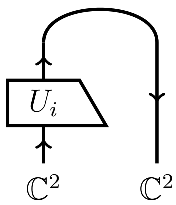
Examples

The category **FHilb** can be given the structure of a dagger compact category. In $\mathbf{FHilb_{ss}}$ this can be done in a canonical way, since every object has a canonical basis, and hence a canonical self-duality. The duality functor then computes transposition of matrices. Since the adjunction functor on this category computes the conjugate transpose matrix, it is clear that this will commute with the duality functor, as we expect from Lemma 3.19. The conjugation functor then computes the conjugate of a matrix.

In **Rel**, every object also has a canonical right dual. The duality functor gives transposition of relations.

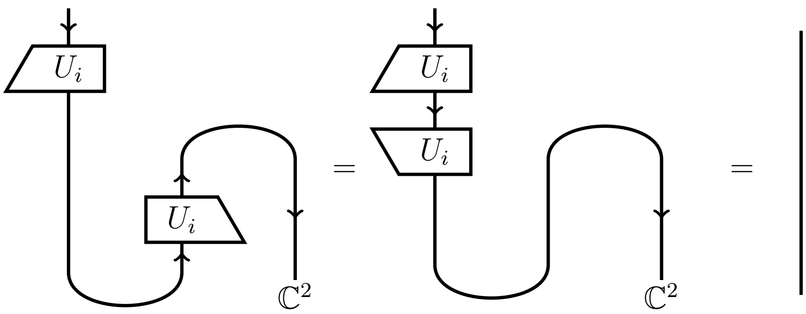
3.5 Quantum teleportation

We can find a basis of costates of $\mathbb{C}^2 \otimes \mathbb{C}^2$ of the form $\llcorner U_i \lrcorner$, where the $U_i : \mathbb{C}^2 \to \mathbb{C}^2$ are unitaries:



(3.24)

We then prepare an initial state composed of the unit of our compact structure, along with our initial qubit. Measuring in the basis $\llcorner U_i \lrcorner$, we are guaranteed for the effect to be the following, for some i:



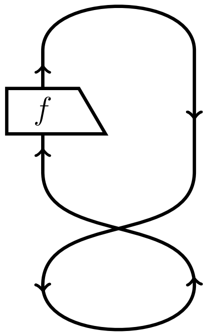
(3.25)

3.6 Traces and dimensions

Introduction and basic properties

In a symmetric monoidal category, we can use the existence of duals to define *traces* of morphisms.

Definition 3.21. In a symmetric monoidal category, for an object L with a right dual and a morphism $L \xrightarrow{f} L$, its *trace* $\mathrm{Tr}_L(f)$ is defined as the following scalar:



(3.26)

Definition 3.22. In a symmetric monoidal category, for an object L with a right dual, we define its *dimension* as $\dim(A) := \mathrm{Tr}_L(\mathrm{id}_L)$.

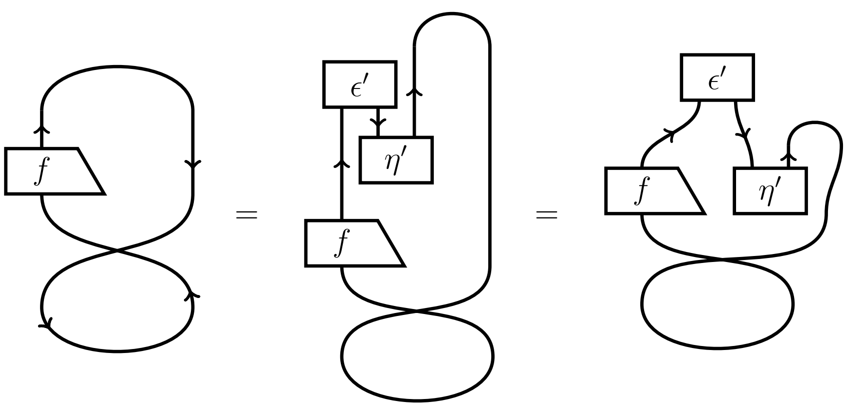
For traces and dimensions to be useful notions, we need the following lemma.

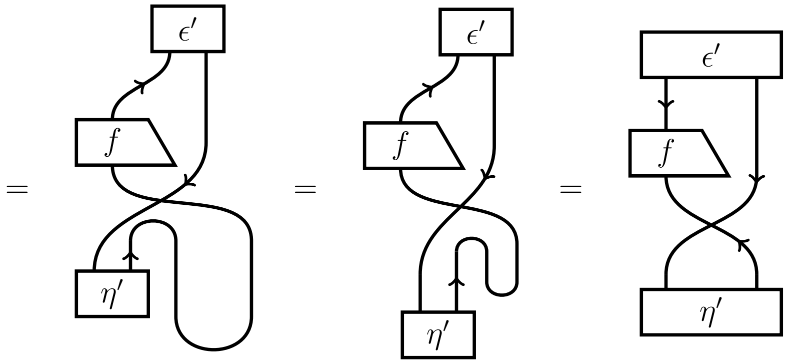
Lemma 3.23. *The trace of a morphism is well-defined.*

Proof. We must show that the same value is obtained whichever choice we make of right dual and unit and counit maps. Suppose then that we have dualities (L, R, η, ϵ) and $(L, R', \eta', \epsilon')$, where we draw the first duality using the conventions of equations (3.3-3.4). We draw η' and ϵ' as follows:

(3.27)

We then make the following argument:





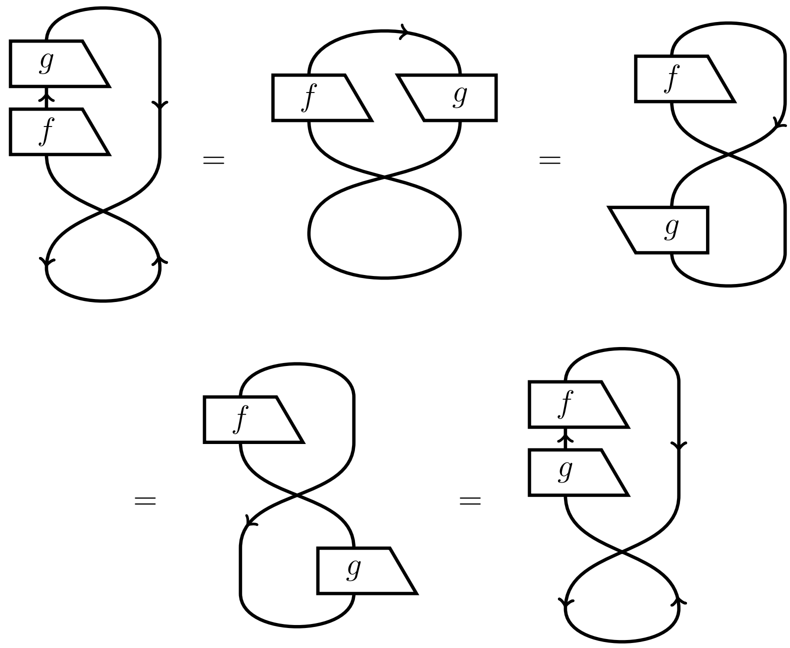
(3.28)

The nontrivial step is the third equality, where we make use of naturality of the symmetry. □

This abstract trace operation has the cyclic property which we are familiar with from ordinary linear algebra.

Lemma 3.24. *For all morphisms* $A \xrightarrow{f} B$ *and* $B \xrightarrow{g} A$*, when* A *and* B *have a right dual, then* $\mathrm{Tr}_A(g \circ f) = \mathrm{Tr}_B(f \circ g)$*.*

Proof.



We make use of naturality of the symmetry twice, for the second and fourth equalities. The second and fourth equalities use properties of dual morphisms. □

Further properties

Lemma 3.25. *Traces have the following properties in a symmetric monoidal category:*

1. $\mathrm{Tr}_{A\otimes B}(f \otimes g) = \mathrm{Tr}_A(f) \circ \mathrm{Tr}_B(g)$ *for morphisms* $A \xrightarrow{f} A$ *and* $B \xrightarrow{g} B$*;*

2. $\mathrm{Tr}_A(f+g) = \mathrm{Tr}_A(f) + \mathrm{Tr}_A(g)$ *for* $A \xrightarrow{f,g} B$*;*

3. $\mathrm{Tr}_{A\oplus B}\begin{pmatrix} f & g \\ h & j \end{pmatrix} = \mathrm{Tr}_A(f) + \mathrm{Tr}_B(j)$ *when the biproduct* $A \oplus B$ *exists, for morphisms* $A \xrightarrow{f} A$*,* $B \xrightarrow{g} A$*,* $A \xrightarrow{h} B$ *and* $B \xrightarrow{j} B$*;*

4. $\mathrm{Tr}_I(s) = s$ *for a scalar* $I \xrightarrow{s} I$*;*

5. $\mathrm{Tr}_A(0_{A,A}) = 0_{I,I}$ *for a zero morphism* $0_{A,A}$*;*

6. $\big(\mathrm{Tr}_A(f)\big)^\dagger = \mathrm{Tr}_A(f^\dagger)$ *for a morphism* $A \xrightarrow{f} A$ *in a strongly compact closed category.*

Proof. See Exercise Sheet 2. □

This immediately gives us some properties of dimensions of objects:

1. $\dim(A \otimes B) = \dim(A) \circ \dim(B)$;

2. $\dim(A \oplus B) = \dim(A) + \dim(B)$ when the biproduct $A \oplus B$ exists;

3. $\dim(I) = \mathrm{id}_I$;

4. $\dim(0) = 0_{I,I}$ for a zero object 0.

Notes and further reading

Compact categories were first introduced by Kelly in 1972 as a class of examples in the context of the coherence problem [43]. They were subsequently studied first from the perspective of categorical algebra [24, 45], and later in relation to linear logic [64, 9].

The terminology "compact category" is historically explained as follows. If G is a Lie group, then its finite-dimensional representations form a compact category. The group G can be reconstructed from the category when it is compact [39]. Thus the name "compact" transferred from the group to categories resembling those of finite-dimensional representations. Compact categories and closely related nonsymmetric variants are known under an abundance of different names in the literature: rigid, pivotal, autonomous, sovereign, spherical, ribbon, tortile, balanced, and category with conjugates [68].

Abstract traces in monoidal categories were introduced by Joyal, Street and Verity in 1996 [41]. Definition 3.21 is one instance. In fact, Hasegawa proved in 2008 that abstract traces in a compact category are unique [32]. The link between abstract traces and traces of matrices was made explicit by Abramsky and Coecke in 2005 [5].

The use of dagger compact categories in foundations of quantum mechanics was initiated in 2004 by Abramsky and Coecke [4]. This was the article that initiated the study of categorical quantum mechanics.

The graphical calculus for dagger compact categories was worked out in detail by Selinger, who proved its soundness [68]. In 2008 [66], he also proved that an equation holds in the graphical calculus of dagger compact categories if and only if it holds in every possible instantiation in **FHilb**.

The quantum teleportation protocol was discovered in 1993 by Bennett, Brassard, Crépeau, Jozsa, Peres, and Wootters [11], and has been performed experimentally many times since, over distances as large as 16 kilometers.

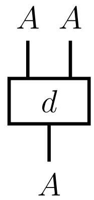
Chapter 4

Classical structures

As we will see later, compact categories model quantum mechanics, in the sense that a no-cloning theorem holds. This means the tensor product cannot be a (categorical) product. So, if we want to consider classical information within our 'quantum' categories, we can only consider copying and deleting operations individually, per object. This chapter does precisely that.

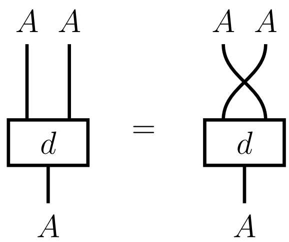
4.1 Monoids and comonoids

Let's start by making the notions of copying and deleting more precise in our setting of symmetric monoidal categories. Clearly, copying should be an operation of type $A \xrightarrow{d} A \otimes A$. We draw it in the following way:



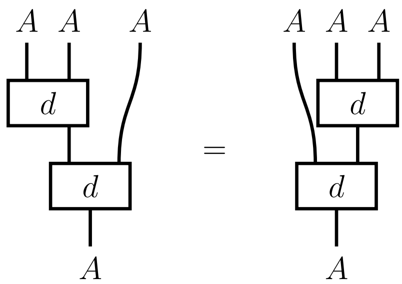
(4.1)

What does it mean that d 'copies' information? First, it shouldn't matter if we switch both output copies, corresponding to the requirement that $\sigma_{A,A} \circ d = d$.



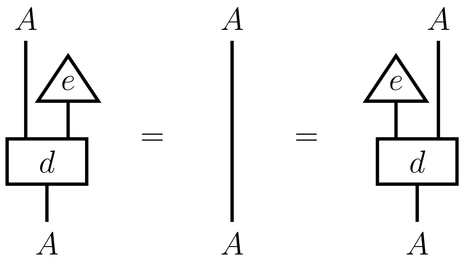
(4.2)

Secondly, if we make a third copy, if shouldn't matter if we make it from the first or the second copy. We can formulate this abstractly as $\alpha_{A,A,A} \circ (d \otimes \mathrm{id}_A) \circ d = (\mathrm{id}_A \otimes d) \circ d$, with the following graphical representation:



(4.3)

Finally, remember that we think of I as the empty system. So deletion should be an operation of type $A \xrightarrow{e} I$. With this in hand, we can formulate what it means that both output copies should equal the input: that $\rho_A \circ (\mathrm{id}_A \otimes e) \circ d = \mathrm{id}_A$ and $\lambda_A \circ (e \otimes \mathrm{id}_A) \circ d$.



(4.4)

These three properties together constitute the structure of a *comonoid* on A.

Definition 4.1 (Comonoid)**.** A *comonoid* in a monoidal category is a triple (A, d, e) of an object A and morphisms $A \xrightarrow{d} A \otimes A$ and $A \xrightarrow{e} I$ satisfying equations (4.3) and (4.4). If the monoidal category is symmetric and equation (4.2) holds, the comonoid is called *cocommutative*.

The map d is called the *comultiplication*, and e is called the *counit*. Properties (4.3) and (4.6) are *coassociativity* and *counitality*.

Some examples of comonoids:

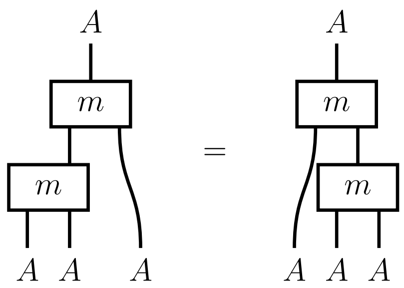
- In **Set**, the tensor product is in fact a Cartesian product, so any object A carries a unique commutative comonoid structure with comultiplication $A \xrightarrow{d} A \times A$ given by $d(a) = (a, a)$, and the unique function $A \to 1$ as counit.
- In **Rel**, any group G forms a comonoid with comultiplication $g \sim (h, h^{-1}g)$ for all $g, h \in G$, and counit $1 \sim \bullet$. The comonoid is cocommutative when the group is abelian.
- In **FHilb**, any choice of basis $|i\rangle$ for a Hilbert space H provides it with comonoid structure, with comultiplication $A \xrightarrow{d} A \otimes A$ defined by $|i\rangle \mapsto |i\rangle \otimes |i\rangle$ and counit $A \xrightarrow{e} I$ defined by $|i\rangle \mapsto 1$.

The comonoids in a monoidal category can be made into a category themselves. The morphisms in this category are morphisms in the original category satisfying the *comonoid homomorphism* property.

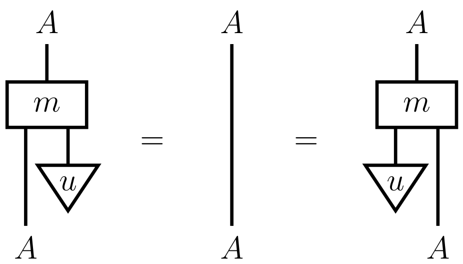
Definition 4.2 (Comonoid homomorphism)**.** A *comonoid homomorphism* from a monoid (A, d, e) to a monoid (A', d', e') is a morphism $A \xrightarrow{f} A'$ such that $(f \otimes f) \circ d = d' \circ f$ and $e' \circ f = e$.

You might be growing tired of "co" before every other word. Indeed, dualizing everything gives the more well-known notion of a *monoid*. In fact, this notion is so important, that one can almost say the entire reason for defining monoidal categories is that one can define monoids in them.

Definition 4.3 (Monoid)**.** A *monoid* in a monoidal category is a triple (A, m, u) of an object A, a morphism $A \otimes A \xrightarrow{m} A$, and a point $I \xrightarrow{u} A$, satisfying the following two equations called *associativity* and *unitality*:

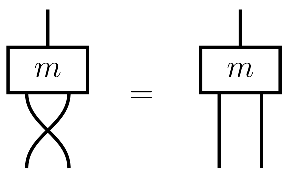


(4.5)



(4.6)

In a symmetric monoidal category, a monoid is called *commutative* when the following equation holds.



(4.7)

There are many examples of monoids:

- The tensor unit I in any monoidal category can be equipped with the structure of a monoid, with $d = \rho_I$ $(= \lambda_I)$ and $e = \mathrm{id}_I$.

- A monoid in **Set** gives the ordinary mathematical notion of a monoid. Any group is an example.
- A monoid **Hilb** is called an *algebra*. The multiplication is a linear function $A \otimes A \xrightarrow{m} A$, corresponding to a bilinear function $A \times A \to A$. Hence an algebra is a set where we can not only add vectors and multiply vectors with scalars, but also multiply vectors with each other in a bilinear way. For example, $\mathbb{C}^n$ forms an algebra under pointwise multiplication; the unit is the point $(1, 1, \ldots, 1)$.

The monoids in a monoidal category can be made into a category themselves. The morphisms in this category are morphisms in the original category satisfying the *monoid homomorphism* property.

Definition 4.4 (Monoid homomorphism). A *monoid homomorphism* from a monoid (A, m, u) to a monoid (A', m', u') is a morphism $A \xrightarrow{f} A'$ such that $f \circ m = m' \circ (f \otimes f)$ and $u' = f \circ u$.

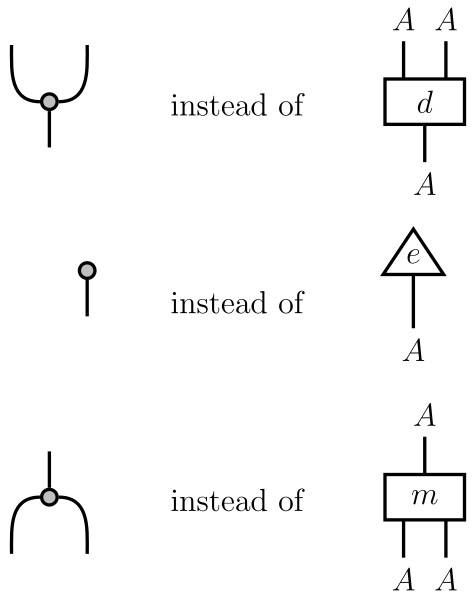
In a monoidal dagger-category, there is a duality between monoids and comonoids.

Lemma 4.5. *If* (A, d, e) *is a comonoid in a monoidal dagger-category, then* $(A, d^\dagger, e^\dagger)$ *is a monoid.*

Proof. Equations (4.5) and (4.6) are just (4.3) and (4.4) vertically reflected. □

As we saw above, any group G gives a comonoid in **Rel** with $d = \{(g, (h, h^{-1}g) \mid g, h \in G\}$. The dagger-functor on **Rel** constructs converse relations, and applying this turns our example into a monoid in **Rel** with multiplication $G \times G \xrightarrow{m} G$ given by $(g, h) \sim gh$ and unit $1 \xrightarrow{u} G$ given by $\bullet \sim 1$.

For the rest of this section, we will simplify our graphical notation for monoids and comonoids in he following way:



 instead of 

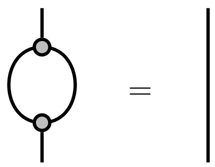
However, although our notations for d and m are related by flipping about a horizontal axis, as are our notations for e and u, these will not necessarily be related to each other by a dagger-functor, so some care must be taken reading this notation.

4.2 Frobenius algebras

There are various ways in which a comonoid and a monoid on the same object can interact. In this chapter we will study one such way, namely *Frobenius algebras*. This turns out to be the right notion to capture classical information. We show how this enables us to model protocols such as state transfer and teleportation.

There are two ways in which a multiplication and a comultiplication can be composed. Let's start with the easiest one: comultiplication followed multiplication.

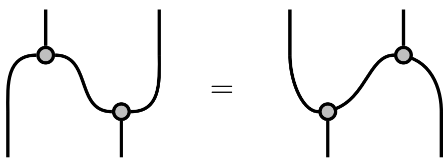
Definition 4.6. A pair of a comonoid (A, d, e) and a monoid (A, m, u) in a monoidal category is called *special* when d is a retraction of m: $m \circ d = \mathrm{id}_A$.



(4.8)

Now we move on to the other order of composition: first multiplication and then comultiplication. One way to do this that we will consider is the Frobenius law.

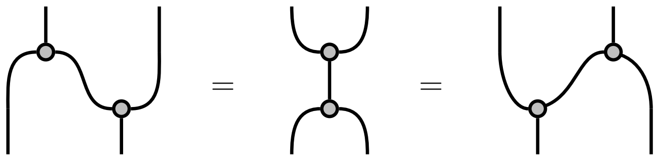
Definition 4.7 (Frobenius algebra via diagrams)**.** In a monoidal category, a *Frobenius algebra* is a comonoid (A, d, e) and a monoid (A, m, u) satisfying the following equation, called the *Frobenius law*:



(4.9)

In a monoidal dagger-category, when $m = d^\dagger$ and $u = e^\dagger$, we call this a *dagger-Frobenius algebra*.

Lemma 4.8. *For a Frobenius algebra, the following equalities hold:*



(4.10)

Proof. See Exercise Sheet 3. □

Examples 4.9. For some examples of Frobenius algebras:

- Let A be an object in the monoidal category **FHilb**. Any choice of orthogonal basis $\{|i\rangle\}_{i=1,\ldots,n}$ for A endows it with the structure of a Frobenius algebra as follows. Define $A \xrightarrow{d} A \otimes A$ by linearly extending $d(|i\rangle)) = |i\rangle \otimes |i\rangle$, define $A \xrightarrow{e} \mathbb{C}$ by linearly extending $e(|i\rangle) = 1$. Then $e^\dagger(z) = z \sum_{i=1}^{n} |i\rangle$, $d^\dagger(|i\rangle \otimes |i\rangle) = 1$ and $d^\dagger(|i\rangle \otimes |j\rangle) = 0$ when $i \neq j$. This algebra is special when the basis is orthonormal instead of just orthogonal.

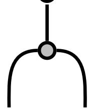
- Any finite group G induces a Frobenius algebra in **FHilb**. Let $A = \mathbb{C}[G]$ be the Hilbert space of linear combinations of elements of G with its standard inner product. In other words, A has G as an orthonormal basis. Define $A \otimes A \xrightarrow{d^\dagger} A$ by linearly extending $d^\dagger(g,h) = gh$, and define $\mathbb{C} \xrightarrow{e^\dagger} A$ by $e^\dagger(z) = z \cdot 1_G$ — this gives an algebra structure called the *group algebra*. Then define $d(g) = \sum_{h \in G} gh^{-1} \otimes h = \sum_{h \in G} h \otimes h^{-1}g$.

- Any group G also induces a Frobenius algebra in **Rel**. Define $d^\dagger = \{((g,h), gh) \mid g,h \in G\} \colon G \times G \to G$ and $e^\dagger = \{(*, 1_G)\} \colon 1 \to G$.

  More generally, recall that a *groupoid* is a category whose every morphism is an isomorphism. Any groupoid $\mathbf{G}$ induces a Frobenius algebra in **Rel** on the set G of all morphisms in $\mathbf{G}$. Define $d^\dagger = \{((g,f), g \circ f) \mid \mathrm{dom}(g) = \mathrm{cod}(f)\}$, $e^\dagger = \{(*, \mathrm{id}_x) \mid x \in \mathrm{Ob}(\mathbf{G})\}$.

Frobenius algebras can also be defined in a different way, closer to the way in which they were originally conceived.

Definition 4.10 (Frobenius algebra via form)**.** A *Frobenius algebra* is a monoid (A, m, u) equipped with a *form* $e : A \to I$, such that the composite



(4.11)

forms part of a self-duality $A \dashv A$. Such a form is sometimes called *non-degenerate*.

Lemma 4.11. *Definitions 4.7 and 4.10 are equivalent.*

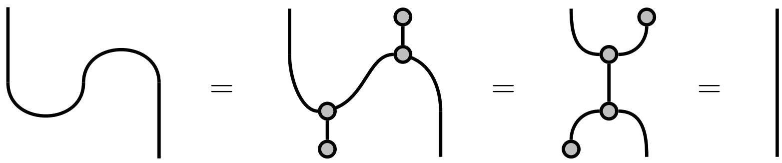
Proof. See Exercise Sheet 3. □

Carrying Frobenius algebra structure is essentially a finite-dimensional property. As the following theorem shows, Frobenius algebras always have dual objects.

Theorem 4.12 (Frobenius algebras have duals)**.** *If an object (A, d, e, m, u) is a Frobenius algebra in a monoidal category, then $A = A^*$ is self-dual (in the sense of Definition 3.1) by $\eta_A = d \circ u$ and $\varepsilon_A = e \circ m$.*

(4.12)

Proof. We have to verify the snake equations (3.5).

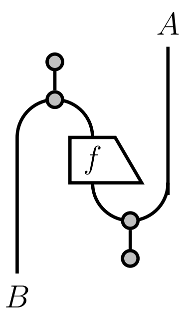


The first equality is the definition (4.12), the second equality is the Frobenius law (4.9), and the third equality follows from unitality (4.6) and counitality (4.4). Similarly, the other snake equation holds. □

Definition 4.13. A *homomorphism of Frobenius algebras* is a morphism that is simultaneously a monoid homomorphism and a comonoid homomorphism.

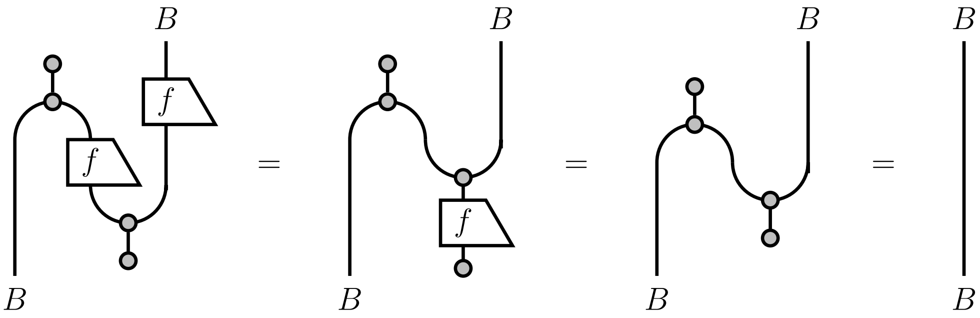
Lemma 4.14. *In a monoidal category, a homomorphism of Frobenius algebras is invertible.*

Proof. Given Frobenius algebras on objects A and B and a Frobenius algebra homomorphism $A \xrightarrow{f} B$, we construct an inverse to f as follows:



(4.13)

We can demonstrate that the composite of this with f gives the identity in one direction:



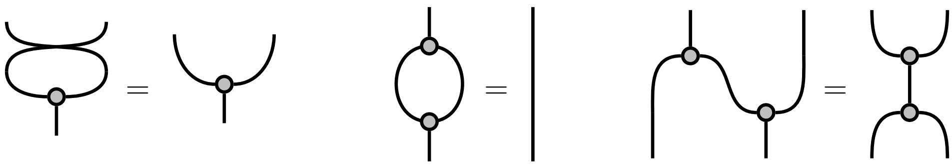
(4.14)

Here the first equality uses the comonoid homomorphism property, the second equality uses the monoid homomorphism property, and the third equality follows from Theorem 4.12. The other composite is also the identity by a similar argument. □

We will see later that in a monoidal category with duals, the no-cloning theorem prevents us from choose copying and deleting maps uniformly. But we can use this contrapositively: instead of stating something negative about *quantum* objects ("you cannot copy them uniformly"), we state something positive about *classical* objects ("you can equip them with a non-uniform copying operation").

Definition 4.15 (Classical structure)**.** A *classical structure* in a dagger–symmetric monoidal category is a commutative special dagger-Frobenius algebra.

Because of cocommutativity (4.2), we only need to require one half of counitality (4.4) and one half of the Frobenius law (4.9). In fact, we need not have mentioned (co)associativity, because it is implied by speciality (4.8) and the Frobenius law (4.9). Also, in compact categories, the Frobenius law (4.9) implies unitality (4.4). Hence to check that (A, d, e) is a classical structure, we only need to verify the following properties:

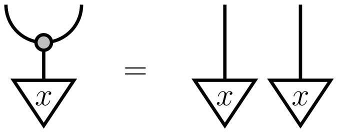


(4.2, 4.8, 4.9)

Classical structures in Hilbert spaces

As we saw in Example 4.9, any choice of orthonormal basis for a finite-dimensional Hilbert space A induces a Frobenius algebra structure on A. In fact, this makes A into a classical structure, as is easy to verify. As it turns out, every classical structure in **FHilb** is of this form. Given a classical structure (A, d, e), we retrieve an orthonormal basis for A by its set of copyable states.

Definition 4.16 (Copyable state)**.** A state $I \xrightarrow{x} A$ of a comonoid (A, d, e) is *copyable* when $(x \otimes x) \circ \rho_I = d \circ x$.



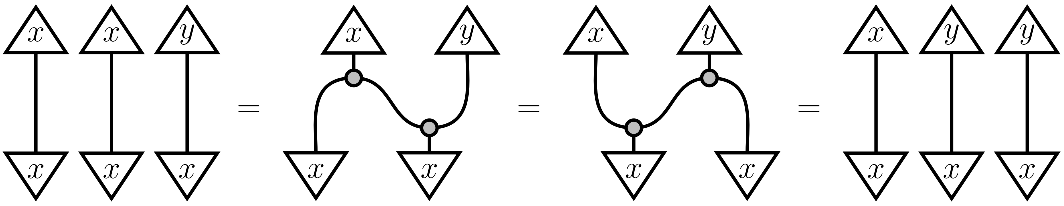
(4.15)

Lemma 4.17. *Nonzero copyable states of a classical structure in* **FHilb** *are orthonormal.*

Proof. It follows from speciality that any nonzero copyable state x has unit norm:

$$\|x\| = \langle x \,|\, x\rangle^{1/2} = \langle d^\dagger \circ d(x) \,|\, x\rangle^{1/2} = \|d(x)\| = \|x \otimes x\| = \|x\|^2.$$

Let x, y be nonzero copyable states and assume that $\langle x \,|\, y\rangle \neq 0$. Then:



In other words, $\langle x \,|\, x\rangle\langle x \,|\, x\rangle\langle y \,|\, x\rangle = \langle x \,|\, x\rangle\langle y \,|\, x\rangle\langle y \,|\, x\rangle$. Since $x \neq 0$ also $\langle x \,|\, x\rangle \neq 0$. So we can divide to get $\langle x \,|\, x\rangle = \langle x \,|\, y\rangle$. Similarly we can find $\langle y \,|\, x\rangle = \langle y \,|\, y\rangle$. Hence these inner products are all in $\mathbb{R}$, and are all equal. But then

$$\langle x - y \,|\, x - y\rangle = \langle x \,|\, x\rangle - \langle x \,|\, y\rangle - \langle y \,|\, x\rangle + \langle y \,|\, y\rangle = 0,$$

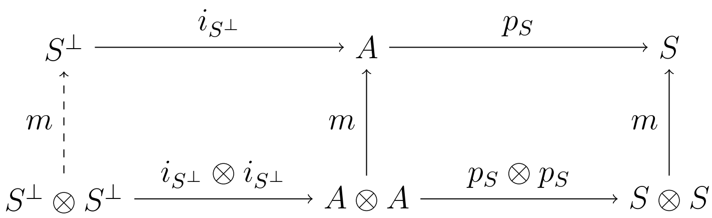
so $x - y = 0$. □

To show that any classical structure comes from an orthonormal basis, we need to show that there are 'enough copyable states'. Unfortunately, this relies on some mathematical results that are too deep to cover here, so we state the following lemma without proof.

Lemma 4.18. *Let* (A, d, e) *be a classical structure in* **FHilb**. *Unless* $A = 0$*, it has a nonzero copyable state.* □

Theorem 4.19 (Classical structures are bases)**.** *Classical structures on* A *in* **FHilb** *correspond to orthonormal bases of* A.

Proof. Let a classical structure be given. By Lemma 4.17, its nonzero copyable states are orthonormal. Suppose that their linear span S is not all of A. Then we can write $A = S \oplus S^\perp$ with $S^\perp \neq 0$. Consider the following diagram.



Because $S^\perp = \ker(p_S)$ and $p_S \circ i_{S^\perp} = 0$, the dashed arrow exists making the left square commute. In other words, $A \otimes A \xrightarrow{m} A$ restricts to a function $S^\perp \otimes S^\perp \xrightarrow{m} S^\perp$, making $S^\perp$ into a classical structure in its own right. But then Lemma 4.18 gives a nonzero vector in $S^\perp \cap S$, which is a contradiction. □

We have justified our definition of classical structure by intuitively thinking of it as copying and deleting operations. As we will see later on, the axioms of a classical structure indeed enable one to work as if handling flows of classical information. The previous theorem really vindicates that our definition was on the right track.

Classical structures in sets and relations

As we saw in Example 4.9, any groupoid **G** forms a Frobenius algebra in **Rel**. This is a classical structure, except that (co)commutativity is perhaps not satisfied.

Definition 4.20. An *abelian groupoid* is a category in which:

- every morphism is an isomorphism;
- if f and g are composable morphisms, then $g \circ f = f \circ g$. (For this equation to make sense, f and g must necessarily be endomorphisms $A \to A$ on some object A.)

As it turns out, every classical structure (G, d, e, m, u) in **Rel** is of this form! First, let's unravel what the conditions mean. From speciality (4.8), it quickly follows that m is single-valued: $((g, f), h) \in m$ and $((g, f), h') \in m$ imply $h = h'$. So we can write $m(g, f) = h$ instead of $((g, f), h) \in m$ without any problems. We will even write $g \circ_m f = h$ instead to suggest that G is going to be a category in its own right — but don't confuse $\circ_m$ in G with the composition $\circ$ in **Rel**. However, a priori there might not always be an h with $g \circ_m f = h$! Therefore, we will use the so-called *Kleene equality*: $x \doteq y$ asserts that if either side of the equality is defined, then so is the other, and they are equal.

Lemma 4.21. *Every classical structure (G, d, e, m, u) in* **Rel** *has a set $U \subseteq G$ such that for every $f \in G$ there are unique $x, y \in U$ with $y \circ_m f \doteq f \doteq f \circ_m x$, namely*

$$U = \{g \in G \mid (*, g) \in u\}.$$

Proof. Unitality (4.6) means that

$$\begin{aligned} &\forall f \in G \exists x \in U \colon f \circ_m x \doteq f, \\ &\forall f \in G \forall x \in U \colon (\exists g \in G \colon f \circ_m x \doteq g) \Rightarrow f \circ_m x \doteq f. \end{aligned}$$

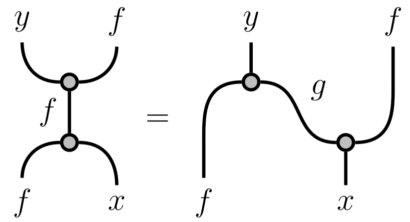
So if $f \circ_m x \doteq f$, then also $(f \circ_m x) \circ_m x \doteq f$. By associativity (4.5) then $x \circ_m x$ is defined, and $x \circ_m x \doteq x$. Now suppose that $f \circ_m x \doteq f \doteq f \circ_m x'$. It follows from the Frobenius law (4.9) that $x \doteq e \circ_m x'$ for some $e \in G$, so that $x \circ_m x' \doteq e \circ_m x' \circ_m x'$ exists. But then $x \doteq x \circ_m x' \doteq x'$ by unitality again. □

Theorem 4.22. *Classical structures on G in* **Rel** *correspond to abelian groupoids whose set of morphisms is G.*

Proof. Given a classical structure G, we define a category $\mathbf{G}$ as follows.

- As objects, we take $x \in U$ as defined by Lemma 4.21.
- As morphisms $x \to y$, we take those $f \in G$ with $y \circ_m f \doteq f \doteq f \circ_m x$.
- As identity on x we just take x itself.
- Composition is given by $\circ_m$. This is well-defined and associative by (4.5): if $x \xrightarrow{f} y$ and $y \xrightarrow{g} z$, then $g \circ_m f \doteq g \circ_m y \circ_m f$ exists.

Let $x \xrightarrow{f} y$ be a morphism. Then $y \circ_m f \doteq f \doteq f \circ_m x$, so $((y, f), (f, x)) \in d \circ m$. We indicate this by annotating the left-hand side diagram below.

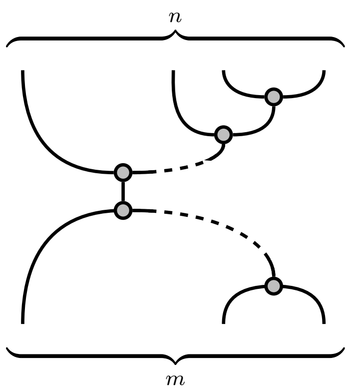


Hence, there must be $g \in G$ on the right-hand side. But that means that $g \circ_m f \doteq x$ and $f \circ_m g \doteq y$, so g and f are inverses. □

4.3 Normal forms

As you might expect, there are only so many ways you can copy (using d), forget (using e), compare (using $d^\dagger$) and create (using $e^\dagger$) classical information. In fact, as long as we are talking about connected diagrams of classical information flows, there is only one! That is, we can prove the following theorem, which reminds one of the Coherence Theorem 1.2.

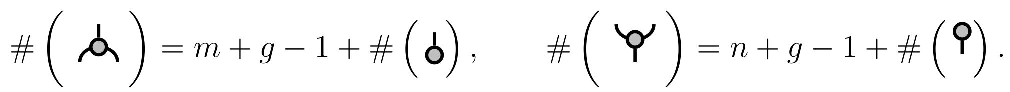
Theorem 4.23 (Spider theorem)**.** *Let (A, d, e) be a classical structure. Any connected morphism $A^{\otimes m} \to A^{\otimes n}$ built out of $d, e, \mathrm{id}, \sigma, \otimes$ and $\dagger$ equals the following* normal form.



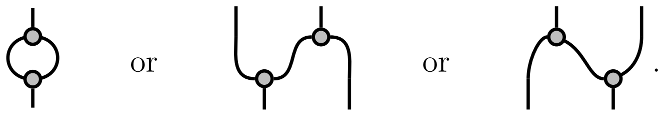
(4.16)

So any morphism built from $d, e, \mathrm{id}, \sigma, \otimes, \dagger$ can be built from normal forms with $\otimes$ and σ.

Proof. We start by ignoring the swap σ, and consider a morphism $A^{\otimes m} \to A^{\otimes n}$ built out of , , , , and . Take one of the building blocks . Our strategy will be to push it down, until it comes before any . What can we meet on our way down? If we meet a , we can use unitality (4.6), and the vanishes. Using the Euler characteristic of the diagram (regarded as a planar graph with g inner faces), we find

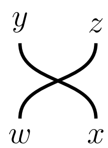


In particular, there are enough copies of to spend on getting rid of all the . We can also meet another . In this case we can use associativity (4.5) to push our chosen one below the one we meet. Finally, we can meet a . This can happen in three ways:



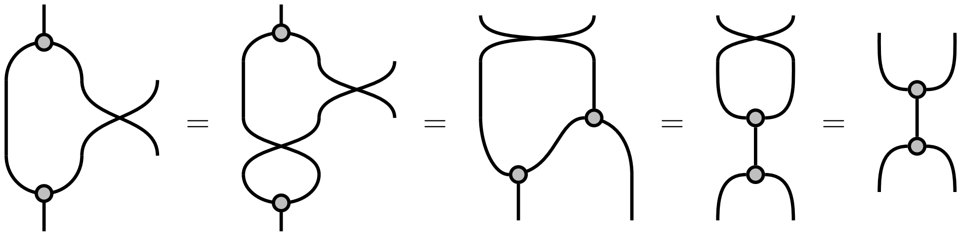
The first case vanishes by speciality, and in the second and third cases we use the Frobenius law (4.9) to push the below the . In the same way, we can push up all the , getting rid of all in the process, and end up with the desired normal form.

Next, consider diagrams that may involve swap maps as well. Pick one of them. By naturality, we can make sure that only pieces are parallel with our swap map:



Since the diagram is connected, some of the other regions w, x, y, z of the diagram must be connected to each other. Suppose w and x are connected to each other. Then they are connected by a diagram involving strictly less swap maps than the original, so by induction we can assume it can be brought on normal form. But then, perhaps by using coassociativity, we can make sure that our chosen swap map comes directly above a . So by cocommutativity, our swap map vanishes, and we are done. The same argument holds when if y and z are interconnected.

We're down to the case where w and y are connected to each other. Then each of the subdiagrams w and y contain strictly less swaps than the original, and we may assume them to be on normal form. So the direct neighbourhood of our swap map looks as follows.

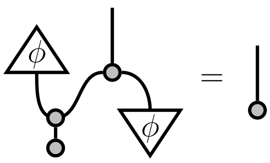


Hence we can make our swap map vanish. The first equality is cocommutativity, the second is naturality of the swap, the third is the Frobenius law, and the fourth equality is cocommutativity again. □

4.4 Phases

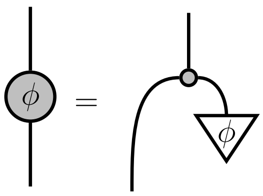
In quantum information theory, an interesting family of maps are *phase gates*: diagonal matrices whose diagonal entries are complex numbers of norm 1. For a particular Hilbert space equipped with a basis, these form a group under composition, which we will call the *phase group*. This turns out to work fully abstractly: any classical structure in any dagger compact category gives rise to a phase group.

Definition 4.24 (Phase). Let (A, m, u) be a classical structure. A state $I \xrightarrow{\phi} A$ is called a *phase* when the following equation holds.



(4.17)

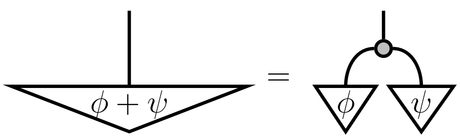
Its *phase shift* is the morphism $d \circ (\phi \otimes \mathrm{id}) \colon A \to A$, which we denote as follows.



(4.18)

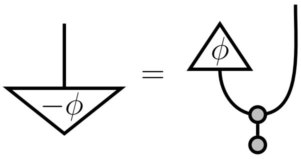
Notice that the unit  of a classical structure is always a phase.

Proposition 4.25. *Let (A, m, u) be a classical structure in a dagger symmetric monoidal category. Its phases form an abelian group under $\phi + \psi := m \circ (\phi \otimes \psi)$ with unit u.*



(4.19)

Proof. It follows from the Spider Theorem 4.23 that $\phi + \psi$ is again a phase when ϕ and ψ are phases. Since m is commutative, the phases thus form a commutative monoid.



(4.20)

By definition (4.17), it is in fact an abelian group, with inverse $-\phi = (\phi \otimes \mathrm{id}) \circ \eta$. □

The group of the previous proposition is called the *phase group*. Equivalently, the phase shifts form an abelian group under composition. For example, let a classical structure on A in **FHilb** be given by an orthonormal basis $\{|i\rangle\}_{i=1,\ldots,n}$. Its phases are the vectors in A of the form

$$\begin{pmatrix} e^{i\phi_1} \\ \vdots \\ e^{i\phi_n} \end{pmatrix}$$

when written on basis $\{|i\rangle\}$, for real numbers ϕ_i. The group operations are simply

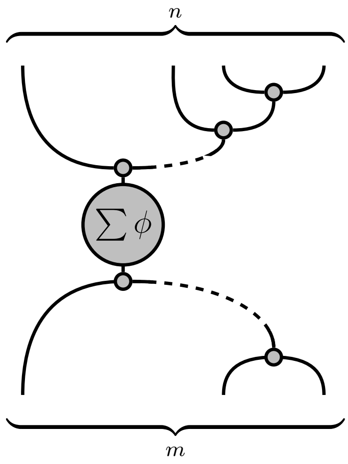
$$\begin{pmatrix} e^{i\phi_1} \\ \vdots \\ e^{i\phi_n} \end{pmatrix} + \begin{pmatrix} e^{i\psi_1} \\ \vdots \\ e^{i\psi_n} \end{pmatrix} = \begin{pmatrix} e^{i(\phi_1+\psi_1)} \\ \vdots \\ e^{i(\phi_n+\psi_n)} \end{pmatrix}, \qquad 0 = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} e^{i0} \\ \vdots \\ e^{i0} \end{pmatrix}$$

The phase shift accompanying a phase is the unitary matrix

$$\begin{pmatrix} e^{i\phi_1} & 0 & \cdots & 0 \\ 0 & e^{i\phi_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i\phi_n} \end{pmatrix}.$$

In **Rel**, the phase group of a classical structure induced by an abelian group G as in Example 4.9, is G itself. More generally, consider an abelian groupoid $\mathbf{G}$ and the classical structure in **Rel** it induces. Its phase group is the product group $\prod_{x \in \mathrm{Ob}(\mathbf{G})} \mathbf{G}(x, x)$.

Theorem 4.26 (Generalized spider theorem)**.** *Let (A, d, e) be a classical structure. Any* *connected morphism* $A^{\otimes m} \to A^{\otimes n}$ *built out of* $d, e, \mathrm{id}, \sigma, \otimes, \dagger$ *and phase shifts equals*



(4.21)

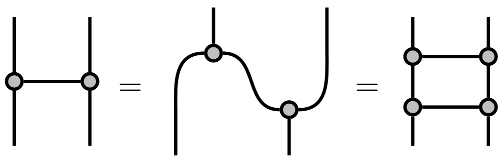
where ϕ *ranges over all the phases used in the diagram.*

Proof. Adapting the proof of the Spider Theorem 4.23, we can get to a normal form of the form (4.16), with phases dangling at the bottom. But then we can propogate those phases upwards, by the very definition of the phase group operation (4.19). When we reach the "middle" of our diagram, all phases will have been incorporated, and we end up with the desired form (4.21). □

4.5 State transfer

Given: two qubits, one in an unknown state and one in the state $|+\rangle = |0\rangle + |1\rangle$.
Goal: transfer the unknown state from the first qubit to the second.
Extra challenge: apply a phase gate ϕ to the first qubit in the process.

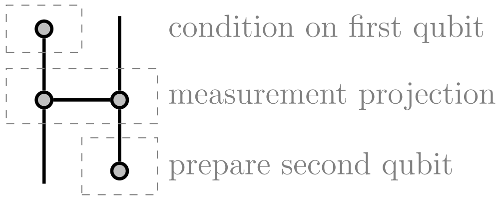
We now study a protocol called state transfer. It operates by using two projections. The first is used to condition on measurement outcomes, and the second is the "measurement projection" (4.22) below. To be precise, consider the computational basis $\{|0\rangle, |1\rangle\}$ on $\mathbb{C}^2$ and the classical structure this induces. By virtue of the spider theorem, we can be quite lax when drawing wires connected by classical structures. They are all the same morphism anyway. For example:



(4.22)

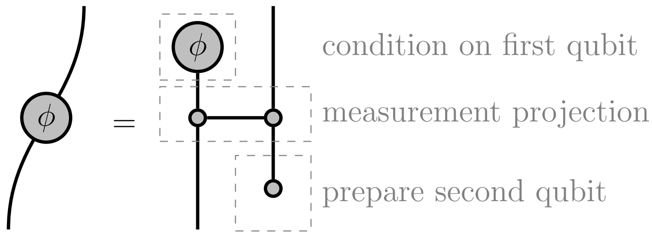
is a projection $\mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$.

The protocol consists of three steps. First, prepare the second qubit in $|+\rangle$. Second, apply the measurement projection to the compound system of both qubits. Third, condition on the first qubit.



By the Spider Theorem 4.23, this equals the identity! Hence this protocol indeed achieves the goal of transfering the first qubit to the second. To appreciate the power of the graphical calculus, one only needs to compute the same protocol using matrices.

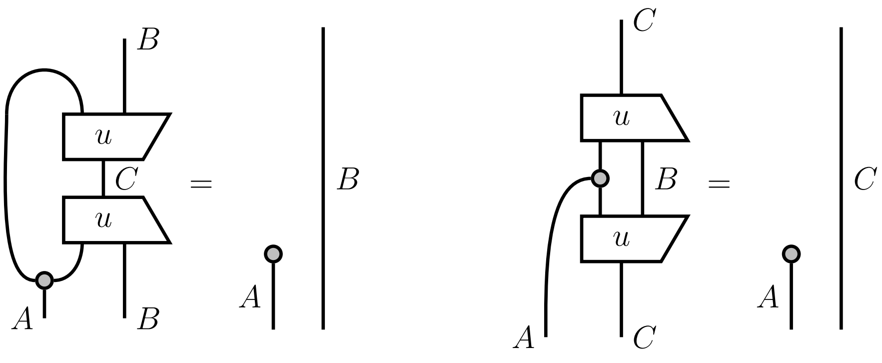
By using the generalized Spider Theorem 4.26, we can also easily achieve the extra challenge, by the following adapted protocol.



This protocol is important in *measurement-based quantum computing.*

4.6* Controlled unitaries

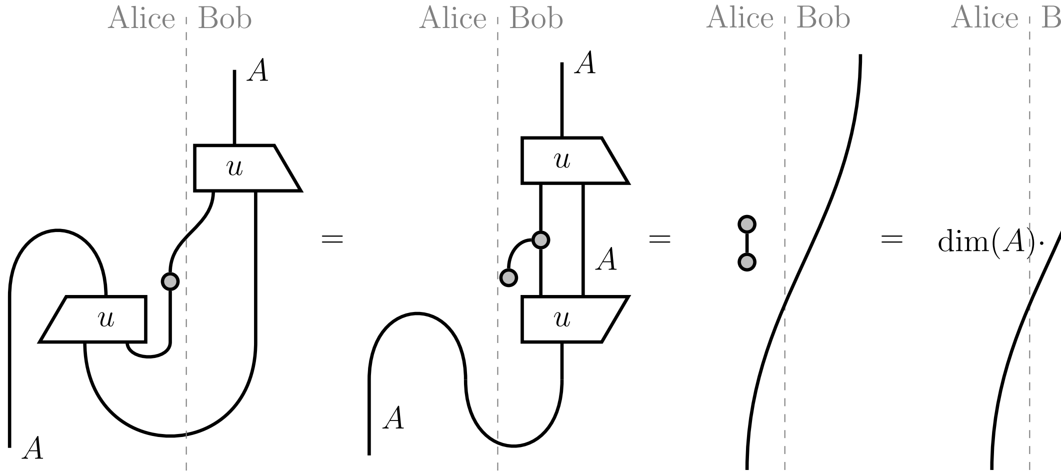
Definition 4.27 (Controlled unitary)**.** Let (A, d, e) be a classical structure. A *controlled unitary* is a morphism $A \otimes B \xrightarrow{u} C$ satisfying the following equations.



(4.23)

In the category **FHilb**, consider the classical structure on $\mathbb{C}^n$ induced by a basis $\{|1\rangle, \ldots, |n\rangle\}$. Then $\mathbb{C}^n \otimes H \xrightarrow{u} K$ is a controlled unitary precisely when the n maps $H \to K$ given by $x \mapsto u(|i\rangle \otimes x)$ are unitaries. In particular, in this way we can make quantum computation gates like CNOT.

We can model quantum teleportation by making use of a classical structure on the object A we want to teleport, a controlled unitary, and the assumption that the scalar $\dim(A)$ is invertible.



(4.24)

4.7 Exercises

Notes and further reading

The Frobenius law (4.9) is named after F. Georg Frobenius, who first studied the requirement that $A \cong A^*$ as right A-modules for a ring A in the context of group representations in 1903 [30]. The formulation with multiplication and comultiplication we use is due to Lawvere in 1967 [47], and was rediscovered by Quinn in 1995 [60] and Abrams in 1997 [1]. Dijkgraaf realized in 1989 that the category of commutative Frobenius algebras is equivalent to that of 2-dimensional topological quantum field theories [26]. For a comprehensive treatment, see the monograph by Kock [46].

Coecke and Pavlović first realized in 2007 that commutative Frobenius algebras could be used to model the flow of classical information [22].

Theorem 4.19, that classical structures in **FHilb** correspond to orthonormal bases, was proved in 2009 by Coecke, Pavlović and Vicary [23]. In 2011, Abramsky and Heunen adapted Definition 4.15 to generalize this correspondence to infinite dimensions in **Hilb** [6].

Theorem 4.22, that classical structures in **Rel** are groupoids, was proved by Heunen, Contreras and Cattaneo in 2012 [34], generalizing earlier work on the commutative case by Pavlović in 2009 [57].

The phase group was made explicit by Coecke and Duncan in 2008 [17], and later Edwards in 2009 [28, 19].

The state transfer protocol is important in efficient measurement-based quantum computation. It is due to Perdrix in 2005 [59].

Chapter 5

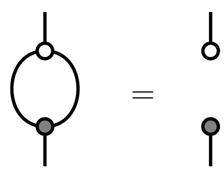
Complementarity

In this chapter we will study what happens when we have *two* interacting classical structures. Specifically, we are interested in they are 'maximally incompatible', or *complementary.* In the case of qubits, such *mutually unbiased bases* play a pivotal role in quantum information theory. We will show how this gets us Hadamard gates, and hence universal quantum computation. Graphically, we will distinguish between the two (co)units and (co)multiplications by colouring their dots differently.

5.1 Bialgebras

It turns out that complementarity can be modelled by letting the multiplication of one observable interact with the comultiplication of the other in a way that is in many ways opposite to the way the multiplication and the comultiplication of a single classical structure interact.

Definition 5.1. A pair of a comonoid (A, d, e) and a monoid (A, m, u) is called *disconnected* when $m \circ d = u \circ e$.

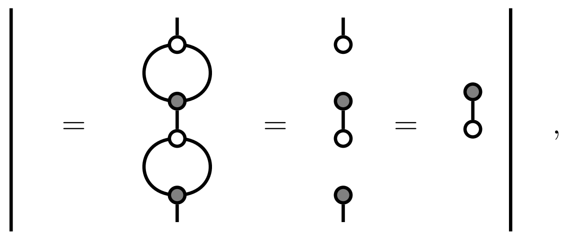


(5.1)

As far as interaction between monoids and comonoids goes, speciality and disconnectedness are opposite extremes. As the following proposition shows, both cannot happen simultaneously under reasonable conditions.

Proposition 5.2. *If a comonoid* (A, d, e) *and a monoid* (A, m, u) *are simultaneously special and disconnected, and* $(e \circ u) \bullet \mathrm{id}_A = \mathrm{id}_A$ *implies* $e \circ u = \mathrm{id}_I$*, then* $A \cong I$*.*

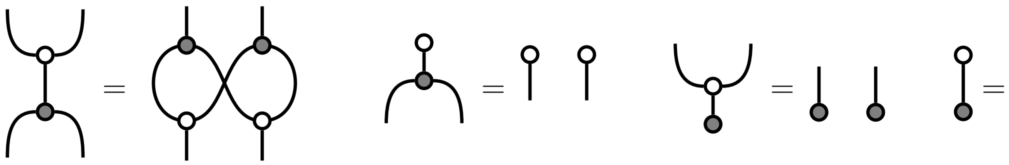
Proof. We will show that e and u are each others' inverses. Applying equation (5.1) and then equation (4.8) establishes $e \circ u = \mathrm{id}_A$. Conversely,



which by assumption implies that $e \circ u = \mathrm{id}_I$. □

There is another way in which we can compose first multiplication and then comultiplication, called the bialgebra laws.

Definition 5.3 (Bialgebra)**.** A *bialgebra* in a monoidal category consists of a monoid (A, m, u) and a comonoid (A, d, e) on the same object, satisfying the following equations, called the *bialgebra laws*.



(5.2)

The last equation $u \circ e = \mathrm{id}_I$ is not missing a picture, because we are drawing id_I as the empty picture (1.10). The following concise formulation is a good way to remember the bialgebra laws.

Lemma 5.4. *A comonoid (A, d, e) and monoid (A, m, u) form a bialgebra if and only if d and e are monoid homomorphisms.*

Proof. Just unfold the definitions. This involves showing that $A \otimes A$ carries a monoid structure when A does, which we leave as an exercise. □

Examples 5.5.

- Considering **Hilb** as a monoidal category under biproducts, any object A has a bialgebra structure given by its copying and deleting maps: $d = \begin{pmatrix}1\\1\end{pmatrix} : A \to A \oplus A$, $e = !_A \colon 0 \to A$, $u = !_A \colon A \to 0$, $m = \begin{pmatrix}1 & 1\end{pmatrix} : A \oplus A \to A$.
- Any finite monoid G (in **Set**) induces a bialgebra in $(\mathbf{Hilb}, \otimes, \mathbb{C})$ as follows. Let $A = \mathbb{C}[G]$ be the Hilbert space of linear combinations of elements of G with its standard inner product. In other words, A has G as an orthonormal basis. Define $A \otimes A \xrightarrow{m} A$ by linearly extending $m(g, h) = gh$, define $\mathbb{C} \xrightarrow{u} A$ by $u(z) = z \cdot 1_G$, and define d and e by linearly extending $d(g) = g \otimes g$ and $e(g) = 1$.

Notice that m and u can also make A into a Frobenius algebra as in Example 4.9, but with different d and e. Indeed, by the following theorem, they *have* to be different unless G is the trivial monoid.

- Any monoid G is a bialgebra in the monoidal category **Set**, by $d(g) = (g, g)$, $e(g) = *$, $u(*) = 1_G$, $m(g, h) = gh$.

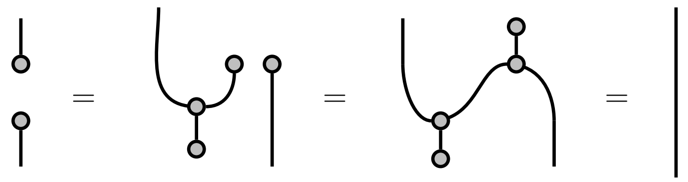
  Notice again that m and u can also make G into a Frobenius algebra in **Rel** as in Example 4.9, but again, with different d and e.

- Fock space?

As far as interaction between monoids and comonoids is concerned, Frobenius algebras and bialgebras are opposite extremes. The following theorem shows that both cannot happen simultaneously, except in the trivial case. The crux is that the Frobenius law (4.9) equates *connected* diagrams, whereas the bialgebra laws (5.2) equate connected diagrams with *disconnected* ones. As we saw with special and disconnected algebras in Proposition 5.2, the only object that is both connected and disconnected is the tensor unit.

Theorem 5.6 (Bialgebras cannot be Frobenius). *If $(A, d, e, d^\dagger, e^\dagger)$ is both a Frobenius algebra and a bialgebra in a monoidal category, then $A \cong I$.*

Proof. We will show that $u = e^\dagger$ and e are each others' inverses. The bialgebra laws (5.2) already require that $e \circ u = \mathrm{id}_I$.



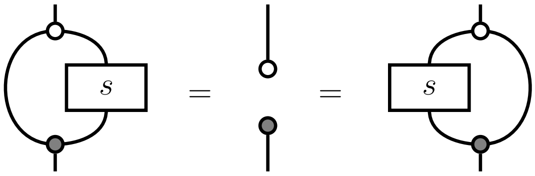
The first equality is counitality (4.4), the second equality is one of the bialgebra law (5.2), and the last equality follows from Theorem 4.12. □

The previous theorem is not all that surprising when we realize that $e \circ u$ is the dimension of A. Equation (5.2) says that A and I have the same dimension. But notice that the proof of the previous theorem holds equally well when we had merely required $e \circ u$ to be positive and invertible, instead of $e \circ u = \mathrm{id}_I$. We will in fact do this soon, but first we consider Hopf algebras.

5.2 Hopf algebras and complementarity

A property that often goes together with bialgebras is the so-called Hopf law.

Definition 5.7 (Hopf law)**.** Let (A, d, e) be a comonoid and (A, m, u) a monoid, and $A \xrightarrow{s} A$ a morphism. The *Hopf law* states $m \circ (\mathrm{id}_A \otimes s) \circ d = \mathrm{id}_A = m \circ (s \otimes \mathrm{id}_A) \circ d$. The morphism s is called the *antipode.*



(5.3)

The example we gave of a bialgebra $\mathbb{C}[G]$ induced by a finite monoid G in fact satisfies the Hopf law if and only if the monoid is a group. The antipode $\mathbb{C}[G] \xrightarrow{s} \mathbb{C}[G]$ is the linear extension of $s(g) = g^{-1}$, and the algebra is then called the *group algebra.* In this sense bialgebras satisfying the Hopf law are the quantum version of groups.

Proposition 5.8. *Bialgebras algebras in* **Set** *satisfying the Hopf law are precisely groups.*

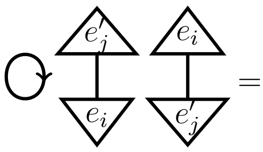
Proof. Given a bialgebra (G, d, e, m, u, s) in **Set** satisfying the Hopf law, define a multiplication on G by $gh := m(g, h)$, define inverses by $g^{-1} := s(g)$, and set $1 := u(*) \in G$. It follows from the Hopf law (5.3) that $g^{-1}g = 1 = gg^{-1}$, and hence that G is a group.

Conversely, let G be a group. Define $G \xrightarrow{d} G \times G$ by $d(g) = (g, g)$. Similarly, define $e(g) = *$, $u(*) = 1_G$, $m(g, h) = gh$, and $s(g) = g^{-1}$. It is a quick exercise to verify that these data satisfy the bialgebra laws (5.2) and the Hopf law (5.3). □

Now, suppose we have not just a pair of a monoid and a comonoid, but a pair of classical structures. In **FHilb**, this means we have chosen two bases of a single space. Then there is a canonical choice for an antipode, and the Hopf law encodes that the two bases are *mutually unbiased.*

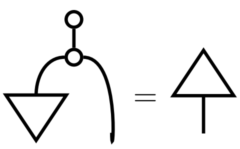
Definition 5.9. Two bases $\{e_i\}, \{e'_i\}$ of a Hilbert space H are *mutually unbiased* when $|\langle e_i \mid e'_j \rangle|^2 = 1/\dim(H)$ for all i, j.

The idea is that each of the elements of one basis make maximal angles with each of the elements of the other basis. In other words, having perfect information about the system in one basis reveals nothing at all in the other basis. For example, in the case of qubits, the bases $\{|0\rangle, |1\rangle\}$ and $\{|+\rangle, |-\rangle\}$ are mutually unbiased. We can reformulate this graphically as follows.



(5.4)

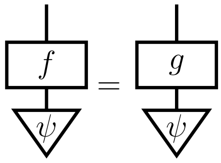
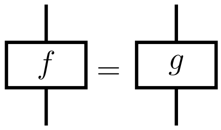
In **FHilb**, basis vectors correspond to copyable states, and satisfy the following equation.



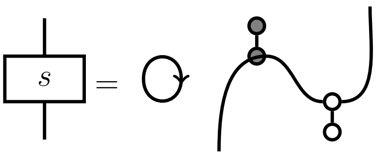
(5.5)

Moreover, they form a basis, which gives a stronger version of well-pointedness.

Definition 5.10. A classical structure on A has *enough copyable states* when two morphisms $A \xrightarrow{f,g} B$ are equal as soon as $f \circ \psi = g \circ \psi$ for all copyable states $I \xrightarrow{\psi} A$.

If  for all copyable states ψ, then . (5.6)

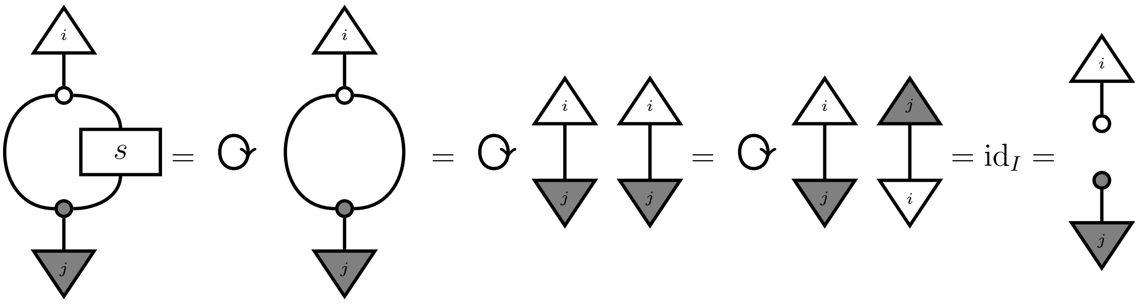
Definition 5.11 (Complementarity)**.** Two classical structures are called *complementary* when they satisfy the Hopf law (5.3) for the following antipode.



(5.7)

Proposition 5.12. *Suppose equations* (5.5) *and* (5.6) *hold. Then the Hopf law* (5.3) *is equivalent to equation* (5.4)*. Hence two orthonormal bases on a Hilbert space are mutually unbiased if and only if the classical structures they induce are complementary.*

Proof. Assume equations (5.4) and (5.5), and draw copyable states in the same color as the classical structure that copies them. Then:

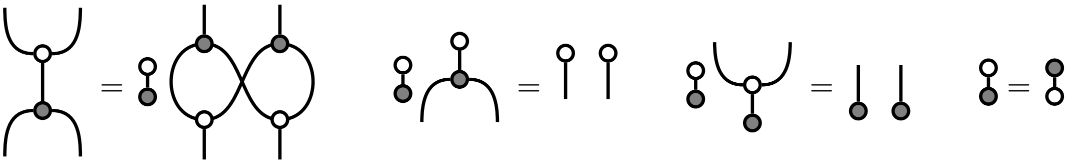


Equation (5.6) now establishes the Hopf law (5.3). The converse is similar. □

5.3 Strong complementarity

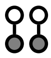
We will now investigate a strong version of complementarity, where not just the Hopf law holds, but also the bialgebra laws. In fact, the latter will imply the former. However, as we saw in Proposition 5.2 and Theorem 5.6, we will need to scale by an appropriate dimension factor. This leads to a *scaled* version of the bialgebra laws.

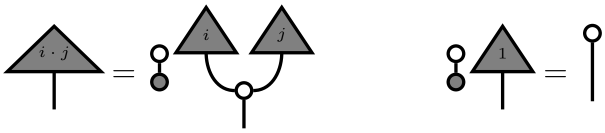
Definition 5.13 (Scaled bialgebra, strong complementarity)**.** A *scaled bialgebra* is a pair of a monoid (A,  , ) and a comonoid (A,  , ) satisfying the following equations.



(5.8)

Two classical structures are called *strongly complementary* when the monoid of one forms a scaled bialgebra with the comonoid of the other.

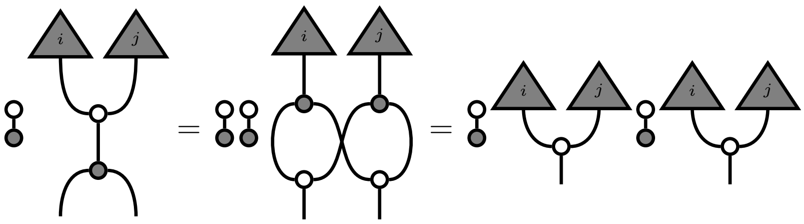
Lemma 5.14. *Suppose that the scalar*  *is invertible. For two strongly complementary classical structures, the following defines a monoid structure on the copyable states of*  *.*



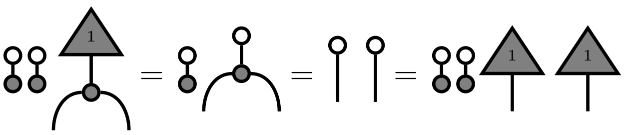
(5.9)

In fact, this defines a submonoid of the phase group for ( *,* *).*

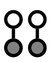
Proof. Associativity and unitality are clear, but we have to prove that $i \cdot j$ and 1 are again copyable states. For $i \cdot j$:

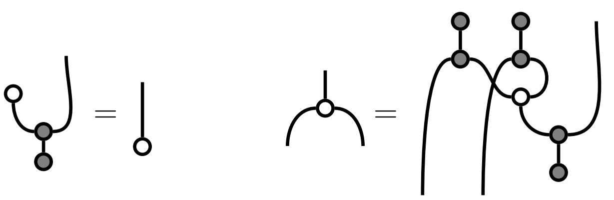


And for 1:

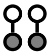


Since the scalar  is invertible, 1 is a copyable state. □

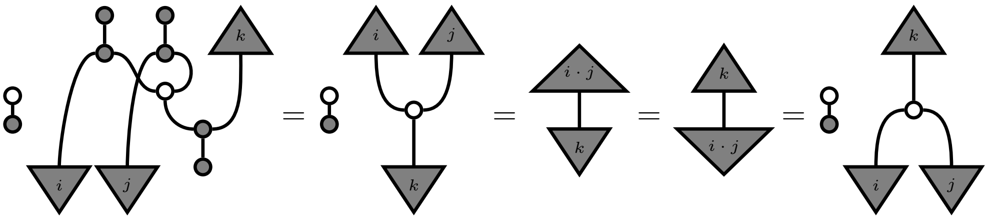
Lemma 5.15. *Suppose that the scalar*  *is invertible. If they have enough copyable states, then strongly complementary classical structures satisfy the following equation.*



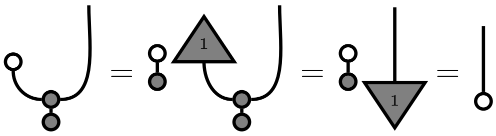
(5.10)

Hence  $= \dim(A)$.

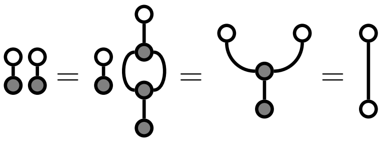
Proof. First, use Lemma 5.14:



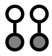
Then, we can conclude the right equation of (5.10) from property (5.6). Similarly, for the left equation of (5.10):



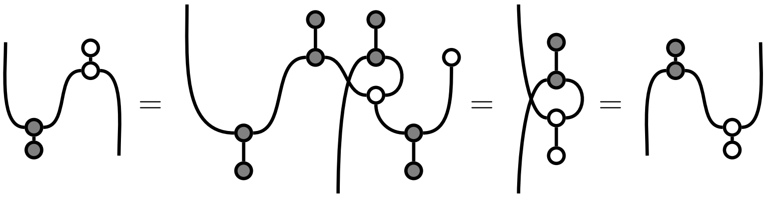
Finally:



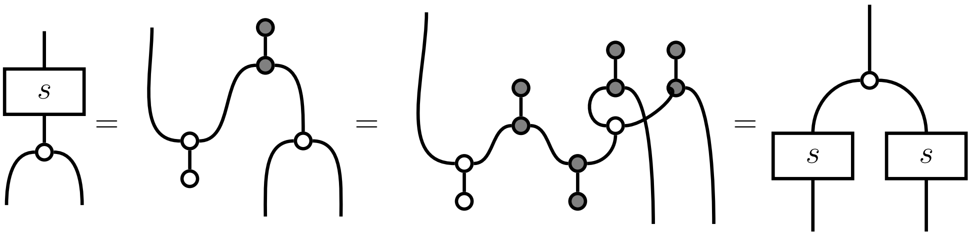
and the latter equals $\dim(A)$ by the Spider Theorem 4.23 and Theorem 4.12. □

Lemma 5.16. *Suppose that the scalar*  *is invertible. If two strongly complementary classical structures have enough copyable states, then the antipode* (5.7) *is self-adjoint, and is an automorphism for both classical structures.*

Proof. First we prove that $s = s^\dagger$ using Lemma 5.15.



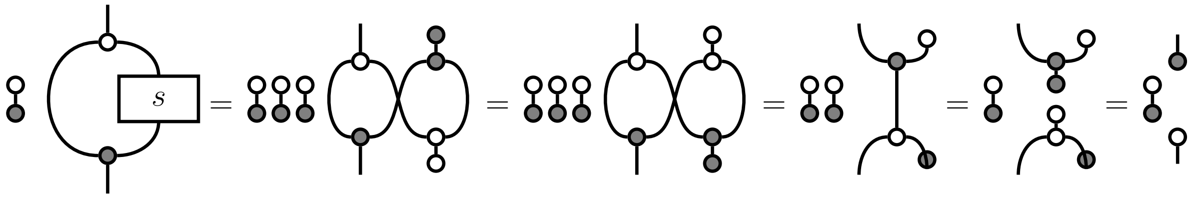
Consequently, s preserves units. Using Lemma (5.10) again:



Therefore s is a homomorphism of Frobenius algebras, and must be an isomorphism by Lemma 4.14. □

Proposition 5.17. *Suppose that the scalar*  *is invertible. If two strongly complementary classical structures have enough copyable states, then they are complementary.*

Proof.



The first equality is the definition of s, the second equality is Lemma 5.16, the third equality is the scaled bialgebra law (5.8), the fourth equation uses the Spider Theorem 4.23 and the scaled bialgebra law (5.8), and the last equation follows from Lemma 5.15. □

The classification of pairs of complementary classical structures (i.e. mutually unbiased bases) on a finite-dimensional Hilbert space is an open problem. But we can classify strong complementarity completely.

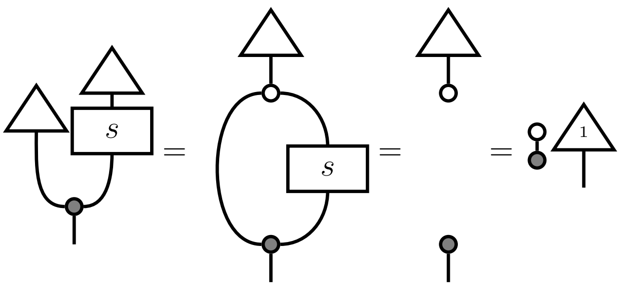
Theorem 5.18. *Pairs of strongly complementary classical structures on H in* **FHilb** *correspond to abelian groups of order* $\dim(H)$.

Proof. Let G be an abelian group of order n. Its elements form a basis $\{|g\rangle\}$ for $H = \mathbb{C}^n$. Defining

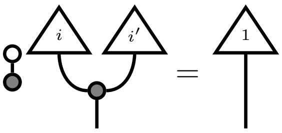
$$d\colon |g\rangle \mapsto |g\rangle \otimes |g\rangle, \qquad e\colon |g\rangle \mapsto 1$$
$$m\colon |g\rangle \otimes |h\rangle \mapsto \frac{1}{\sqrt{n}}|g+h\rangle \qquad u\colon 1 \mapsto \sum_{g\in G} |g\rangle$$

gives classical structures $(A, d, e, d^\dagger, e^\dagger)$ and $(A, m^\dagger, u^\dagger, m, u)$. Moreover, (A, d, e, m, u) is a scaled version of the group algebra, and hence forms a scaled bialgebra. Therefore these two classical structures are strongly complementary.

For the converse, let two strongly complementary classical structures be given. By Lemma 5.14 the copyable states of  form a monoid under , and in fact a submonoid of the phase group. But the phase group is finite, and any submonoid of a finite group is a (sub)group itself. This already establishes the theorem, but let's work out what inverses look like anyway. The following equation now follows from Proposition 5.17 for any state that is copyable under .



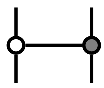
By Lemma 5.16 the antipode s is a homomorphism of Frobenius algebras and therefore an isomorphism by Lemma 4.14. Thus s permutes classical points. Hence the previous equation implies that each copyable state i has a copyable state i' such that:



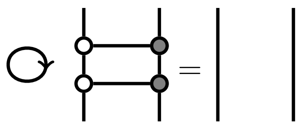
Therefore all copyable states of  have inverses, and  is isomorphic to the group algebra $\mathbb{C}[G]$ for that abelian group G. □

5.4 Applications

We can now consider some applications to quantum computation. We start by defining CNOT gates. This gate performs a NOT operation on the second qubit if the first (control) qubit is $|1\rangle$, and does nothing if the first qubit is $|0\rangle$. But the definition itself makes sense for arbitrary pairs of classical structures.

CNOT :=  (5.11)

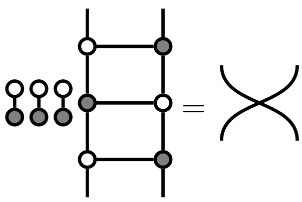
Proposition 5.19. *Two classical structures (*  *,*  *) and (*  *,*  *) are complementary if and only if the following equation holds.*



(5.12)

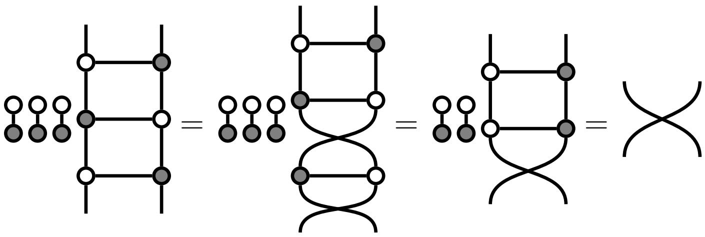
Proof. Both implications follow from one application of the Spider Theorem (4.23) and one application of the Hopf law (5.3). □

Theorem 5.20. *Two complementary classical structures (*  *,*  *) and (*  *,*  *) are strongly complementary if and only if the following equation holds.*



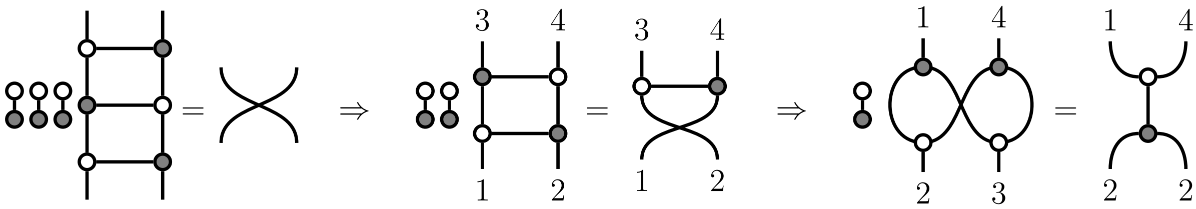
(5.13)

Proof. First, assume strong complementarity. Then:



By naturality of the swap, the scaled bialgebra law (5.8) and Proposition 5.19.

Conversely:



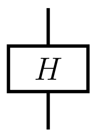
The first implication follows from postcomposing with CNOT and Proposition 5.19. The second implication follows from the Spider Theorem 4.23; for convenience, we have labeled the wires to make the idenfication. The other scaled bialgebra laws follow similarly. □

For the rest of this section, we work in the category **FHilb**, fix $A = \mathbb{C}^2$, let (,) be defined by the Z basis $\{|0\rangle, |1\rangle\}$, and define (,) to copy the X basis $\{|+\rangle, |-\rangle\}$.

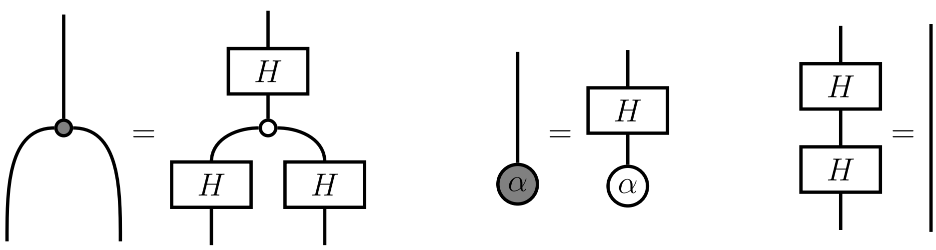
Equation (5.11) now indeed reduces to the CNOT gate.

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{5.14}$$

The relationship between the two classical structures is $|+\rangle = |0\rangle + |1\rangle$, and $|-\rangle = |0\rangle - |1\rangle$. Hence they are transformed into each other by the *Hadamard gate*.

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} =$$  (5.15)

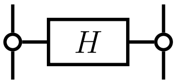
Thus the following equations are satisfied.



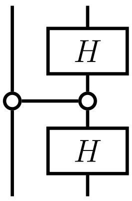
(5.16)

In addition to the CNOT gate, we can now also define the CZ gate abstractly. This gate performs a Z phase shift on the second qubit when the first (control) qubit is $|1\rangle$, and leaves it alone when the first qubit is $|0\rangle$.

Lemma 5.21. *The CZ gate can be defined as follows.*

$CZ :=$  (5.17)

Proof. We can rewrite equation (5.17) as follows.

CZ = 

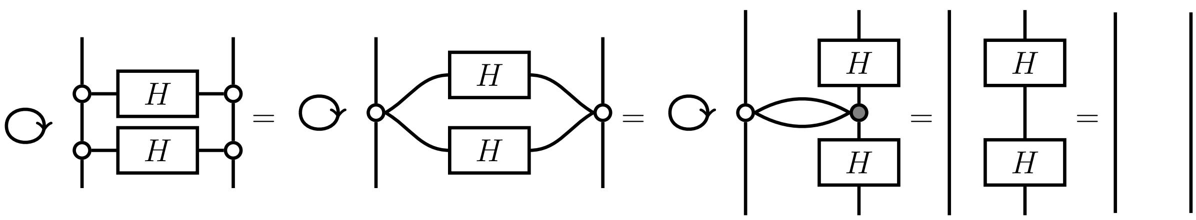
Hence

$$\mathrm{CZ} = (\mathrm{id} \otimes H) \circ \mathrm{CNOT} \circ (\mathrm{id} \otimes H) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

This is indeed the controlled Z gate. □

Proposition 5.22. $2 \bullet CZ \circ CZ = \mathrm{id}$.

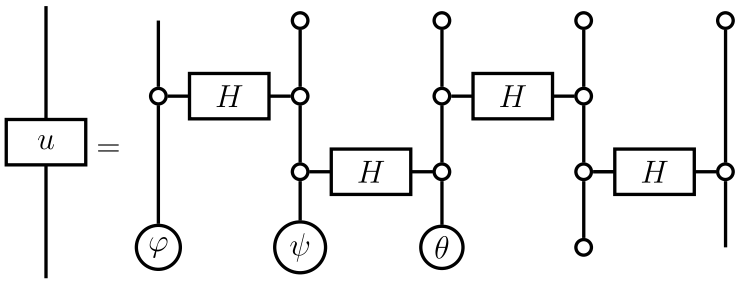
Proof.



□

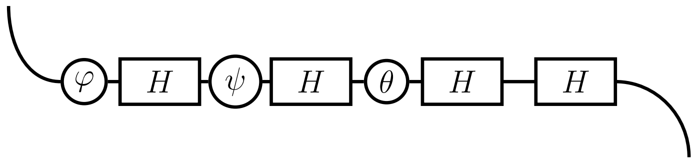
Qubits have the nice property that any unitary on them can be implemented via its *Euler angles*. More precisely: for any unitary $\mathbb{C}^2 \xrightarrow{u} \mathbb{C}^2$, there exist phases φ, ψ, θ such that $u = Z_\theta \circ X_\psi \circ Z_\varphi$. Therefore we can implement such unitaries abstractly using just CZ-gates and Hadamard gates.

Theorem 5.23. *If a unitary* $\mathbb{C}^2 \xrightarrow{u} \mathbb{C}^2$ *in* **FHilb** *has Euler angles* φ, ψ, θ*, then:*

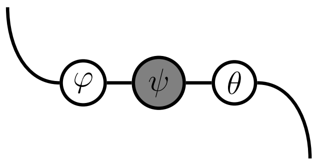


(5.18)

Proof. By using the Generalized Spider Theorem 4.26 equation (5.18) reduces to



But by equation (5.16), this is just:



which equals u, by definition of the Euler angles. □

5.5 Exercises

Notes and further reading

Complementarity has been a basic principle of quantum theory from very early on. It was proposed by Niels Bohr in the 1920s, and is closely identified with the Copenhagen interpretation [62]. Its mathematical formulation in terms of mutually unbiased bases is due to Schwinger in 1960 [63]. The abstract formulation in terms of classical structures we used was first given by Coecke and Duncan in 2008 [17]. Strong complementarity was first discussed in that article, and the ensuing Theorem 5.18 is due to Coecke, Duncan, Kissinger and Wang in 2012 [18].

The applications in Section 5.4 are basic properties in quantum computation [54], and are especially important to measurement based quantum computing [61]. See [27] for more abstract results on Euler angles.

Bialgebras and Hopf algebras are the starting point for the theory of quantum groups [42, 70]. They have been around in algebraic form since the 1960s, when Heinz Hopf first studied them [35]. Graphical notation for them is becoming more standard now, with so-called Sweedler notation as a middle ground [14].

Chapter 6

Copying and deleting

Our running examples of compact categories involved tensor products rather than products or direct sums. This chapter shows there is a good reason for doing so: categorical products might give a perfectly good example of a monoidal category, but they cannot give examples of compact categories except in degenerate cases.

This sets "classical" categories like **Set** apart from more "quantum" categories like **Rel** and **Hilb**. To see the difference between, for example, **Set** and **Rel**, we have to think about classical and quantum information. Recall the famous *no-cloning theorem*, and its slightly less well-known sibling the *no-deleting theorem*. They show that quantum information is distinguished by the fact that it cannot be copied or deleted. Conversely, we will show that tensor products equipped with uniform copying and deleting operators are (categorical) products. But before we go into these matters, we have to review the issue of *closure*.

6.1 Closure

Up to now we have mostly considered objects and morphisms up to "first order": we think of morphisms as a transformation of the *input* type into the *output* type. But sometimes we would like to talk about transformations of morphisms into morphisms. For example, when we have a superposition rule as in **Vect**, addition of matrices yields a new matrix.

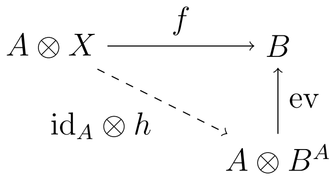
Indeed, the monoidal category **Vect** is able to handle "higher order" morphisms. Namely, if V and W are vector spaces, then the set

$$W^V = \{f : V \to W \mid f \text{ linear}\} \tag{6.1}$$

is again a vector space, with pointwise operations such as $(f + g)(x) = f(x) + g(x)$. (In fact, this is the homset $\mathbf{Vect}(V, W)$ itself!) Thus we can talk about transformations of morphisms as being just ordinary morphisms by encoding morphisms as vectors in function spaces.

The vector space W^V comes with nice property we might expect from such a function space. If we have $f \in W^V$ and $x \in V$, then there is $f(x) \in W$. Moreover, this assignment is linear in both f and x. In other words, there is a bilinear function $V \times W^V \to W$ given by $(f, x) \mapsto f(x)$. Hence, there is an *evaluation* map $\text{ev}\colon V \otimes W^V \to W$. Objects that stand in such a relation to the tensor product are called *exponentials* in general.

Definition 6.1 (Exponential)**.** Let A and B be objects in a symmetric monoidal category. Their *exponential* is an object B^A together with a map $\text{ev}\colon A \otimes B^A \to B$ such that every morphism $f\colon A \otimes X \to B$ allows a unique morphism $h\colon X \to B^A$ with $f = \text{ev} \circ (\text{id}_A \otimes h)$.



(6.2)

The category is called *closed* when every pair of objects has an exponential.

For the monoidal category **Hilb**, equation (6.1) does not obviously give a well-defined object: what would the inner product be? Indeed, **Hilb** is not closed. In finite dimension, however, we can take the so-called *Hilbert-Schmidt inner product* $\langle f \mid g \rangle = \text{Tr}(f^\dagger \circ g)$. In general, objects that have duals always have exponentials!

Lemma 6.2. *If an object A in a symmetric monoidal category has a dual A^*, and B is any object, then $B^A := A^* \otimes B$ is an exponential.*

Proof. Define the evaluation map by

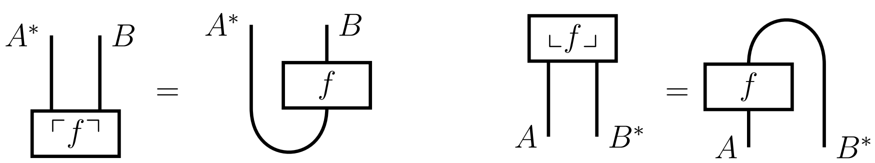
$$\text{ev} = \lambda_B \circ (\eta_A \otimes \text{id}_B) \circ \alpha_{A,A^*,B}\colon A \otimes (A^* \otimes B) \to B.$$

It is now trivial to check equation (6.2). □

Hence we can think of an object A in a compact category as an *output* type, and its dual A^* as the corresponding *input* type. According to our definitions, the previous lemma says that compact categories are always closed. Regardless, compact categories are sometimes also called compact closed categories.

Taking $B = I$ in Lemma 6.2 gives an especially nice setting. We can encode morphisms as states in this way. We repeat the definition of names and conames from Definition 3.3.

Definition 6.3 (Name, coname)**.** The *name* of a morphism $f\colon A \to B$ in a compact category is the morphism $\ulcorner f \urcorner = (\text{id}_{A^*} \otimes f) \circ \eta_A\colon I \to A^* \otimes B$. Its *coname* is the morphism $\llcorner f \lrcorner = \varepsilon_B \circ (f \otimes \text{id}_{B^*})\colon A \otimes B^* \to I$.



(6.3)

This is also called *map-state duality* or the *Choi-Jamiołkowski isomorphism.* With this preparation, we can get back to thinking about copying and deleting.

6.2 Uniform deleting

The counit $A \xrightarrow{e} I$ of a comonoid A tells us we can 'forget' about A if we want to. In other words, we can delete the information contained in A. It is perfectly possible to delete individual systems like this. The no-deleting theorem only prohibits a systematic way of deleting arbitrary systems.

What happens when *every* object in our category can be deleted *systematically*? In our setting, deleting systematically means that the deleting operations respect the categorical structure of composition and tensor products. This means that deleting is *uniform*, in the sense that it doesn't matter if we delete something right away, or first process it for a while and then delete the result. In that case, we can say something quite dramatic.

Definition 6.4 (Uniform deleting)**.** A monoidal category has *uniform deleting* if there is a natural transformation $A \xrightarrow{e_A} I$ with $e_I = \mathrm{id}_I$, making the following diagram commute for all objects A and B:

$$\begin{array}{ccc} & A \otimes B & \\ {}^{e_A \otimes e_B}\swarrow & & \searrow^{e_{A\otimes B}} \\ I \otimes I & \xrightarrow[\lambda_I]{} & I \end{array} \tag{6.4}$$

We now show that uniform deleting has significant effects in a compact category.

Definition 6.5 (Preorder)**.** A *preorder* is a category that has at most one morphism $A \to B$ for any pair of objects A, B.

Theorem 6.6 (Deleting collapse)**.** *If a compact category has uniform deleting, then it must be a preorder.*

Proof. Let $A \xrightarrow{f,g} B$ be morphisms. Naturality of e makes the following diagram commute.

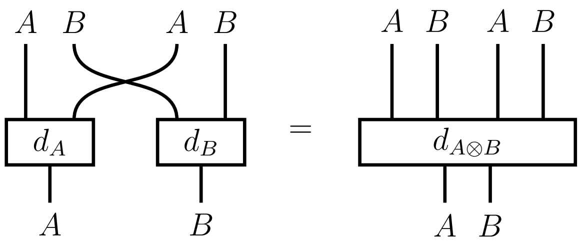
$$\begin{array}{ccc} A \otimes B^* & \xrightarrow{e_{A\otimes B^*}} & I \\ {\scriptstyle \llcorner f \lrcorner}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_I} \\ I & \xrightarrow{e_I} & I \end{array} \tag{6.5}$$

But because deleting is uniform, $e_I = \mathrm{id}_I$. So $\llcorner f \lrcorner = e_{A\otimes B^*}$, and similarly $\llcorner g \lrcorner = e_{A\otimes B^*}$. Hence $f = g$. □

6.3 Uniform copying

We now move to uniform copying. The comultiplication $A \xrightarrow{d} A \otimes A$ of a comonoid lets us copy the information contained in one object A. What happens if we have this ability for all objects, systematically?

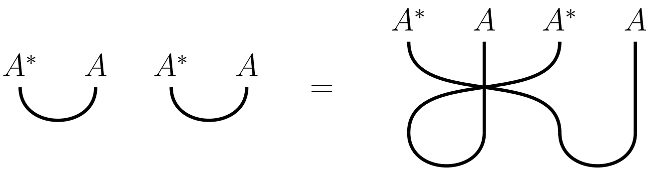
Definition 6.7 (Uniform copying)**.** A symmetric monoidal category has *uniform copying* if there is a natural transformation $A \xrightarrow{d_A} A \otimes A$ with $d_I = \rho_I$, satisfying equations (4.2) and (4.3), and making the following diagram commute for all objects A, B.



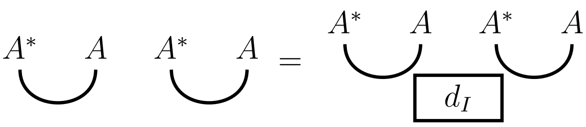
(6.6)

This turns out to be a drastic restriction on the category, as we will see in the Copying collapse theorem below. First we need some preparatory lemmas.

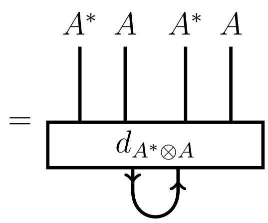
Lemma 6.8. *If a compact category has uniform copying, then*



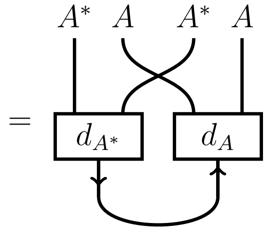
Proof. First, consider the following equalities.



(because $d_I = \rho_I$)

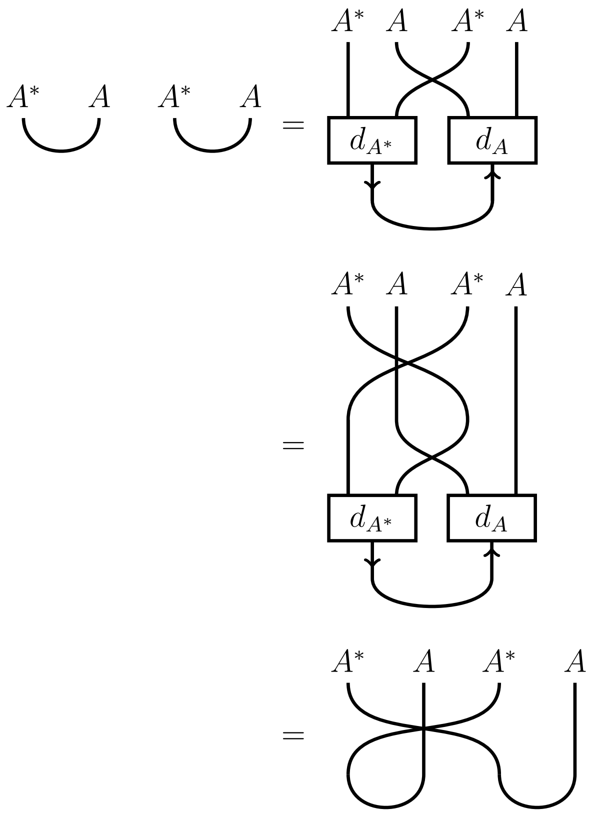


(by naturality of d)



(by equation (6.6))

Let's temporarily call this equation (∗). Then:



(by equation (∗))

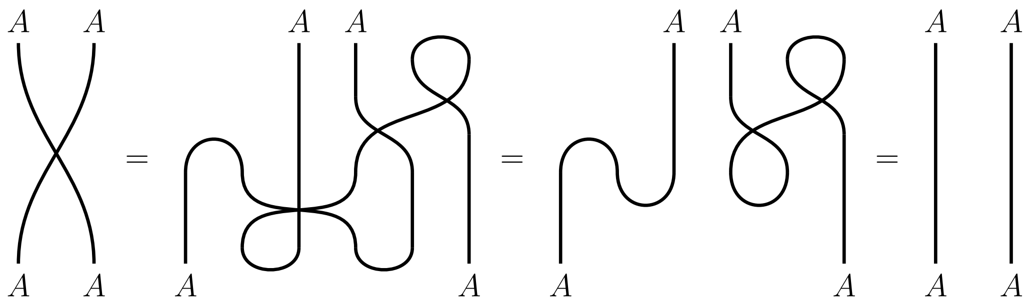
(by equation (4.2))

(by equation (∗))

□

Lemma 6.9. *If a compact category has uniform copying, then* $\sigma_{A,A} = \mathrm{id}_{A\otimes A}$.

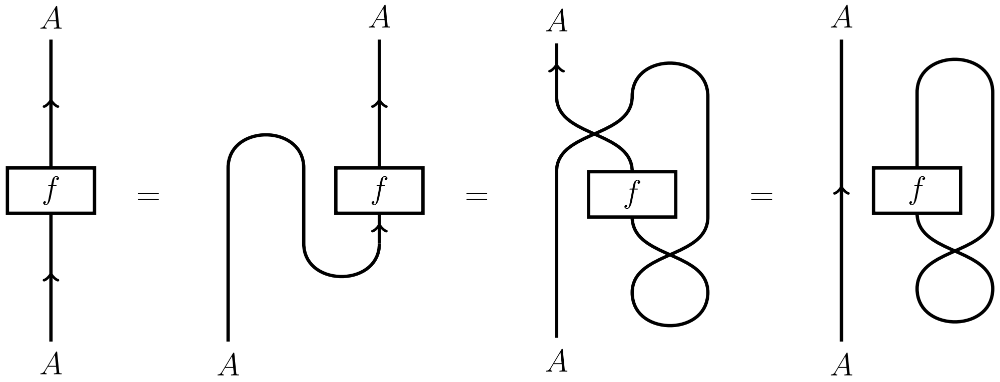
Proof.



The middle equation is Lemma 6.8, and the outer equations are standard operations in a symmetric monoidal category. □

Theorem 6.10 (Copying collapse)**.** *If a compact category has uniform copying, then every endomorphism is a scalar multiple of the identity. In fact,* $f = \mathrm{Tr}(f) \bullet \mathrm{id}_A$ *for any* $A \xrightarrow{f} A$.

Proof.



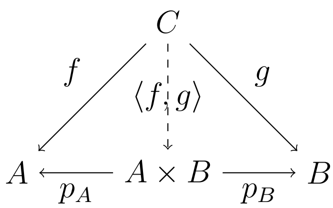
The central equality makes use of Lemma 6.9. □

Thus, if a compact category has uniform copying, all endo-homsets are 1-dimensional, in the sense that they are in bijection with the scalars. Hence, in this sense, all objects are 1-dimensional, and the category degenerates.

6.4 Products

Let's forget about compact structure for this section. What happens when a symmetric monoidal category has uniform copying and deleting? When we phrase the latter property right, it turns out to imply that the tensor product is an actual (categorical) product. First recall what products are.

Definition 6.11 (Products)**.** A *product* of two objects A, B in a category is an object $A \times B$ together with morphisms $A \times B \xrightarrow{p_A} A$ and $A \times B \xrightarrow{p_B} B$, such that every diagram as below has a unique morphism $\langle f, g \rangle$ making both triangles commute.



(6.7)

An object I is *terminal* when there is a unique morphism $A \xrightarrow{!_A} I$ for every object A. A category that has a terminal object and products for all pairs of objects is called *cartesian.*

For an example, let's temporarily go back to the compact case. There, uniform deleting implies that I is terminal. But in general, I being terminal is strictly stronger than uniform deleting.

Lemma 6.12. *Let* **C** *be a monoidal category.*

1. *If the tensor unit I is terminal, then* **C** *has uniform deleting.*

2. *If* **C** *is compact and has uniform deleting, then its tensor unit I is terminal.*

Proof. If I is terminal, we can define $!_A = e_A \colon A \to I$. This will automatically satisfy naturality as well as equation (6.4). For the second part, notice that any object A has at least one morphism $A \to I$, namely e_A. By the deleting collapse theorem 6.6, this must be the only morphism of that type. □

Now we can make precise when tensor products are (categorical) products. We will clearly need uniform copying and deleting. Additionally, the copying and deleting operators have to form comonoids, and the tensor unit has to be terminal.

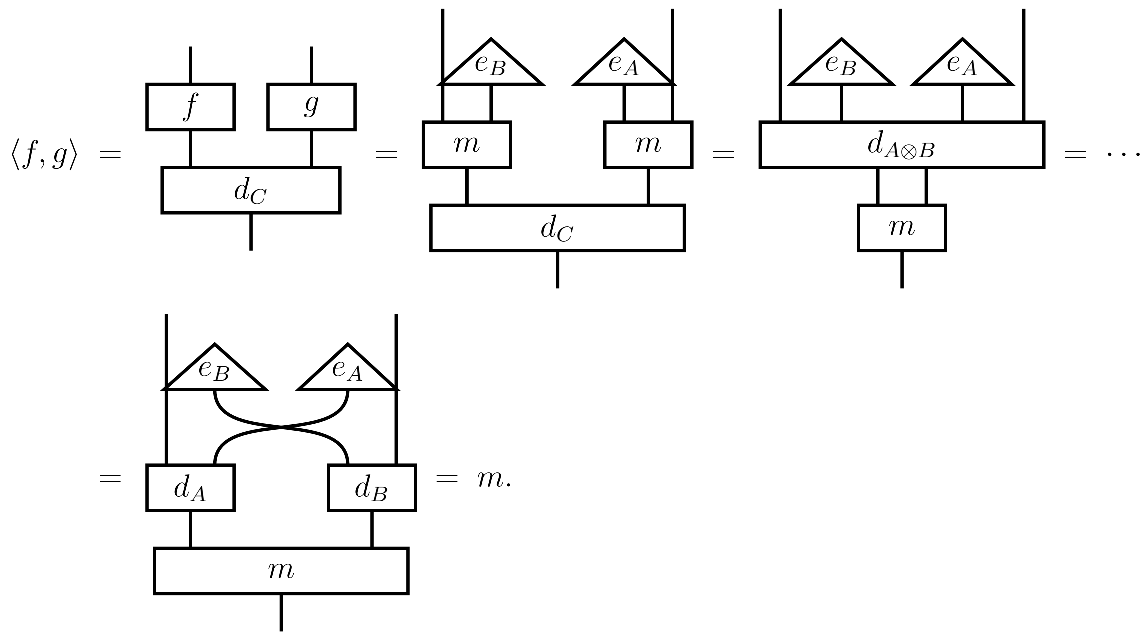
Theorem 6.13. *The following are equivalent for a symmetric monoidal category:*

- *it is Cartesian; more precisely, tensor products are products;*
- *it has uniform copying and deleting, I is terminal, and equation* (4.4) *holds.*

Proof. If the category is Cartesian, it is trivial to see that $e_A = !_A$ and $d_A = \langle \mathrm{id}_A, \mathrm{id}_A \rangle$ provide uniform copying and deleting operators that moreover satisfy (4.4). Moreover, I is by definition terminal.

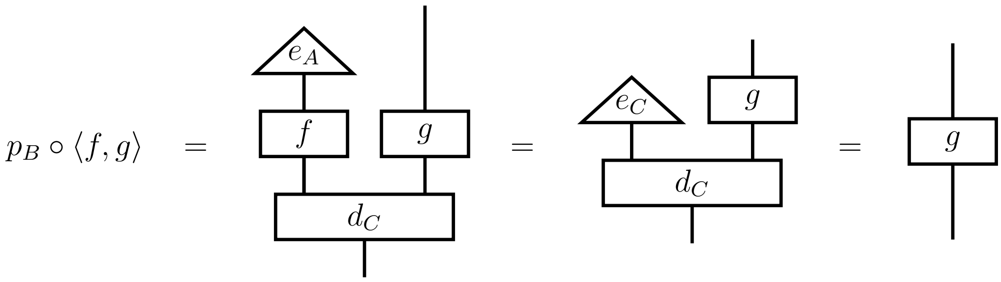
For the converse, we need to prove that $A \otimes B$ is a product of A and B. Define $p_A = \rho_A \circ (\mathrm{id}_A \otimes !_B) \colon A \otimes B \to A$ and $p_B = \lambda_B \circ (!_A \otimes \mathrm{id}_B) \colon A \otimes B \to I$. For given $C \xrightarrow{f} A$ and $C \xrightarrow{g} B$, define $\langle f, g \rangle = (f \otimes g) \circ d$.

First, suppose $C \xrightarrow{m} A \otimes B$ satisfies $p_A \circ m = f$ and $p_B \circ m = g$; we show that $m = \langle f, g \rangle$.



The second equality is our assumption, the third equality is naturality of d, the fourth equality is equation (6.6), and the last equality follows from equation (4.4). Hence mediating morphisms, if they exist, are unique: they all equal $\langle f, g\rangle$.

Finally, we show that $\langle f, g\rangle$ indeed satisfies (6.7).



The first equality holds by definition, the second equality is naturality of e, and the last equality is equation (4.4). Similarly $p_A \circ \langle f, g\rangle = f$. □

6.5 Exercises

Notes and further reading

The no-cloning theorem was proved in 1982 independently by Wootters and Zurek, and Dieks [72, 25]. The categorical version we presented here is due to Abramsky in 2010 [3]. The no-deleting theorem we presented is due to Coecke and was also published in that paper.

Theorem 6.13 is "folklore": it has long been known by category theorists, but seems never to have been published. Jacobs gave a logically oriented account in 1994 [37]. It should be mentioned here that, in compact categories, products are automatically biproducts, which was proved by Houston in 2008 [36].

The notion of closure of monoidal categories is the starting point for a large area called *enriched* category theory [44]. Exponentials also play an important role in categorical logic, namely that of implications between logical formulae.

Chapter 7

Complete positivity

In Chapter 6 we have seen that the kind of categories we consider do not support uniform copying and deleting. However, that does not yet guarantee they model quantum mechanics. Classical mechanics might have copying, and quantum mechanics might not, but statistical mechanics, for example, has no copying either. What really sets quantum mechanics apart is the fact that uniform broadcasting is impossible. This means we have to add another layer of structure to our categories. This chapter studies a beautiful construction with which we don't have to step outside the realm of dagger compact categories after all. As a result, we show that broadcasting is impossible, finishing our categorical setup capturing quantum mechanics.

The key point is that in quantum mechanics, we often do not know precisely what pure state a system is in, but we do know that it is in one of several pure states with certain probability. This leads to general states being convex sums of pure states, which can conveniently be captured using *density matrices* — positive matrices with unit trace. We will not concern ourselves with the trace condition. Recall that unlike superposition, which is inherent to the physical system, these probabilities only represent our own (lack of) knowledge about the system.

7.1 Complete positivity

We have defined *states* as morphisms $I \xrightarrow{\psi} A$. Such a state is *normal* when $\psi^\dagger \circ \psi = \mathrm{id}_I$. In the category **Hilb**, normal states thus correspond to normal vectors, i.e. vectors ψ on the unit sphere, i.e. $\|\psi\| = 1$. However, in this chapter it will be more convenient to think of the rank 1 map $\psi \circ \psi^\dagger \colon A \to A$ induced by a (pure) state.

Definition 7.1 (Pure state)**.** A *pure state* of an object A is a morphism $A \to A$ of the form $\psi \circ \psi^\dagger$ for a morphism $\psi \colon I \to A$ with $\psi^\dagger \circ \psi = \mathrm{id}_I$.

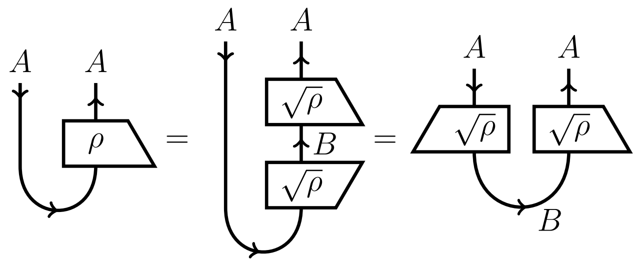
Hence pure states are by definition positive maps. Then, abstracting from the category **Hilb**, general states, also called mixed states, are convex sums of pure states.

Definition 7.2 (Mixed state)**.** A *mixed state* of an object A is a positive morphism $A \xrightarrow{\rho} A$.

When working in compact categories, instead of morphisms $A \to B$, we can equivalently work with matrices $I \to A^* \otimes B$ by taking names (see Definition 3.3).

Definition 7.3 (Positive matrix)**.** A *positive matrix* is a morphism $I \xrightarrow{\ulcorner \rho \urcorner} A^* \otimes A$ that is the name of a positive morphism $A \xrightarrow{\rho} A$.

Graphically, positive matrices are morphisms of the following form.

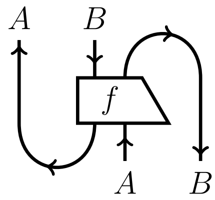


(7.1)

The morphism $\sqrt{\rho}$ and the object B are by no means unique.

Next, we of course want processes to send (mixed) states to (mixed) states. In other words, we are only interested in morphisms $A^* \otimes A \to B^* \otimes B$ that preserve positive matrices. Once again taking our cue from the situation in **FHilb**, these turn out to be the following sort of morphisms.

Definition 7.4 (Completely positive morphism)**.** A morphism $A^* \otimes A \xrightarrow{f} B^* \otimes B$ is *completely positive* when the following morphism $B \otimes A^* \to B \otimes A^*$ is positive.

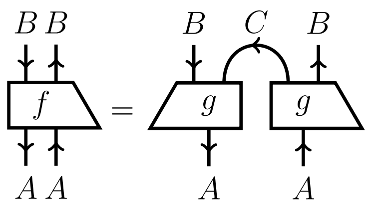


(7.2)

This definition looks fairly abstract, so let's unpack it.

Theorem 7.5 (Stinespring Dilation Theorem)**.** *The following are equivalent:*

1. $A^* \otimes A \xrightarrow{f} B^* \otimes B$ *is completely positive;*
2. *there is an object C and a morphism $A \xrightarrow{g} C \otimes B$ making the following equation true.*



(7.3)

Given a completely positive map f as in the previous theorem, the morphisms g are called its *Kraus morphisms*. Similarly, the object C is called the *ancilla* of f. These are not unique.

7.2 The CP construction

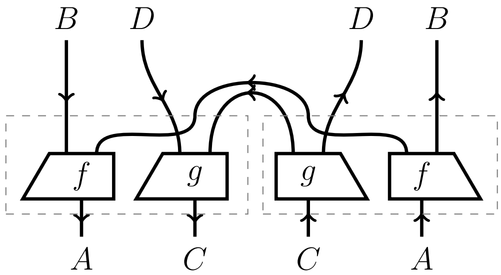
We will now see that completely positive morphisms behave well under our categorical operations, and hence form a well-behaved category in their own right. Thus we will assign to any dagger compact category $\mathbf{C}$ a new one called $\mathrm{CP}(\mathbf{C})$.

Lemma 7.6 (CP respects structure)**.** *In a dagger compact category:*

(i) the identity map $A^* \otimes A \xrightarrow{\mathrm{id}} A^* \otimes A$ *is completely positive;*

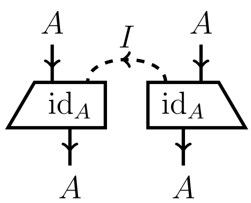
(ii) if $A^* \otimes A \xrightarrow{f} B^* \otimes B$ *and* $B^* \otimes B \xrightarrow{g} C^* \otimes C$ *are completely positive, then so is* $A^* \otimes A \xrightarrow{g \circ f} C^* \otimes C$*;*

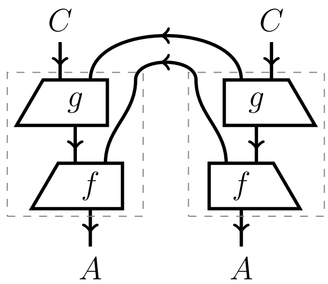
(iii) if $A^* \otimes A \xrightarrow{f} B^* \otimes B$ *and* $C^* \otimes C \xrightarrow{g} D^* \otimes D$ *are completely positive, then so is*



(7.4)

Proof. This is obvious from the graphical calculus and Theorem 7.5.





□

Definition 7.7 (The CP construction)**.** Given a dagger compact category $\mathbf{C}$, we define a new category $\mathrm{CP}(\mathbf{C})$. Its objects are the same as those of $\mathbf{C}$. A morphism $A \to B$ in $\mathrm{CP}(\mathbf{C})$ is a completely positive morphism $A^* \otimes A \xrightarrow{f} B^* \otimes B$ in $\mathbf{C}$. Composition and identities in $\mathrm{CP}(\mathbf{C})$ are as in $\mathbf{C}$.

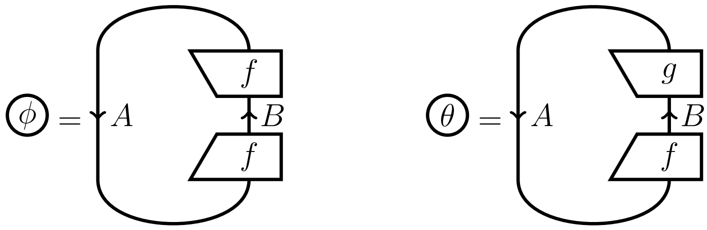
Notice that $\mathrm{CP}(\mathbf{C})$ is indeed a well-defined category by Lemma 7.6.

Lemma 7.8 (CP kills phases)**.** *Let* $\mathbf{C}$ *be a dagger compact category.*

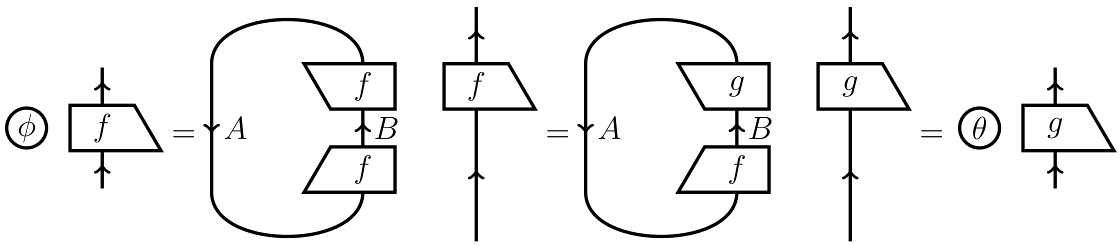
(i) There is a functor $F\colon \mathbf{C} \to \mathrm{CP}(\mathbf{C})$*, defined by* $F(A) = A^* \otimes A$ *and* $F(f) = f_* \otimes f$*.*

(ii) The functor F is faithful up to global phases. More precisely: if $F(f) = F(g)$ for $A \xrightarrow{f,g} B$, then there are scalars $I \xrightarrow{\phi,\theta} I$ with $\phi \bullet f = \theta \bullet g$ and $\phi^\dagger \bullet \phi = \theta^\dagger \bullet \theta$.

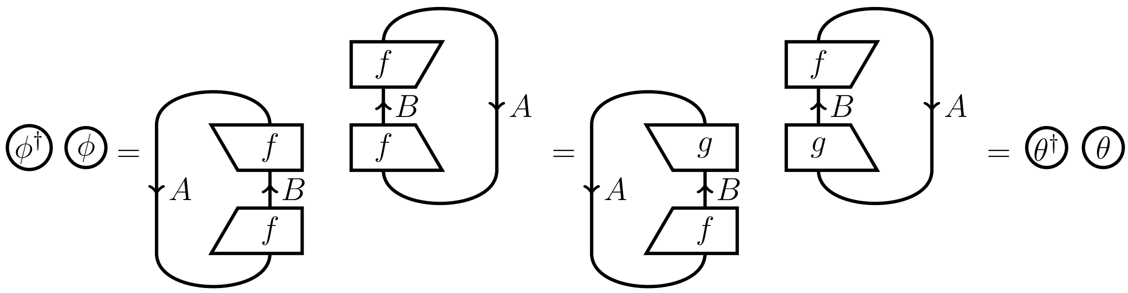
Proof. Part (i) is clear. Let f, g as in part (ii) be given. Define



Then:



And:

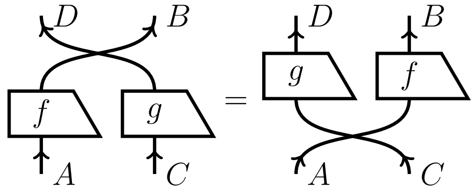


This proof is completely graphical and does not depend on anything like angles. □

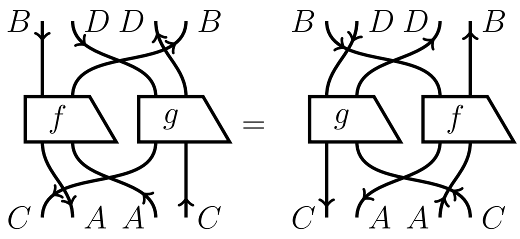
In fact, CP(**C**) is not just a category, but again a dagger compact category.

Theorem 7.9 (CP is dagger compact)**.** *If* **C** *is a dagger compact category, so is* CP(**C**).

Proof. The proof consists of verifying a lot of equations, but the graphical calculus makes them all easy. See Table 7.1 for a dictionary. We check one equation as an example: naturality of σ. To prove that



holds in CP(**C**), we must prove the following equation in **C**.



But this is clearly satisfied. □

Question. What would go wrong if we insisted that morphisms in CP(**C**) preserve trace?

Examples

By spelling out the definition, we see that a morphism $X \times X \xrightarrow{R} Y \times Y$ in **Rel** is completely positive when the following two properties hold for all $x, x' \in X$ and $y, y' \in Y$:

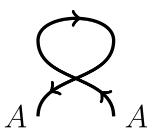
$$(x', x)R(y', y) \iff (x, x')R(y, y'), \tag{7.5}$$
$$(x', x)R(y', y) \implies (x, x)R(y, y). \tag{7.6}$$

In the category **Hilb**, we can identify $(\mathbb{C}^n)^* \otimes \mathbb{C}^n$ with the Hilbert space M_n of n-by-n matrices, with inner product $\langle f \mid g \rangle = \mathrm{Tr}(f^\dagger g)$. By Choi's theorem, completely positive morphisms $\mathbb{C}^m \to \mathbb{C}^n$ in **Hilb** are then precisely what are usually called completely positive maps: a linear map $M_m \xrightarrow{T} M_n$ is called *positive* when it preserves positive matrices, and *completely positive* when $M_m \otimes M_k \xrightarrow{T \otimes \mathrm{id}_{M_k}} M_n \otimes M_k$ is positive. The idea behind this usual definition is that not only T should send states to states, but also regarding T as a local operation on a larger system should send states to states. We can now recognize Theorem 7.5 as the Stinespring Dilation Theorem, and the CP construction of Definition 7.7 as lifting that characterization to a definition.

We can regard the ancilla system C as the "amount of probabilistic mixing" inherent in the completely positive morphism f. Indeed, morphisms in image of the functor $\mathbf{C} \to \mathrm{CP}(\mathbf{C})$ have ancilla system I, and hence no mixing at all. In the case of **Hilb**, the minimal dimension of C make this amount more precise.

7.3 Environment structures

In categories of the form CP(**C**), any object A allows a morphism $A \xrightarrow{\top_A} I$, namely $A^* \otimes A \xrightarrow{\sigma_{A^*,A}} A \otimes A^* \xrightarrow{\varepsilon_A} I \cong I^* \otimes I$.



(7.7)

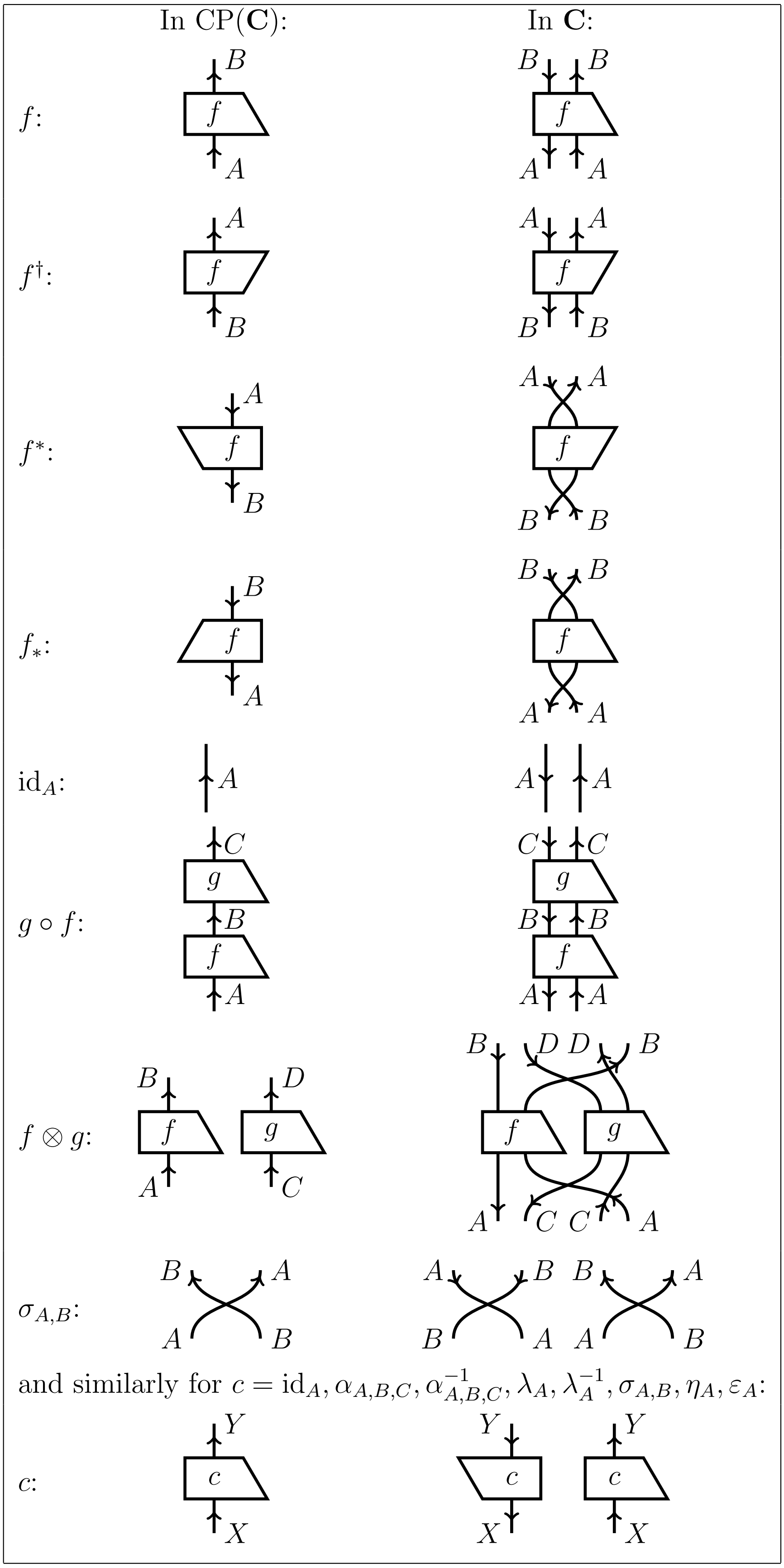


Table 7.1: The CP construction, graphically.

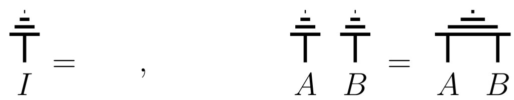
We can think of this morphism as *tracing out* the system A: if $I \xrightarrow{\ulcorner \rho \urcorner} A^* \otimes A$ is the matrix of a map $A \xrightarrow{\rho} A$, then $\top_A \circ \ulcorner \rho \urcorner = \mathrm{Tr}(\rho)\colon I \to I$ by Definition 3.21. As it turns out, we can axiomatize whether a given abstract category is of the form $\mathrm{CP}(\mathbf{C})$ in this way.

Definition 7.10 (Environment structure)**.** An *environment structure* for a dagger compact category $\mathbf{C}$ consists of the following data:

- a dagger compact category $\widehat{\mathbf{C}}$ of which $\mathbf{C}$ is a dagger compact subcategory, that satisfies $\mathrm{Ob}(\widehat{\mathbf{C}}) = \mathrm{Ob}(\mathbf{C})$;
- for each object A, a morphism $A \xrightarrow{\top_A} I$, depicted as ;

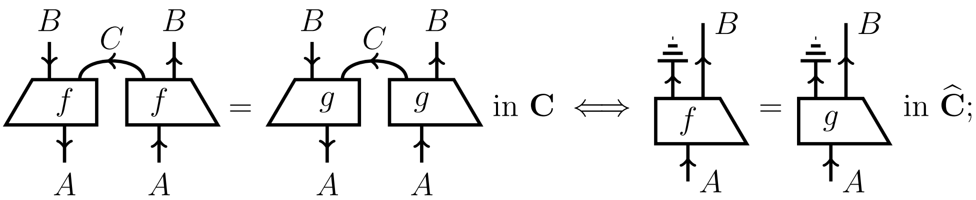
satisfying the following properties:

(i) $\top_I = \mathrm{id}_I$ and $\top_{A \otimes B} = (\top_A \otimes \top_B) \circ \lambda_I$;



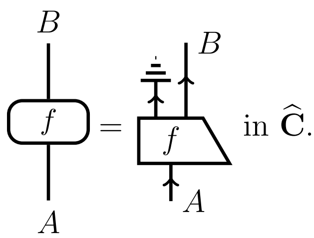
(7.8)

(ii) for all $A \xrightarrow{f,g} C \otimes B$ in $\mathbf{C}$:



(7.9)

(iii) for each $A \xrightarrow{\widehat{f}} B$ in $\widehat{\mathbf{C}}$ there is $A \xrightarrow{f} C \otimes B$ such that



(7.10)

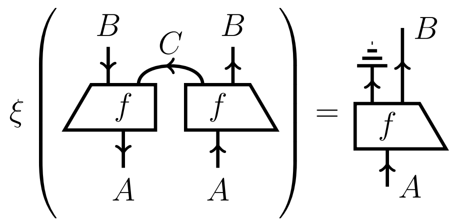
Morphisms in $\widehat{\mathbf{C}}$ are depicted with round corners.

Intuitively, we think of $\mathbf{C}$ as consisting of pure states, and the supercategory $\widehat{\mathbf{C}}$ of containing mixed states. Condition (7.10) then reads that every mixed state can be regarded as a pure state in an extended system. The idea behind the ground symbol is that the ancilla system becomes the 'environment', into which our system is plugged.

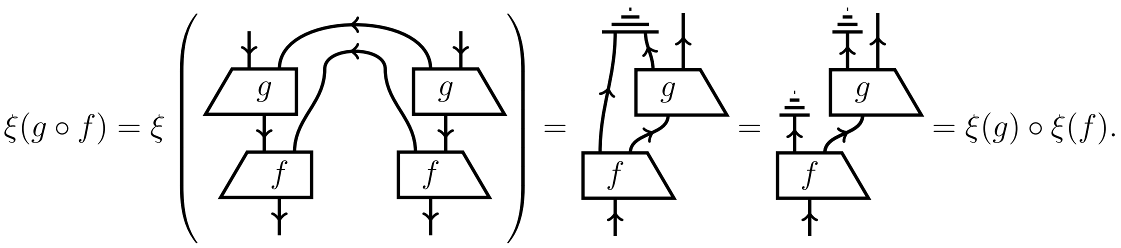
Starting with a dagger compact category **C**, write **D** for the image of the functor **C** $\to$ CP(**C**. Explicitly, **D** is the subcategory of CP(**C**) whose morphisms can be written with ancilla I. (Don't forget that **C** $\to$ CP(**C**) is not faithful, see Lemma 7.8!) This category **D** is clearly dagger compact again. Then **D** has an environment structure with $\widehat{\mathbf{D}}$ = CP(**C**), and $\top_A$ given by (7.7). Conversely, having an environment structure is essentially the same as working with a category of completely positive morphisms, as the following theorem shows.

Theorem 7.11. *If a dagger compact category* **C** *comes with an environment structure, then there is an invertible functor* $\xi\colon$ CP(**C**) $\to \widehat{\mathbf{C}}$ *that satisfies* $\xi(A) = A$ *on objects and* $\xi(f \otimes g) = \xi(f) \otimes \xi(g)$ *on morphisms.*

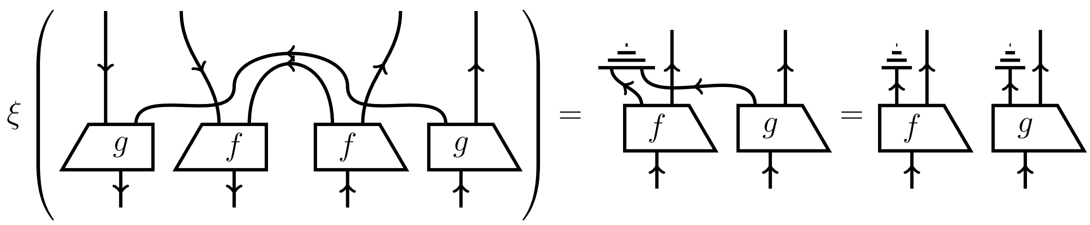
Proof. Define ξ by $\xi(A) = A$ on objects, and as follows on morphisms.



This is indeed functorial by (7.8):



It is obvious that the functor ξ is invertible: (7.9) shows that it faithful, and (7.10) shows that it is full. Finally, by (7.8):



So $\xi(f \otimes g) = \xi(f) \otimes \xi(g)$. □

Environment structures give us a convenient way to graphically handle categories of completely positive maps, because we do not have to "double" the pictures all the time.

7.4 Exercises

Notes and further reading

The use of completely positive maps originated for algebraic reasons in operator algebra theory, and dates back at least to 1955, when Stinespring proved his dilation theorem [69]. Quantum information theory could be said to have grown out of operator algebra theory, and repurposed completely positive maps. See also the textbooks [56, 12].

The CP construction is due to Selinger in 2007 [65]. Coecke and Heunen subsequently realized in 2011 that compactness is not necessary for the construction, and it therefore also works for infinite dimensional Hilbert spaces [20].

Environment structures are due to Coecke [15, 21].

The no-broadcasting theorem was proved in 1996 and is due to Barnum, Caves, Jozsa, Fuchs and Schumacher [8].

Bibliography

[1] Lowell Abrams. *Frobenius algebra structures in topological quantum field theory and quantum cohomology.* PhD thesis, Johns Hopkins University, 1997.

[2] Samson Abramsky. Abstract scalars, loops, and free traced and strongly compact closed categories. In *Algebra and Coalgebra in Computer Science*, CALCO'05, pages 1–30. Springer, 2005.

[3] Samson Abramsky. No-cloning in categorical quantum mechanics. In Simon Gay and Ian Mackey, editors, *Semantic Techniques in Quantum Computation*, pages 1–28. Cambridge University Press, 2010.

[4] Samson Abramsky and Bob Coecke. A categorical semantics of quantum protocols. In *Logic in Computer Science 19*, pages 415–425. IEEE Computer Society, 2004.

[5] Samson Abramsky and Bob Coecke. Abstract physical traces. *Theory and Applications of Categories*, 14(6):111–124, 2005.

[6] Samson Abramsky and Chris Heunen. H*-algebras and nonunital Frobenius algebras: first steps in infinite-dimensional categorical quantum mechanics. *Clifford Lectures, AMS Proceedings of Symposia in Applied Mathematics*, 2011.

[7] John Baez and James Dolan. Higher-dimensional algebra and topological quantum field theory. *Journal of Mathematical Physics*, 36:6073–6105, 1995.

[8] Howard Barnum, Carlton M. Caves, Chris A. Fuchs, Richard Jozsa, and Benjamin Schumacher. Noncommuting mixed states cannot be broadcast. *Physical Review Letters*, 76(15):2818–2821, 1996.

[9] Michael Barr. **-autonomous categories*, volume 752 of *Lecture Notes in Mathematics.* Springer, 1979.

[10] Jean Bénabou. Categories avec multiplication. *Comptes Rendus de l'Acadmie des Sciences. Série I. Mathmatique*, pages 1887–1890, 1963.

[11] Charles H. Bennett, Giles Brassard, Claue Crépeau, Richard Jozsa, Asher Peres, and William K. Wootters. Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels. *Physical Review Letters*, 70:1895–1899, 1993.

[12] Rejandra Bhatia. *Positive definite matrices.* Princeton University Press, 2007.

[13] Francis Borceux. *Handbook of Categorical Algebra.* Encyclopedia of Mathematics and its Applications 50–52. Cambridge University Press, 1994.

[14] Stephen U. Chase and Moss Sweedler. *Hopf algebras and Galois theory.* Number 97 in Lecture Notes in Mathematics. Springer, 1969.

[15] Bob Coecke. De-linearizing linearity: projective quantum axiomatics from strong compact closure. In Peter Selinger, editor, *QPL 5*, volume 170 of *Electronic Notes in Theoretical Computer Science*, pages 49–72, 2007.

[16] Bob Coecke, editor. *New structures for physics.* Number 813 in Lecture Notes in Physics. Springer, 2011.

[17] Bob Coecke and Ross Duncan. Interacting quantum observables. In *International Colloquium on Automata, Languages and Programming*, volume 5126 of *Lecture Notes in Computer Science*, pages 298–310. Springer, 2008.

[18] Bob Coecke, Ross Duncan, Aleks Kissinger, and Quanlong Wang. Strong complementarity and non-locality in categorical quantum mechanics. *to appear*, 2012.

[19] Bob Coecke, Bill Edwards, and Robert W. Spekkens. Phase groups and the origin of non-locality for qubits. In Bob Coecke, Prakash Panangaden, and Peter Selinger, editors, *QPL 9*, volume 270 of *Electronic Notes in Theoretical Computer Science*, pages 15–36, 2011.

[20] Bob Coecke and Chris Heunen. Pictures of complete positivity in arbitrary dimension. *Proceedings of QPL 2011*, 2011.

[21] Bob Coecke, Éric O. Paquette, and Simon Perdrix. Bases in diagrammatic quantum protocols. In *Mathematical Foundations of Programming Semantics 24*, volume 218 of *Electronic Notes in Theoretical Computer Science*, pages 131–152. Elsevier, 2008.

[22] Bob Coecke and Duško Pavlović. Quantum measurements without sums. In *Mathematics of Quantum Computing and Technology.* Taylor and Francis, 2007.

[23] Bob Coecke, Duško Pavlović, and Jamie Vicary. A new description of orthogonal bases. *Mathematical Structures in Computer Science*, 2009.

[24] Brian J. Day. Note on compact closed categories. *Journal of the Australian Mathematical Society*, Series A 24(3):309–311, 1977.

[25] Dennis Dieks. Communication by EPR devices. *Physics Letters A*, 92(6):271–272, 1982.

[26] Robbert Dijkgraaf. *A geometric approach to two dimensional conformal field theory.* PhD thesis, University of Utrecht, 1989.

[27] Ross Duncan and Simon Perdrix. Graph states and the necessity of euler decomposition. In Klaus Ambos-Spies, Benedikt Löwe, and Wolfgang Merkle, editors, *Computability in Europe*, volume 5635 of *Lecture Notes in Computer Science*, pages 167–177. Springer, 2009.

[28] Bill Edwards. *Non-locality in categorical quantum mechanics*. PhD thesis, Oxford University, 2009.

[29] Peter Freyd. *Abelian Categories: An introduction to the theory of functor*. Harper and Row, 1964.

[30] F. Georg Frobenius. Theorie der hyperkomplexen grössen. *Sitzungsberichte der Koniglich Preussischen Akademie Der Wissenschaften*, 24:504–537; 634–645, 1903.

[31] Alexandre Grothendieck. Pursuing stacks. *Documents Mathématiques, Société Mathétique de France*, 1983. Letter to Daniel Quillen.

[32] Masahito Hasegawa. On traced monoidal closed categories. *Mathematical Structures in Computer Science*, 19:217–244, 2008.

[33] Chris Heunen. An embedding theorem for Hilbert categories. *Theory and Applications of Categories*, 22(13):321–344, 2009.

[34] Chris Heunen, Ivan Contreras, and Alberto S. Cattaneo. Relative Frobenius algebras are groupoids. *to appear in Journal of Pure and Applied Algebra*, 2012.

[35] Heinz Hopf. Über die Topologie der Gruppen-Mannigfaltigkeiten und ihrer Verallgemeinerungen. *Annals of Mathematics*, 42:22–52, 1941.

[36] Robin Houston. Finite products are biproducts in a compact closed category. *Journal of Pure and Applied Algebra*, 212(2):394–400, 2008.

[37] Bart Jacobs. Semantics of weakening and contraction. *Annals of Pure and Applied Logic*, 69:73–106, 1994.

[38] André Joyal and Ross Street. The geometry of tensor calculus I. *Advances in Mathematics*, 88:55–113, 1991.

[39] André Joyal and Ross Street. An introduction to Tannaka duality and quantum groups. In *Category Theory, Part II*, volume 1488 of *Lecture Notes in Mathematics*, pages 411–492. Springer, 1991.

[40] André Joyal and Ross Street. Braided tensor categories. *Advances in Mathematics*, 102:20–78, 1993.

[41] André Joyal, Ross Street, and Dominic Verity. Traced monoidal categories. *Mathematical Proceedings of the Cambridge Philosophical Society*, 3(447–468), 1996.

[42] Christian Kassel. *Quantum Groups.* Springer, 1995.

[43] G. Max Kelly. Many variable functorial calculus (I). In *Coherence in Categories*, volume 281 of *Lecture Notes in Mathematics*, pages 66–105. Springer, 1970.

[44] G. Max Kelly. *Basic Concepts of Enriched Category Theory.* Cambridge University Press, 1982.

[45] G. Max Kelly and Miguel L. Laplaza. Coherence for compact closed categories. *Journal of Pure and Applied Algebra*, 19:193–213, 1980.

[46] Joachim Kock. *Frobenius algebras and 2-D Topological Quantum Field Theories.* Number 59 in London Mathematical Society Student Texts. Cambridge University Press, 2003.

[47] F. William Lawvere. Ordinal sums and equational doctrines. In Beno Eckmann, editor, *Seminar on triples and categorical homology theory*, number 80 in Lecture Notes in Mathematics, pages 141–155, 1967.

[48] Tom Leinster. *Higher operads, higher categories.* Number 298 in London Mathematical Society Lecture Note Series. Cambridge University Press, 2004.

[49] Saunders Mac Lane. Duality for groups. *Bulletin of the American Mathematical Society*, 56(6):485–516, 1950.

[50] Saunders Mac Lane. An algebra of additive relations. *Proceedings of the National Academy of Sciences*, 47:1043–1051, 1961.

[51] Saunders Mac Lane. Natural associativity and commutativity. *Rice University Studies*, 49:28–46, 1963.

[52] Saunders Mac Lane. *Categories for the Working Mathematician.* Springer, 2nd edition, 1971.

[53] Barry Mitchell. *Theory of Categories.* Academic Press, 1965.

[54] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information.* Cambridge University Press, 2000.

[55] Paul H. Palmquist. Adjoint functors induced by adjoint linear transformations. *Proceedings of the American Mathematical Society*, 44(2):251–254, 1974.

[56] Vern Paulsen. *Completely bounded maps and operators algebras.* Cambridge University Press, 2002.

[57] Duško Pavlović. Quantum and classical structures in nondeterministic computation. In P. Bruza et al., editor, *Third International symposium on Quantum Interaction*, volume 5494 of *Lecture Notes in Artificial Intelligence*, pages 143–157. Springer, 2009.

[58] Roger Penrose. Applications of negative dimensional tensors. In *Combinatorial mathematics and its applications*, pages 221–244, 1971.

[59] Simon Perdrix. State transfer instead of teleportation in measurement-based quantum computation. *International journal of quantum information*, 3(1):219–223, 2005.

[60] Frank Quinn. Lectures on axiomatic topological quantum field theory. In *Geometry and quantum field theory*, pages 323–453. American Mathematical Society, 1995.

[61] Robert Raussendorf and Hans J. Briegel. A one-way quantum computer. *Physical Review Letters*, 86(22):5188, 2001.

[62] Léon Rosenfeld. *Foundations of Quantum Physics II*, chapter Complementarity: Bedrock of the quantal description, pages 284–285. Number 7 in Niels Bohr, collected works. Elsevier, 1996.

[63] Julian Schwinger. Unitary operator bases. *Proceedings of the National Academy of Sciences: Physics*, 46:570–579, 1960.

[64] Robert A. G. Seely. Linear logic, $*$-autonomous categories and cofree coalgebras. In *Categories in Computer Science and Logic*, volume 92, pages 371–382. American Mathematical Society, 1989.

[65] Peter Selinger. Dagger compact closed categories and completely positive maps. In *Quantum Programming Languages*, volume 170 of *Electronic Notes in Theoretical Computer Science*, pages 139–163. Elsevier, 2007.

[66] Peter Selinger. Finite dimensional Hilbert spaces are complete for dagger compact closed categories. In *Quantum Physics and Logic*, Electronic Notes in Theoretical Computer Science, 2008.

[67] Peter Selinger. Idempotents in dagger categories. In *Quantum Programming Languages*, volume 210 of *Electronic Notes in Theoretical Computer Science*, pages 107–122. Elsevier, 2008.

[68] Peter Selinger. A survey of graphical languages for monoidal categories. In *New Structures for Physics*, Lecture Notes in Physics. Springer, 2009.

[69] W. Forrest Stinespring. Positive functions on C*-algebras. *Proceedings of the American Mathematical Society*, pages 211–216, 1955.

[70] Ross Street. *Quantum Groups: a path to current algebra*. Number 19 in Australian Mathematical Society Lecture Series. Cambridge University Press, 2007.

[71] Jamie Vicary. Completeness of †-categories and the complex numbers. *Journal of Mathematical Physics*, 52:082104, 2011.

[72] William K. Wootters and Wojciech H. Zurek. A single quantum cannot be cloned. *Nature*, 299:802–803, 1982.

Index