# (Non-)Succinctness of Uniform Interpolants of General Terminologies in the Description Logic $\mathcal{EL}$[☆]

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**Abstract**

$\mathcal{EL}$ is a popular description logic, used as a core formalism in large existing knowledge bases. Uniform interpolants of knowledge bases are of high interest, e.g. in scenarios where a knowledge base is supposed to be partially reused. However, to the best of our knowledge no procedure has yet been proposed that computes uniform $\mathcal{EL}$ interpolants of general $\mathcal{EL}$ terminologies. Up to now, also the bound on the size of uniform $\mathcal{EL}$ interpolants has remained unknown. In this article, we propose an approach to computing a finite uniform interpolant for a general $\mathcal{EL}$ terminology if it exists. Further, we show that, if such a finite uniform $\mathcal{EL}$ interpolant exists, then there exists one that is at most triple exponential in the size of the original TBox, and that, in the worst-case, no smaller interpolants exist, thereby establishing tight worst-case bounds on their size.

*Keywords:* Ontologies, Knowledge Representation, Automated Reasoning, Description Logis, Uniform Interpolation, Forgetting, $\mathcal{EL}$

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[☆]This is a revised and extended version of previous work [1].

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## 1. Introduction

With the wide-spread adoption of ontological modeling by means of the W3C-specified OWL Web Ontology Language [2], description logics (DLs, [3, 4]) have developed into one of the most popular family of formalisms employed for knowledge representation and reasoning [5, 6, 7, 8]. For application scenarios where scalability of reasoning is of utmost importance, specific tractable sublanguages (the so-called *profiles* [9]) of OWL have been put into place, among them OWL 2 EL which in turn is based on DLs of the $\mathcal{EL}$ family [10, 11].

In view of the practical deployment of OWL and its profiles [12, 13, 14], non-standard reasoning services for supporting modeling activities gain in importance. An example of such reasoning services supporting knowledge engineers in different tasks is that of *uniform interpolation*: given a theory using a certain vocabulary, and a subset of "relevant terms" of that vocabulary, find a theory (referred to as a *uniform interpolant*, short: UI) that uses only the relevant terms and gives rise to the same consequences (expressible via relevant terms) as the original theory. Intuitively, this provides a view on the ontology where all irrelevant (asserted as well as implied) statements have been filtered out.

Uniform interpolation has many applications within ontology engineering. For instance, it can help ontology engineers understand existing ontological specifications by visualizing implicit dependencies between relevant concepts and roles, as used, for instance, for interactive ontology revision [15]. In particular for understanding and developing complex knowledge bases, e.g., those consisting of *general concept inclusions* (GCIs), appropriate tool support of this kind would be beneficial. Another application of uniform interpolation is ontology reuse: given an ontology that is to be reused in a different scenario, most likely not all aspects of this ontology are relevant to the new usage requirements. In combination with module extraction, uniform interpolation can be used to reduce the amount of irrelevant information within an ontology employed in a new context.

For DL-Lite, the problem of uniform interpolation has been investigated [16, 17] and a tight exponential bound on the size of uniform interpolants has been shown. Lutz and Wolter [18] propose an approach to uniform interpolation in expressive description logics such as $\mathcal{ALC}$ featuring general terminologies showing a tight triple-exponential bound on the size of uniform interpolants. For the lightweight description logic $\mathcal{EL}$, the problem of uniform interpolation has, however, not been solved. To the best of our knowledge, the only existing approach [19] to uniform interpolation in $\mathcal{EL}$ is restricted to terminologies containing each concept symbol at most once on the left-hand side of concept inclusions and additionally satisfying particular acyclicity conditions which are sufficient, but not necessary for the existence of a uniform interpolant. Recently, Lutz, Seylan and Wolter [20] proposed an ExpTime procedure for deciding, whether a finite uniform $\mathcal{EL}$ interpolant exists for a particular general terminology and a particular set of relevant terms. However, the authors do not address the actual computation of such a uniform interpolant. Up to now, also the bounds on the size of uniform $\mathcal{EL}$ interpolants have remained unknown.

In this paper, we propose a worst-case-optimal approach to computing a finite uniform $\mathcal{EL}$ interpolant for a general terminology. We compute uniform interpolants by rewriting the terminology, i.e., exchanging explicitly given axioms by other axioms which are logically entailed. Since our rewriting approach operates on the syntactic structure of terminologies, the task can be significantly facilitated by converting the terminology into a simplified form in a semantics-preserving way. For this purpose, we make use of a normalization strategy, wherein fresh vocabulary elements are introduced in order to obtain a syntactically simple terminology that provides for vocabulary elements finite sets of so-called subsumees and subsumers. We show via a proof-theoretic analysis that this representation does indeed capture all consequences of the initial terminology expressed using the set of relevant terms.

This specific normalized form can then be transformed into regular tree grammars, whose corresponding tree languages are used to represent (possibly infinite) sets of consequences. We show that certain finite subsets of the languages generated by these grammars can be transformed into a uniform $\mathcal{EL}$ interpolant of at most triple exponential size, if such a finite uniform $\mathcal{EL}$ interpolant exists for the given terminology and a set of terms. Further, we show that, in the worst-case, no shorter interpolants exist, thereby establishing tight bounds on the size of uniform interpolants in $\mathcal{EL}$.

The paper is structured as follows: In Section 2, we recall the necessary preliminaries on $\mathcal{EL}$ and regular tree languages/grammars. In Section 3, we introduce a calculus for deriving general subsumptions in $\mathcal{EL}$ terminologies, which is used as a major tool in the proofs of this work. Section 4 formally introduces the notion of inseparability and defines the task of uniform interpolation. Section 5 demonstrates that the smallest uniform interpolants in $\mathcal{EL}$ can be triple exponential in the size of the original knowledge base. In the first part of Section 6, we show that applying flattening to terminologies simplifies tracking of entailed subsumption dependencies. In Section 6.2, we introduce regular tree grammars representing subsumees and subsumers of concept symbols, which are the basis for computing uniform $\mathcal{EL}$ interpolants as shown in Section 6.3. In the same section, we also show the upper bound on the size of uniform interpolants. After giving an overview of related work in Section 7, we summarize the contributions in Section 8 and discuss some ideas for future work. This is a revised and extended version of our previous paper [1], containing a more detailed argumentation, examples and the full proofs.

## 2. Preliminaries

In this section, we formally introduce the description logic $\mathcal{EL}$, and recall some of its well-known properties. Furthermore, we introduce tree grammars, which we will later use as a formal tool to represent infinite sets of $\mathcal{EL}$ concept expressions.

### *2.1. The Description Logic $\mathcal{EL}$*

Let $N_C$ and $N_R$ be countably infinite and mutually disjoint sets called *concept symbols* and *role symbols*, respectively. $\mathcal{EL}$ *concepts* $C$ are defined by

$$C ::= A \mid \top \mid C \sqcap C \mid \exists r.C$$

where $A$ and $r$ range over $N_C$ and $N_R$, respectively. In the following, we use symbols $A, B$ to denote concept symbols (i.e., concepts from $N_C$) or $\top$ and $C, D, E$ to denote arbitrary concepts. We use the term *simple concept* to refer to a simpler form of $\mathcal{EL}$ concepts defined by $C_s ::= A \mid \exists r.A$, where $A$ and $r$ range over $N_C \cup \{\top\}$ and $N_R$, respectively.

A *terminology* or *TBox* consists of *concept inclusion* axioms $C \sqsubseteq D$ and *concept equivalence* axioms $C \equiv D$, the latter used as a shorthand for the mutual inclusion $C \sqsubseteq D$ and $D \sqsubseteq C$.[1] The *signature* of an $\mathcal{EL}$ concept $C$, an axiom $\alpha$ or a TBox $\mathcal{T}$, denoted by $\text{sig}(C)$, $\text{sig}(\alpha)$ or $\text{sig}(\mathcal{T})$, respectively, is the set of concept and role symbols occurring in it. To distinguish between the set of concept symbols and the set of role symbols, we use $\text{sig}_C(\cdot)$ and $\text{sig}_R(\cdot)$, respectively. Further, we use $\text{sub}(\mathcal{T})$ to denote the set of all subformulae of concepts in $\mathcal{T}$.

For a concept $C$, let the *role depth* of $C$ (denoted by $d(C)$) be the maximal nesting depth of existential restrictions within $C$. For instance, $d(\exists r.(\exists s.A \sqcap B) \sqcap$

[1] While knowledge bases in general can also include a specification of individuals with the corresponding concept and role assertions (ABox), in this paper we do not consider ABoxes, but concentrate on TBoxes.

$\exists s.B) = 2$. For a TBox $\mathcal{T}$, the role depth is given by $d(\mathcal{T}) = \mathtt{max}\{d(C) \mid C \in \text{sub}(\mathcal{T})\}$.

Next, we recall the semantics of the DL constructs introduced above, which is defined by the means of interpretations. An *interpretation* $\mathcal{I}$ is given by a set $\Delta^{\mathcal{I}}$, called the *domain*, and an *interpretation function* $\cdot^{\mathcal{I}}$ assigning to each concept $A \in N_C$ a subset $A^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$ and to each role $r \in N_R$ a subset $r^{\mathcal{I}}$ of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. The interpretation of $\top$ is fixed to $\Delta^{\mathcal{I}}$. The interpretation of arbitrary $\mathcal{EL}$ concepts is defined inductively via $(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}$ and $(\exists r.C)^{\mathcal{I}} = \{x \mid (x, y) \in r^{\mathcal{I}}$ and $y \in C^{\mathcal{I}}$ for some $y\}$. An interpretation $\mathcal{I}$ *satisfies* an axiom $C \sqsubseteq D$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. $\mathcal{I}$ is a *model* of a TBox $\mathcal{T}$, if it satisfies all axioms in $\mathcal{T}$. We say that $\mathcal{T}$ entails an axiom $\alpha$ (in symbols, $\mathcal{T} \models \alpha$), if $\alpha$ is satisfied by all models of $\mathcal{T}$. The *deductive closure* of a TBox $\mathcal{T}$ is the set of all axioms entailed by $\mathcal{T}$. For two arbitrary $\mathcal{EL}$ concepts $C, D$ such that $\mathcal{T} \models C \sqsubseteq D$, we call $C$ a *subsumee* of $D$ and $D$ a *subsumer* of $C$.

### *2.2. Model-Theoretic Properties of $\mathcal{EL}$ Concepts*

In the following, we provide some results concerning model-theoretic properties of $\mathcal{EL}$ concept expressions, which are essentially common knowledge. Nevertheless, to make the paper self-contained, we include the proofs in the appendix. We first define pointed interpretations as well as homomorphisms between them. Moreover we define the notion of a characteristic interpretation of an $\mathcal{EL}$ concept expression. Intuitively, a concept's characteristic interpretation describes a partial model with one distinguished element which represents necessary and sufficient conditions for a domain element to be an instance of this concept.

**Definition 1.** *A* pointed interpretation *is a pair $(\mathcal{I}, x)$ with $x \in \Delta^{\mathcal{I}}$. Given two pointed interpretations $(\mathcal{I}_1, x_1)$ and $(\mathcal{I}_2, x_2)$, a* homomorphism *from $(\mathcal{I}_1, x_1)$ to $(\mathcal{I}_2, x_2)$ is a mapping $\varphi : \Delta^{\mathcal{I}_1} \to \Delta^{\mathcal{I}_2}$ such that*

- $\varphi(x_1) = x_2$,
- $x \in A^{\mathcal{I}_1}$ *implies* $\varphi(x) \in A^{\mathcal{I}_2}$ *for all* $A \in N_C$,
- $(x, y) \in r^{\mathcal{I}_1}$ *implies* $(\varphi(x), \varphi(y)) \in r^{\mathcal{I}_2}$ *for all* $r \in N_R$.

*Given an $\mathcal{EL}$ concept expression $C$, we define its* characteristic pointed interpretation *$(\mathcal{I}_C, x_C)$ inductively over the structure of $C$ as follows:*

- *For $A \in N_C \cup \{\top\}$ we let $\Delta^{\mathcal{I}_A} = \{x_A\}$ with*

- $A^{\mathcal{I}_A} = \{x_A\}$,
- $B^{\mathcal{I}_A} = \emptyset$ *for all* $B \in N_C \setminus \{A\}$, *and*
- $r^{\mathcal{I}_A} = \emptyset$ *for all* $r \in N_R$.

- *For* $C = C_1 \sqcap C_2$, *we define* $\Delta^{\mathcal{I}_C} = \{x_C\} \cup \bigcup_{\iota \in \{1,2\}} (\Delta^{\mathcal{I}_{C_\iota}} \setminus \{x_{C_\iota}\}) \times \{\iota\}$ *with*
  - $A^{\mathcal{I}_C} = \{x_C \mid x_{C_1} \in A^{\mathcal{I}_{C_1}}$ *or* $x_{C_2} \in A^{\mathcal{I}_{C_2}}\} \cup \bigcup_{\iota \in \{1,2\}} (A^{\mathcal{I}_{C_\iota}} \setminus \{x_{C_\iota}\}) \times \{\iota\}$ *for all* $A \in N_C$, *and*
  - $r^{\mathcal{I}_C} = \{(x_C, (y, \iota)) \mid (x_{C_\iota}, y) \in r^{\mathcal{I}_{C_\iota}}\} \cup \bigcup_{\iota \in \{1,2\}} \{((y, \iota), (y', \iota)) \mid (y, y') \in r^{\mathcal{I}_{C_\iota}}, y \neq x_{C_\iota}\}$ *for all* $r \in N_R$.

- *For* $C = \exists r.C'$, *we define* $\Delta^{\mathcal{I}_C} = \{x_C\} \cup \Delta^{\mathcal{I}_{C'}}$ *with*
  - $A^{\mathcal{I}_C} = A^{\mathcal{I}_{C'}}$ *for all* $A \in N_C$, *and*
  - $(r')^{\mathcal{I}_C} = \{(x_C, x_{C'}) \mid r' = r\} \cup (r')^{\mathcal{I}_{C'}}$ *for all* $r' \in N_R$.

The subsequent lemma shows that characteristic interpretations indeed characterize $\mathcal{EL}$ concept membership via the existence of appropriate homomorphisms.

**Lemma 1** (structurality of validity of $\mathcal{EL}$ concepts)**.** *For any* $\mathcal{EL}$ *concept expression* $C$ *and any interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ *and* $x \in \Delta^{\mathcal{I}}$ *it holds that* $x \in C^{\mathcal{I}}$ *if and only if there is a homomorphism from* $(\mathcal{I}_C, x_C)$ *to* $(\mathcal{I}, x)$.

The next lemma shows that $\mathcal{EL}$ concept subsumption in the absence of terminological background knowledge can as well be characterized via homomorphisms between characteristic interpretations.

**Lemma 2** (Structurality of $\mathcal{EL}$ concept subsumption)**.** *Let* $C$ *and* $C'$ *be two* $\mathcal{EL}$ *concept expressions. Then* $\emptyset \models C \sqsubseteq C'$ *if and only if there is a homomorphism from* $(\mathcal{I}'_C, x'_C)$ *to* $(\mathcal{I}_C, x_C)$.

The proofs of both lemmas can be found in Appendix A.

### *2.3. Regular Tree Grammars*

We briefly recall the basics of tree languages and regular tree grammars. A *ranked alphabet* is a pair ($\mathcal{F}$, Arity) where $\mathcal{F}$ is a finite set and Arity is a mapping from $\mathcal{F}$ into $\mathbb{N}$. We use superscripts to denote the arity $> 0$ of alphabet symbols, e.g., $f^2(g^1(a), a)$. The set of ground terms over the alphabet $\mathcal{F}$ (which are also simply referred to as *trees*) is denoted by $T(\mathcal{F})$. Let $\mathcal{X}_n$ be a set of $n$ variables. Then, $T(\mathcal{F}, \mathcal{X}_n)$ denotes the set of terms over the alphabet $\mathcal{F}$ and the set of variables $\mathcal{X}_n$. A term $C \in T(\mathcal{F}, \mathcal{X}_n)$ containing each variable from $\mathcal{X}_n$ at most once is called a *context*.

**Example 1.** *Let $\mathcal{F} = \{f^2, g^1, a\}$ with non-zero arities of symbols denoted by the subscripts and $X, Y$ two variables. Terms $f^2(g^1(a), X), f^2(g^1(Y), X)$ and $f^2(Y, X)$ are contexts obtained by replacing terminal symbols within the term $f^2(g^1(a), a)$ with a variable. The term $f^2(g^1(X), X)$ is not a context, since it contains the variable $X$ more than once.*

A *regular tree grammar* $G = (S, \mathcal{N}, \mathcal{F}, R)$ is composed of a *start symbol* $S$, a set $\mathcal{N}$ of *non-terminal symbols* (non-terminal symbols have arity 0) with $S \in \mathcal{N}$, a ranked alphabet $\mathcal{F}$ of *terminal symbols* with a fixed arity such that $\mathcal{F} \cap \mathcal{N} = \emptyset$, and a set $R$ of derivation rules of the form $N \to \beta$ where $N$ is a non-terminal from $\mathcal{N}$ and $\beta$ is a term from $T(\mathcal{F} \cup \mathcal{N})$. The ranked alphabet $\mathcal{F} \cup \mathcal{N}$ is considered to be disjoint from the set of variables $\mathcal{X}_n$. Given a regular tree grammar $G = (S, \mathcal{N}, \mathcal{F}, R)$, the derivation relation $\to_G$ associated to $G$ is a relation on terms from $T(\mathcal{F} \cup \mathcal{N})$ such that $s \to_G t$ if and only if there is a rule $N \to \alpha \in R$ and there is a context $C$ such that $s = C[N/X]$ and $t = C[\alpha/X]$, where $X$ is a variable from $\mathcal{X}_n$. The subset of $T(\mathcal{F} \cup \mathcal{N})$ which can be generated by successive derivations starting with the start symbol is denoted by $L_u(G) = \{s \in T(\mathcal{F} \cup \mathcal{N}) \mid S \to_G^+ s\}$ where $\to_G^+$ is the transitive closure of $\to_G$. We omit the subscript $G$ when the grammar $G$ is clear from the context. The language generated by $G$ denoted by $L(G) = T(\mathcal{F}) \cap L_u(G)$. For the purpose of this paper, we also consider commutative associative closure $L_u^*(G)$ and $L^*(G)$ of $L_u(G)$ and $L(G)$, respectively.

**Example 2.** *Let $G = (A, \{A, B\}, \{f^2, g^1, a, b\}, R)$ with $R$ given by the following derivation rules:*

- $A \to f^2(B, A) \mid a$
- $B \to g^1(A) \mid b$

*Then, $f^2(g^1(a), a) \in L(G)$, since $A \to f^2(B, A) \to f^2(B, a) \to f^2(g^1(A), a) \to f^2(g^1(a), a)$. While $f^2(a, g^1(a))$ is not in $L(G)$, it is contained in $L^*(G)$ due to commutativity of $f^2$.*

For further details on regular tree grammars, we refer the reader, for instance, to [21].

## 3. A Gentzen-Style Proof System for $\mathcal{EL}$

The aim of this section is to provide a proof-theoretic calculus that is sound and complete for general subsumption in $\mathcal{EL}$. We will use this calculus in the subsequent sections to prove particular properties of TBoxes of a certain form in the context of consequence-preserving rewriting. The Gentzen-style calculus for $\mathcal{EL}$ is shown in Fig. 1 and is a variation of the calculus given by Hofmann [22].

$$\frac{}{C \sqsubseteq C}(\textsc{Ax}) \quad \frac{}{C \sqsubseteq \top}(\textsc{AxTop})$$

$$\frac{D \sqsubseteq E}{C \sqcap D \sqsubseteq E}(\textsc{AndL})$$

$$\frac{C \sqsubseteq E \quad C \sqsubseteq D}{C \sqsubseteq D \sqcap E}(\textsc{AndR})$$

$$\frac{C \sqsubseteq D}{\exists r.C \sqsubseteq \exists r.D}(\textsc{Ex})$$

$$\frac{C \sqsubseteq E \quad E \sqsubseteq D}{C \sqsubseteq D}(\textsc{Cut})$$

Figure 1: Gentzen-style proof system for general $\mathcal{EL}$ terminologies with $C, D, E$ arbitrary concept expressions.

The calculus operates on sequents. A *sequent* is of the form $C \sqsubseteq D$, where $C, D$ are $\mathcal{EL}$ concepts. The rules depicted in Fig. 1 can be used to derive new sequents from sequents that have already been derived. For instance, if we have derived the sequent $C \sqsubseteq D$, we can derive the sequent $\exists r.C \sqsubseteq \exists r.D$ using rule (Ex). A *derivation* (or *proof*) of a sequent $C \sqsubseteq D$ is a finite tree with whose nodes are labeled with sequents. The tree root is labeled with the sequent $C \sqsubseteq D$. Within the tree, a parent node is always labeled by the conclusion of a proof rule from Fig. 1 whose antecedent(s) are the labels of the child nodes. The leaves of a derivation are either labeled by axioms from $\mathcal{T}$ or conclusions of (Ax) or (AxTop). We use the notation $\mathcal{T} \vdash C \sqsubseteq D$ to indicate that there is a derivation of $C \sqsubseteq D$. In our calculus, we assume commutativity of conjunction for convenience. Fig. 2 shows an example derivation of the sequent $\exists r.C_1 \sqsubseteq C_2$ in our calculus w.r.t. the $\mathcal{EL}$ TBox $\mathcal{T}_e = \{\exists r.C_1 \sqsubseteq C_1 \sqcap C_2\}$.

$$\dfrac{\exists r.C_1 \sqsubseteq C_1 \sqcap C_2 \qquad \dfrac{\dfrac{}{C_2 \sqsubseteq C_2}\,(\textsc{Ax})}{C_1 \sqcap C_2 \sqsubseteq C_2}\,(\textsc{AndL})}{\exists r.C_1 \sqsubseteq C_2}\,(\textsc{Cut})$$

Figure 2: Example derivation of $\exists r.C_1 \sqsubseteq C_2$ from $\mathcal{T}_e$.

We show that the above calculus is sound and complete for subsumptions between arbitrary $\mathcal{EL}$ concepts.

**Lemma 3** (Soundness and Completeness)**.** *Let $\mathcal{T}$ be an arbitrary $\mathcal{EL}$ TBox, $C, D$ $\mathcal{EL}$ concepts. Then $\mathcal{T} \models C \sqsubseteq D$, iff $\mathcal{T} \vdash C \sqsubseteq D$.*

*Proof.* While the soundness of the proof system (if-direction) can be easily verified for each rule separately, the proof of completeness is more sophisticated. Analogously to other proof-theoretic approaches [11, 23], we show the only-if-direction of the lemma by constructing a model $\mathcal{I}$ for $\mathcal{T}$ wherein *only* the GCIs derivable from $\mathcal{T}$ are valid. The construction of the model is rather standard (a similar construction is, e.g., given by Lutz and Wolter [24]). The model is defined as follows:

- $\Delta^{\mathcal{I}}$ is the set of elements $\delta_C$ where $C$ is an $\mathcal{EL}$ concept expression;
- $A^{\mathcal{I}} := \{\delta_C \in \Delta^{\mathcal{I}} \mid \mathcal{T} \vdash C \sqsubseteq A\}$, where $A$ ranges over concept symbols;
- $r^{\mathcal{I}} := \{(\delta_C, \delta_D) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid \mathcal{T} \vdash C \sqsubseteq \exists r.D\}$ where $r$ ranges over role symbols.

We will show that the following claim holds for $\mathcal{I}$:
*For all $\delta_E \in \Delta^{\mathcal{I}}$ and $\mathcal{EL}$ concepts $F$ it holds that $\delta_E \in F^{\mathcal{I}}$ iff $\mathcal{T} \vdash E \sqsubseteq F$.* *(*)*

This claim can be exploited in two ways: First, we use it to show that $\mathcal{I}$ is indeed a model of $\mathcal{T}$. Let $C \sqsubseteq D \in \mathcal{T}$ and consider an arbitrary concept expression $G$ with $\delta_G \in C^{\mathcal{I}}$. Via (*) we obtain $\mathcal{T} \vdash G \sqsubseteq C$. Further, $\mathcal{T} \vdash C \sqsubseteq D$ due to $C \sqsubseteq D \in \mathcal{T}$. Thus we can derive $\mathcal{T} \vdash G \sqsubseteq D$ via (Cut) and consequently, applying (*) again, we obtain $\delta_G \in D^{\mathcal{I}}$. Thereby modelhood of $\mathcal{I}$ with respect to $\mathcal{T}$ has been proved.

Second, we use (*) to show that $\mathcal{I}$ is a counter-model for all GCIs not derivable from $\mathcal{T}$ as follows: Assume $\mathcal{T} \not\vdash C \sqsubseteq D$. From $\mathcal{T} \vdash C \sqsubseteq C$ and (*) we derive

$\delta_C \in C^{\mathcal{I}}$. From $\mathcal{T} \not\vdash C \sqsubseteq D$ and (*) we obtain $\delta_C \notin D^{\mathcal{I}}$. Hence we get $C^{\mathcal{I}} \not\subseteq D^{\mathcal{I}}$ and therefore $\mathcal{I} \not\models C \sqsubseteq D$.

It remains to prove (*). This is done by an induction over the structure of the concept expression $F$. There are two base cases:

- for $F = \top$, the claim trivially follows from (AxTop),
- for a concept symbol $F$, it is a direct consequence of the definition of our model ($F^{\mathcal{I}} := \{\delta_C \in \Delta^{\mathcal{I}} \mid \mathcal{T} \vdash C \sqsubseteq F\}$).

we now consider the cases where $F$ is a complex concept expression

- for $F = C_1 \sqcap \ldots \sqcap C_n$, we note that $\delta_E \in F^{\mathcal{I}}$ exactly if $\delta_E \in C_i^{\mathcal{I}}$ for all $i \in \{1 \ldots n\}$. By induction hypothesis, this means $\mathcal{T} \vdash E \sqsubseteq C_i$ for all $i \in \{1 \ldots n\}$. Finally, observe that $\{E \sqsubseteq C_i \mid 1 \leq i \leq n\}$ and $E \sqsubseteq C_1 \sqcap \ldots \sqcap C_n$ can be mutually derived from each other:
    - $\{E \sqsubseteq C_i \mid 1 \leq i \leq n\} \vdash E \sqsubseteq C_1 \sqcap \ldots \sqcap C_n$ is a straightforward consequence of (AndR);
    - To derive $E \sqsubseteq C_1 \sqcap \ldots \sqcap C_n \vdash \{E \sqsubseteq C_i \mid 1 \leq i \leq n\}$, we first derive $C_1 \sqcap \ldots \sqcap C_n \sqsubseteq C_i$ from $C_i \sqsubseteq C_i$ (obtained using (Ax)) by applying (AndL) multiple times. Since $\mathcal{T} \vdash E \sqsubseteq C_1 \sqcap \ldots \sqcap C_n$, we can apply (Cut) (with $E \sqsubseteq C_1 \sqcap \ldots \sqcap C_n$ as the left antecedent and $C_1 \sqcap \ldots \sqcap C_n \sqsubseteq C_i$ as the right antecedent) to derive $E \sqsubseteq C_i$.
- for $F = \exists r.G$, we prove the two directions separately. First assuming $\delta_E \in F^{\mathcal{I}}$ we must find $(\delta_E, \delta_H) \in r^{\mathcal{I}}$ for some $H$ with $\delta_H \in G^{\mathcal{I}}$. This implies both $\mathcal{T} \vdash E \sqsubseteq \exists r.H$ (by the definition of the model) and $\mathcal{T} \vdash H \sqsubseteq G$ (via the induction hypothesis). From the latter, we can deduce $\mathcal{T} \vdash \exists r.H \sqsubseteq \exists r.G$ by (Ex) and consequently $\mathcal{T} \vdash E \sqsubseteq \exists r.G$. For the other direction, note that by definition, $\mathcal{T} \vdash E \sqsubseteq \exists r.G$ implies $(\delta_E, \delta_G) \in r^{\mathcal{I}}$. On the other hand, we get $\mathcal{T} \vdash G \sqsubseteq G$ by (Ax) and therefore $\delta_G \in G^{\mathcal{I}}$ by the induction hypothesis which yields us $\delta_E \in F^{\mathcal{I}}$. □

Alternatively, the completeness of the calculus could be shown by a reduction to the calculus of Hofmann [22].

## 4. Uniform Interpolation

Uniform interpolation has many potential applications in ontology engineering due to its ability to reduce the amount of irrelevant information within a terminology while preserving all relevant consequences given the set of relevant signature elements. The task of computing terminologies with such properties is not trivial. For instance, it is not sufficient to simply eliminate axioms containing only irrelevant entities, since it can change the meaning of the relevant entities and cause a loss of relevant information. Example 3 demonstrates the effect of such an elimination.

**Example 3.** *Consider the terminology $\mathcal{T}$ given by*

$$A_{i+1} \sqsubseteq A_i \qquad 0 \leq i \leq 3 \tag{1}$$

$$A_4 \sqsubseteq \exists r.A_4 \tag{2}$$

*If we are only interested in entities $A_1, A_4, r$, then we might consider to eliminate all axioms except for those that contain at least one relevant entity, obtaining $\mathcal{T}' = \mathcal{T} \setminus \{A_3 \sqsubseteq A_2\}$. However, in this way we would lose the information about the connection between the relevant entities, for instance $A_4 \sqsubseteq A_1, A_4 \sqsubseteq \exists r.A_1, A_4 \sqsubseteq \exists r.\exists r.A_1, \ldots$. Indeed, $\mathcal{T}'$ does not entail any of these statements. Thus, by omitting axioms based only on the absence of relevant entities can lead to a loss of relevant information.*

In typical ontology reuse scenarios, it is required to preserve the meaning of the relevant entities while computing a terminology that contains as little irrelevant information as possible. We say that the meaning of relevant entities is preserved, if every logical statement that follows from the original terminology and contains only relevant entities also follows from the resulting terminology. The logical foundation for such a preservation of relevant consequences can be defined using the notion of *inseparability*. Two terminologies, $\mathcal{T}_1$ and $\mathcal{T}_2$, are inseparable w.r.t. a signature $\Sigma$ if they have the same $\Sigma$-consequences, i.e., consequences whose signatures are subsets of $\Sigma$. Depending on the particular application requirements, the expressivity of those $\Sigma$-consequences can vary from subsumption axioms and concept assertions to conjunctive queries. In the following, we consider *concept-inseparability* of general $\mathcal{EL}$ terminologies as given, for instance, in [17, 19, 18]:

**Definition 2.** *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two general $\mathcal{EL}$ terminologies and $\Sigma$ a signature. $\mathcal{T}_1$ and $\mathcal{T}_2$ are* concept-inseparable *w.r.t. $\Sigma$, in symbols $\mathcal{T}_1 \equiv_{\Sigma}^{\mathcal{EL}} \mathcal{T}_2$, if for all $\mathcal{EL}$*

*concepts $C, D$ with sig$(C) \cup$sig$(D) \subseteq \Sigma$ it holds that $\mathcal{T}_1 \models C \sqsubseteq D$, iff $\mathcal{T}_2 \models C \sqsubseteq D$.*

Due to its usefulness for different ontology engineering tasks, concept-inseparability has been investigated by different authors in the last decade. For instance, in the context of ontology reuse, the notion of inseparability can be used to derive a terminology that is inseparable from the initial terminology and is using only terms from $\Sigma$. This is an established non-standard reasoning task called forgetting or uniform interpolation.

**Definition 3.** *Given a signature $\Sigma$ and a terminology $\mathcal{T}$, the task of* uniform interpolation *is to determine a terminology $\mathcal{T}'$ with sig$(\mathcal{T}') \subseteq \Sigma$ such that $\mathcal{T} \equiv^{\mathcal{EL}}_{\Sigma} \mathcal{T}'$. $\mathcal{T}'$ is also called a* uniform $\Sigma$-interpolant *of $\mathcal{T}$.*

For the TBox $\mathcal{T}$ in Example 3, one possible uniform $\Sigma$-interpolant for $\Sigma = \{A_1, A_4, r\}$ would be $\mathcal{T}_\Sigma = \{A_4 \sqsubseteq A_1, A_4 \sqsubseteq \exists r.A_4\}$. We see that, by introducing a shortcut axiom $A_4 \sqsubseteq A_1$, we preserve all relevant logical consequences (those expressed using $\Sigma$) while eliminating all other logical consequences, e.g., $A_{i+1} \sqsubseteq A_i$ for $0 \leq i \leq 3$.

In practice, uniform interpolants are required to be finite, i.e., expressible by a finite set of finite axioms using only the language constructs of a particular DL. It is well-known (e.g., see [19]) that, in the presence of cyclic concept inclusions, a finite uniform $\mathcal{EL}$ $\Sigma$-interpolant might not exist for a particular terminology $\mathcal{T}$ and a particular $\Sigma$.

**Example 4.** *Consider the terminology $\mathcal{T} = \{A' \sqsubseteq A, A \sqsubseteq A'', A \sqsubseteq \exists r.A, \exists s.A \sqsubseteq A\}$ and let $\Sigma = \{s, r, A', A''\}$. As consequences, we obtain infinite sequences $A' \sqsubseteq \exists r.\exists r.\exists r.\ldots A''$ and $\exists s.\exists s.\exists s.\ldots A' \sqsubseteq A''$ which contain nested existential quantifiers of unbounded depth. Those sequences cannot be finitely axiomatized, using only signature elements from $\Sigma$.*

Lutz, Seylan and Wolter [20] give an ExpTime procedure for deciding if a finite uniform $\mathcal{EL}$ interpolant exists. In the following, we extend the results and show that, if a finite uniform $\mathcal{EL}$ interpolant exists for the given terminology and signature, then there exists a uniform $\mathcal{EL}$ interpolant of at most triple exponential size. Further, we show that, in the worst-case, no shorter interpolants exist, thereby establishing tight bounds on the size of uniform interpolants in $\mathcal{EL}$.

## 5. Lower Bound

In this section we will establish the lower bound for the size of uniform interpolants of $\mathcal{EL}$ terminologies, in case they exist. It is interesting that, while deciding the existence of uniform interpolants in $\mathcal{EL}$ [20] is one exponential less complex than the same decision problem for the more complex logic $\mathcal{ALC}$ [18], the size of uniform interpolants remains triple-exponential. An intuitive reason for this rather unexpected result can be seen in the unavailability of disjunction, which would allow for a more succinct representation of the interpolants. We show this lower bound by means of a sequence of terminologies (obtained by a slight modification of the corresponding example given in [27] originally demonstrating a double exponential lower bound in the context of conservative extensions).

We start with an intuitive explanation of what the terminology is supposed to express. Assume, given some $n \in \mathbb{N}$ we want to label domain elements with natural numbers $0 \dots 2^n - 1$ according to the following scheme: domain elements belonging to the concepts $A_1$ or $A_2$ are labeled with $0$. Further, whenever we find a domain element $\delta$ that is linked via an $r$-role to an $\ell$-labeled domain element $\delta_1$ and linked via an $s$-role to an $\ell$-labeled domain element $\delta_2$, then $\delta$ will be labeled with $\ell + 1$ (provided $\ell < 2^n - 1$). Finally, we stipulate that every domain element labeled with $2^n - 1$ will belong to the concept $B$. In order to encode this labeling scheme in a knowledge base whose size is polynomial in $n$, we encode the number-labels in a binary way as a conjunction of $n$ concepts. Thereby, the concept symbols $X_i, \overline{X_i}$ represent the $i^{th}$ bit of $\ell$'s binary representation being clear or set.

**Definition 4.** *The $\mathcal{EL}$ TBox $\mathcal{T}_n$ for a natural number $n$ is given by*

$$A_1 \sqsubseteq \overline{X_0} \sqcap \ldots \sqcap \overline{X_{n-1}} \tag{3}$$

$$A_2 \sqsubseteq \overline{X_0} \sqcap \ldots \sqcap \overline{X_{n-1}} \tag{4}$$

$$\bigsqcap_{\sigma \in \{r,s\}} \exists \sigma.(\overline{X_i} \sqcap X_0 \sqcap \ldots \sqcap X_{i-1}) \sqsubseteq X_i \qquad i < n \tag{5}$$

$$\bigsqcap_{\sigma \in \{r,s\}} \exists \sigma.(X_i \sqcap X_0 \sqcap \ldots \sqcap X_{i-1}) \sqsubseteq \overline{X_i} \qquad i < n \tag{6}$$

$$\bigsqcap_{\sigma \in \{r,s\}} \exists \sigma.(\overline{X_i} \sqcap \overline{X_j}) \sqsubseteq \overline{X_i} \;\; j < i < n \tag{7}$$

$$\bigsqcap_{\sigma \in \{r,s\}} \exists \sigma.(X_i \sqcap \overline{X_j}) \sqsubseteq X_i \;\; j < i < n \tag{8}$$

$$X_0 \sqcap \ldots \sqcap X_{n-1} \sqsubseteq B \tag{9}$$

In the above TBox, Axiom (5) ensures that a clear bit will be set in the successor number, if all lower bits are already set. The subsequent Axiom (6) ensures that a set bit will be clear in the successor number, if all lower bits are also set. Axioms (7) and (8) ensure that in all other cases, bits are not toggled. For instance, Axiom (7) states that, if any of the bits lower than $i$ is clear, then bit $i$ will remain clear also in the successor number.

If we now consider sets $\mathcal{C}_i$ of concept descriptions inductively defined by $\mathcal{C}_0 = \{A_1, A_2\}$, $\mathcal{C}_{i+1} = \{\exists r.C_1 \sqcap \exists s.C_2 \mid C_1, C_2 \in \mathcal{C}_i\}$, then we find that $|\mathcal{C}_{i+1}| = |\mathcal{C}_i|^2$ and consequently $|\mathcal{C}_i| = 2^{(2^i)}$. Thus, the set $\mathcal{C}_{2^n-1}$ contains triply exponentially many different concepts, each of which is doubly exponential in the size of $\mathcal{T}_n$ (intuitively, we obtain concepts having the shape of binary trees of exponential depth, thus having doubly exponentially many leaves, each of which can be $A_1$ or $A_2$, which gives rise to triply exponentially many different such trees). Then we will show that for each concept $C \in \mathcal{C}_{2^n-1}$ it holds that $\mathcal{T}_n \models C \sqsubseteq B$ and that there cannot be a smaller uniform interpolant with respect to the signature $\Sigma = \{A_1, A_2, B, r, s\}$ than the one containing all these GCIs.

Based on the above definition, we now prove the following result.

**Theorem 1.** *There exists a sequence of $\mathcal{EL}$ TBoxes and a fixed signature $\Sigma$ such that for each TBox $(\mathcal{T}_n)$ within this sequence the following hold:*

- *the size of $\mathcal{T}_n$ is at most polynomial in $n$ and*

- *the size of the smallest uniform interpolant of $\mathcal{T}_n$ with respect to $\Sigma$ is at least $2^{(2^{(2^n-1)})}$.*

*Proof.* Obviously, the size of $\mathcal{T}_n$ is polynomially bounded by $n$. We now consider sets $\mathcal{C}_k$ of concepts defined above. Since $|\mathcal{C}_k| = 2^{(2^k)}$, we find that the set $\mathcal{C}_{2^n-1}$ contains triply exponentially many different concepts, each of which is doubly exponential in the size of $\mathcal{T}_n$.

Obviously, for any $k$, every concept description from $\mathcal{C}_k$ contains only signature elements from $A_1, A_2, r, s$.

It is rather straightforward to check that $\mathcal{T}_n \models C \sqsubseteq B$ holds for each concept $C \in \mathcal{C}_{2^n-1}$: by induction on $k$, we can show that for any $C \in \mathcal{C}_k$ with $k < 2^n$ it holds that $\mathcal{T}_n \models C \sqsubseteq Y_0^k \sqcap \ldots \sqcap Y_{n-1}^k$ with

$$Y_i^k = \begin{cases} X_i \text{ if } \lfloor \frac{k}{2^i} \rfloor \mathsf{mod}\ 2 = 1 \\ \overline{X_i} \text{ if } \lfloor \frac{k}{2^i} \rfloor \mathsf{mod}\ 2 = 0 \end{cases},$$

i.e., $Y_i^k$ indicates the $i$th bit of the number $k$ in binary encoding. Then, $C \sqsubseteq B$ follows via the last axiom of $\mathcal{T}_n$.

Toward the claimed triple-exponential lower bound, we now show that every uniform interpolant of $\mathcal{T}_n$ for $\Sigma = \{A_1, A_2, B, r, s\}$ must contain for each $C \in \mathcal{C}_{2^n-1}$ a GCI of the form $C \sqsubseteq B'$ with $B' = B$ or $B' = B \sqcap F$ for some $F$ (where we consider structural variants – i.e., concept expressions whose characteristic interpretations are isomorphic – as syntactically equal). Toward a contradiction, we assume that this is not the case, i.e., there is a uniform interpolant $\mathcal{T}'$ and a $C \in \mathcal{C}_{2^n-1}$ where $C \sqsubseteq B' \notin \mathcal{T}'$ for any $B'$ containing $B$ as a (top-level) conjunct.

Yet, $C \sqsubseteq B$ must be a consequence of $\mathcal{T}'$, since it is a consequence of of $\mathcal{T}_n$ containing only signature elements from $\Sigma$ and $\mathcal{T}'$ is a uniform interpolant of $\mathcal{T}_n$ w.r.t. $\Sigma$ by assumption. Therefore, there must be a derivation of it. Looking at the derivation calculus from the last section, the last derivation step must be (ANDL) or (CUT). We can exclude (ANDL) since neither $\exists r.C' \sqsubseteq B$ nor $\exists s.C' \sqsubseteq B$ is the consequence of $\mathcal{T}'$ for any $C' \in \mathcal{C}_{2^n-2}$ (which can be easily shown by providing appropriate witness models of $\mathcal{T}'$). Consequently, the last derivation step must be an application of (CUT), i.e., there must be a concept $E \neq C$ such that $\mathcal{T}' \models C \sqsubseteq E$ and $\mathcal{T}' \models E \sqsubseteq B$. Without loss of generality, we assume that we consider a derivation tree where the subtree deriving $C \sqsubseteq E$ has minimal depth.

We now distinguish two cases: either $E$ contains $B$ as a conjunct or not.

- First we assume $E = E' \sqcap B$, i.e. the (CUT) rule was used to derive $C \sqsubseteq B$

from $C \sqsubseteq E' \sqcap B$ and $E' \sqcap B \sqsubseteq B$. The former cannot be contained in $\mathcal{T}'$ by assumption, hence it must have been derived itself. We can exclude (AndR) due to the minimality of the proof. Again, it cannot have been derived via (AndL) for the same reasons as given above, which again leaves (Cut) as the only possible derivation rule for obtaining $C \sqsubseteq E' \sqcap B$. Thus, there must be some concept $G$ with $\mathcal{T}' \models C \sqsubseteq G$ and $\mathcal{T}' \models G \sqsubseteq E' \sqcap B$. Once more, we distinguish two cases: either $G$ contains $B$ as a conjunct or not.

  - If $G$ contains $B$ as a conjunct, i.e., $G = G' \sqcap B$, the derivation of $C \sqsubseteq E$ was not depth-minimal since there is a better proof where $C \sqsubseteq B$ is derived from $C \sqsubseteq G' \sqcap B$ and $G' \sqcap B \sqsubseteq B$ via (Cut). Hence we have a contradiction.
  - If $G$ does not contain $B$ as a conjunct, the original derivation of $C \sqsubseteq E$ was not depth-minimal since we can construct a better one that derives $C \sqsubseteq B$ directly from $C \sqsubseteq G$ and $G \sqsubseteq B$ (the latter being derived from $G \sqsubseteq E' \sqcap B$ via (AndR)).

- Now assume $E$ does not contain $B$ as a conjunct.

  We construct a specific interpretation $(\Delta, \cdot^{\mathcal{I}})$ as follows ($\epsilon$ denoting the empty word):

  - $\Delta = \{w \mid w \in \{r, s\}^*, \text{ length}(w) < 2^n\}$
  - We define an auxiliary function $\chi$ associating a concept expression to each domain element: we let $\chi(\epsilon) = C$ (with $\epsilon$ being the empty word) and, for every $wr, ws \in \Delta$ with $\chi(w) = \exists r.C_1 \sqcap \exists s.C_2$, we let $\chi(wr) = C_1$ and $\chi(ws) = C_2$.
  - the concepts and roles are interpreted as follows:
    - $A_\iota^{\mathcal{I}} = \{w \mid \chi(w) = A_\iota\}$ for $\iota \in \{1, 2\}$
    - $B^{\mathcal{I}} = \{\epsilon\}$
    - $X_i^{\mathcal{I}} = \{w \mid \lfloor\frac{\text{length}(w)}{2^i}\rfloor \text{mod } 2 = 0\}$ for $i < n$
    - $\overline{X_i}^{\mathcal{I}} = \{w \mid \lfloor\frac{\text{length}(w)}{2^i}\rfloor \text{mod } 2 = 1\}$ for $i < n$
    - $r^{\mathcal{I}} = \{\langle w, wr\rangle \mid wr \in \Delta\}$
    - $s^{\mathcal{I}} = \{\langle w, ws\rangle \mid ws \in \Delta\}$

  It is straightforward to check that $\mathcal{I}$ is a model of $\mathcal{T}_n$. Furthermore using descending induction on the length of $w$, we can show that for every $w \in \Delta$

it holds that $w \in (\chi(w))^{\mathcal{I}}$, thus, in particular, $\epsilon \in C^{\mathcal{I}}$. Consequently, due to our assumption, $\epsilon \in E^{\mathcal{I}}$ must hold. Now we observe that the restriction of $\mathcal{I}$ to the signature elements $A_1, A_2, r, s$ is isomorphic to $\mathcal{I}_C$ (with $x_C$ corresponding to $\epsilon$). On the other hand, as $\epsilon \in E^{\mathcal{I}}$ we find by Lemma 1 that there must be a homomorphism from $(\mathcal{I}_E, x_E)$ to $(\mathcal{I}, \epsilon)$ and hence to $(\mathcal{I}_C, x_C)$, thus we can invoke Lemma 2 to deduce that $E$ is a proper "structural superconcept" of $C$, i.e., $\emptyset \models C \sqsubseteq E$ and $\emptyset \not\models E \sqsubseteq C$ must hold.

We now obtain $\widetilde{E}$ by enriching $E$ as follows: starting from $k = 0$ and iteratively incrementing $k$ up to $2^n - 1$, every subconcept $G$ of $E$ satisfying $\emptyset \models G \sqsubseteq C'$ for some $C' \in \mathcal{C}_k$ is substituted by $G \sqcap Y_0^k \sqcap \ldots \sqcap Y_{n-1}^k$ where, as before,

$$Y_i^k = \begin{cases} X_i \text{ if } \lfloor \frac{k}{2^i} \rfloor \mathsf{mod}\ 2 = 1 \\ \overline{X_i} \text{ if } \lfloor \frac{k}{2^i} \rfloor \mathsf{mod}\ 2 = 0 \end{cases},$$

i.e., $Y_i^k$ indicates the $i$th bit of the number $k$ in binary encoding.

Then, $\widetilde{E}$'s characteristic pointed interpretation $(\mathcal{I}_{\widetilde{E}}, x_{\widetilde{E}})$ satisfies that $\mathcal{I}_{\widetilde{E}}$ is a model of $\mathcal{T}_n$ (following from structural induction on subconcepts of $\widetilde{E}$) and its root individual $x_{\widetilde{E}}$ is in the extension of $\widetilde{E}$. Still, we find $x_{\widetilde{E}} \notin C^{\mathcal{I}_{\widetilde{E}}}$ for the following reason: $C$ does only contain signature elements from $\{A_1, A_2, B, r, s\}$, and the restriction of $(\mathcal{I}_{\widetilde{E}}, x_{\widetilde{E}})$ to these signature elements is isomorphic to $(\mathcal{I}_E, x_E)$, therefore $x_{\widetilde{E}} \in C_{\mathcal{I}_{\widetilde{E}}}$ iff $x_E \in C^{\mathcal{I}_E}$. The latter is however not the case as this would imply by Lemma 1 that there is a homomorphism from $(\mathcal{I}_C, x_C)$ to $(\mathcal{I}_E, x_E)$ and consequently, via Lemma 2 $\emptyset \models E \sqsubseteq C$, contradicting our above finding.

Yet, the root individual $x_{\widetilde{E}}$ cannot satisfy any other concept expression $C''$ from $\mathcal{C}_{2^n-1} \setminus \{C\}$ either, since this, via $\emptyset \models E \sqsubseteq C''$, would imply $\emptyset \models C \sqsubseteq C''$ which is not the case (by induction on $k$ one can show that there cannot be a homomorphism between the characteristic pointed interpretations of any two distinct concepts from any $\mathcal{C}_k$). In particular, we note that $x_{\widetilde{E}} \notin B^{\mathcal{I}_{\widetilde{E}}}$. Thus, we have found a model of $\mathcal{T}_n$ witnessing $\mathcal{T}_n \not\models E \sqsubseteq B$, contradicting our assumption that $\mathcal{T}' \models E \sqsubseteq B$.

□

Hence we have found a class $\mathcal{T}_n$ of TBoxes giving rise to uniform $\mathcal{EL}$ interpolants of triple-exponential size in terms of the original TBox.

## 6. Upper Bound

Now we discuss the upper bound on the size of uniform $\mathcal{EL}$ interpolants as well as their computation. Since, for a TBox $\mathcal{T}$ and a signature $\Sigma$, there are in general infinitely many $\Sigma$-consequences, in the following, we aim at identifying a subset of such consequences, the deductive closure of which contains the whole set. Interestingly, there exists a bound on the role depth of $\Sigma$-consequences such that, for the set $\mathcal{T}_{\Sigma,N}$ of all $\Sigma$-consequences of $\mathcal{T}$ with the maximal role depth $N$ the following holds: either $\mathcal{T}_{\Sigma,N}$ is a uniform $\mathcal{EL}$ interpolant of $\mathcal{T}$ with respect to $\Sigma$ or such a finite uniform $\mathcal{EL}$ interpolant of $\mathcal{T}$ does not exist. This is an easy consequence of results obtained by Lutz, Seylan and Wolter [20] while investigating the problem of existence of uniform $\mathcal{EL}$ interpolants (proof can be found in Appendix B).

**Lemma 4** (Reformulation of Lemma 55 from [20] )**.** *Let $\mathcal{T}$ be an $\mathcal{EL}$ TBox, $\Sigma$ a signature. The following statements are equivalent:*

1. *There exists a uniform $\mathcal{EL}$ $\Sigma$-interpolant of $\mathcal{T}$.*
2. *There exists a uniform $\mathcal{EL}$ $\Sigma$-interpolant $\mathcal{T}'$ of $\mathcal{T}$ for which holds $d(\mathcal{T}') \leq 2^{4\cdot(|sub(\mathcal{T})|)} + 1$.*

However, an upper bound on the role depth is only sufficient for showing a non-elementary upper bound on the size of uniform interpolants for the following reasons. There are $2^n$ many different conjunctions of $n$ different conjuncts, and, accordingly, for each role, $2^m$ many different existential restrictions of depth $i+1$ if $m$ is the number of existential restrictions of depth $i$. Moreover, for any role depth $i$, we can find a TBox such that $i$ is the corresponding maximal role depth. Subsequently, the upper bound on the role depth does not suffice to obtain an upper bound for the number $i$ of exponents bounding the size of the uniform interpolant.

In order to obtain a tight upper bound, we need to further narrow down the subset of $\Sigma$-consequences required to obtain a uniform interpolant. To this end, we show the following:

- If we "flatten" terminologies, i.e., we reduce the maximal role depth of $\mathcal{T}$ to $1$ by recursively introducing fresh concept symbols for all subconcepts occurring in $\mathcal{T}$, it is sufficient to consider the $\Sigma$-consequences stating subsumees and subsumers of all concept symbols referenced by the flattened terminology $\mathcal{T}'$ in order to preserve all consequences;

- Lemma 4 can be transferred to flattened TBoxes such that it is sufficient to consider subsumees and subsumers of role depth $2^{4\cdot(|\mathsf{sub}(\mathcal{T}')|)}+1$ in order to preserve all consequences of $\mathcal{T}$;
- There is a particular type of subsumees and subsumers that do not add any consequences to the deductive closure, which we call *weak* subsumees and subsumers. These are subsumees obtained by adding arbitrary conjuncts to arbitrary subconcepts of other subsumees and, accordingly, subsumers obtained from other subsumers by omitting conjuncts from arbitrary subconcepts. When included into the uniform interpolant, weak subsumees and subsumers have a negative impact on its size. Given the exponential bound on the role depth, each concept has non-elementary many weak subsumees. Since weak subsumers and subsumees do not add any new consequences, we can safely exclude them.

We show that, in case a finite uniform $\mathcal{EL}$ interpolant of $\mathcal{T}$ with respect to $\Sigma$ exists, there are at most triple-exponentially many such non-weak subsumers and subsumees of role depth up to $2^{4\cdot(|\mathsf{sub}(\mathcal{T})|)}+1$. Moreover, we show that each of them is of at most double-exponential size.

### 6.1. Flattening

Recall that we want to compute the uniform interpolant of a TBox $\mathcal{T}$ by rewriting the latter, ensuring that the part of the deductive closure of $\mathcal{T}$ consisting of $\Sigma$-consequences is preserved throughout the rewriting process. Since rewriting operates on the syntactic structure of $\mathcal{T}$, it is desirable that the syntactic structure has a close relation to the deductive closure of $\mathcal{T}$ such that we can easily manipulate the deductive closure via changes of the syntactic structure. As in other syntax-based approaches ([11, 23, 19], we decompose complex axioms into syntactically simple ones. We refer to this process as *flattening*: assigning a temporary concept symbol to each complex subconcept occurring in $\mathcal{T}$, so that the terminology can be represented without nested expressions, namely using only axioms of the form $A \sqsubseteq B$, $A \equiv B_1 \sqcap \ldots \sqcap B_n$, and $A \equiv \exists r.B$, where $A$ and $B_{(i)}$ are concept symbols or $\top$ and $r$ is a role. For this purpose, we introduce a minimal required set of fresh concept symbols $N_D$ with exactly on equivalence axiom $A' \equiv C'$ for each $A' \in N_D$, where $C'$ is equivalent to a subconcept of $\mathcal{T}$ replaced by $A'$.

In what follows, we assume terminologies to be flattened and all concepts symbols from $N_D$ to be in $\mathsf{sig}_C(\mathcal{T}) \setminus \Sigma$. W.l.o.g., we also assume that $\mathcal{EL}$ concepts do not contain any equivalent concepts in conjunctions and that whenever several concept symbols are equivalent in $\mathcal{T}$, all their occurrences have been replaced by a single representative of the corresponding equivalence class. Concept symbols from $\Sigma$ are preferred to be selected as representatives. Note that this is a preprocessing step that can be performed in polynomial time as $\mathcal{EL}$ allows for polytime reasoning. The following lemma postulates the close semantic relation between a TBox and its flattening.

**Lemma 5** (Model-conservativity)**.** *Any $\mathcal{EL}$ TBox $\mathcal{T}$ can be rewritten into a flattened TBox $\mathcal{T}'$ so that each model of $\mathcal{T}'$ is a model of $\mathcal{T}$ and each model of $\mathcal{T}$ can be extended into a model of $\mathcal{T}'$.*

As a result of flattening, each TBox $\mathcal{T}$ can be represented as a *subsumee/-subsumer relation pair* – a pair of binary relations $\langle P^{\mathcal{T}}_{\sqsupseteq}, P^{\mathcal{T}}_{\sqsubseteq}\rangle$ on concept expressions where $P^{\mathcal{T}}_{\sqsupseteq}$ relates concept symbols $B \in \mathrm{sig}_C(\mathcal{T})$ to their subsumees ($\{C \mid C \bowtie B \in \mathcal{T}, \bowtie \in \{\equiv, \sqsubseteq\}\}$), and $P^{\mathcal{T}}_{\sqsubseteq}$ relates concept symbols to their subsumers ($\{C \mid B \bowtie C \in \mathcal{T}, \bowtie \in \{\equiv, \sqsubseteq\}\}$). If $\mathcal{T}$ is clear from the context, we simply write $\langle P_{\sqsupseteq}, P_{\sqsubseteq}\rangle$. In turn, each subsumee/subsumer relation pair has a corresponding representation by means of a TBox. For the computation of uniform interpolants, we would like to restrict the signature of the resulting TBox constructed from a subsumee/subsumer relation pair. As we will show later on, for the computation of uniform interpolants we use only $\Sigma$-subsumees and $\Sigma$-subsumers. To ensure that the resulting TBox only contains symbols from $\Sigma$, we additionally avoid references to concept symbols not from $\Sigma$ by forming subsumptions between their subsumees and subsumers directly.

**Definition 5.** *Let $\mathcal{T}$ be an $\mathcal{EL}$ TBox and $\Sigma$ a signature. Further, let $\langle P_{\sqsupseteq}, P_{\sqsubseteq}\rangle$ be a subsumee/subsumer relation pair for $\mathcal{T}$. Then,*

$$\begin{aligned}\mathtt{M}(P_{\sqsupseteq}, P_{\sqsubseteq}, \Sigma) = {} & \{C \sqsubseteq A \mid A \in \Sigma, (A, C) \in P_{\sqsupseteq}\} \cup \\ & \{A \sqsubseteq D \mid A \in \Sigma, (A, D) \in P_{\sqsubseteq}\} \cup \\ & \{C \sqsubseteq D \mid \textit{there exists } A \notin \Sigma, \\ & \quad (A, C) \in P_{\sqsupseteq}, (A, D) \in P_{\sqsubseteq}\}.\end{aligned}$$

In the next subsection, we represent the corresponding subsumee/subsumer relation pair of a classified, flattened TBox $\mathcal{T}$ as a pair of regular tree grammars on ranked trees (with concept symbols interpreted as non-terminals and $\exists r, \sqcap$ as functions). We show that all non-weak subsumees and subsumers entailed by $\mathcal{T}$ can be generated by these grammars. To this end, we now analyse the derivation of subsumptions in flattened TBoxes by means of the deduction calculus introduced in Section 3.

First, we consider the derivation of subsumees. We use the auxiliary function $\texttt{Pre} : \text{sig}_C(\mathcal{T}) \to 2^{2^{\text{sig}_C(\mathcal{T})}}$ which allows us for any concept symbol $A$ to refer to its subsumees of the form $B_1 \sqcap ... \sqcap B_n$, where $B_{(i)}$ are concept symbols. For each such conjunction, the set of its conjuncts is an element of Pre.

**Definition 6.** *Let $\mathcal{T}$ be an $\mathcal{EL}$ TBox and $A \in sig_C(\mathcal{T})$. $\texttt{Pre}(A)$ is the smallest set with the following properties:*

- $\{A\} \in \texttt{Pre}(A)$.
- *For each $K \in \texttt{Pre}(A)$ and each $B \in K$, if there is $\mathcal{T} \models B' \sqsubseteq B$, then also $(K/\{B\}) \cup \{B'\} \in \texttt{Pre}(A)$.*
- *For each $K \in \texttt{Pre}(A)$ and each $B \in K$, if there is $B \equiv B_1 \sqcap ... \sqcap B_n \in \mathcal{T}$, then also $(K/\{B\}) \cup \{B_1, ..., B_n\} \in \texttt{Pre}(A)$.*

We can show the following closure property of Pre.

**Lemma 6.** *Let $\mathcal{T}$ be an $\mathcal{EL}$ TBox and $A \in sig_C(\mathcal{T})$. For each $K \in \texttt{Pre}(A)$, each $B \in K$ and each $M \in \texttt{Pre}(B)$, we have $(K/\{B\}) \cup M \in \texttt{Pre}(A)$.*

The above lemma can be shown by an easy induction over the derivation of $M$ from $B$.

In essence, the lemma below implies that, in case of flattened terminologies explicitly containing all elements of Pre, we can derive all subsumees of a concept by (1) applying the rule (EX) to construct existential restrictions from two concepts in a subsumption relation and/or (2) replacing concepts occurring within subsumees by their subsumees.

**Lemma 7.** *Let $\mathcal{T}$ be a flattened $\mathcal{EL}$ TBox and $C, D$ two $\mathcal{EL}$ concepts with $sig(C) \cup sig(D) \subseteq sig(\mathcal{T})$ such that $\mathcal{T} \models C \sqsubseteq D$. Let*

$$C = \bigsqcap_{1 \leq j \leq n} A_j \sqcap \bigsqcap_{1 \leq k \leq m} \exists r_k.E_k$$

*where $A_j$ are concept symbols, $r_k$ are role symbols and $E_k$ are arbitrary $\mathcal{EL}$ concepts. Then, for all conjuncts $D_i$ of $D$, the following is true: If $D_i$ is a concept symbol, there is a set $M \in \texttt{Pre}(D_i)$ of concept symbols from $sig_C(\mathcal{T})$ such that at least one of the conditions [A1]-[A2] holds for each $B \in M$:*

*(A1) There is an $A_j$ in $C$ such that $A_j = B$.*

*(A2) There are $r_k, E_k$ and there exists $B' \in sig_C(\mathcal{T})$ such that $\mathcal{T} \models E_k \sqsubseteq B'$ and $B \equiv \exists r_k.B' \in \mathcal{T}$.*

*If $D_i = \exists r'.D'$ for a role $r'$ and an $\mathcal{EL}$ concept $D'$, at least one of the conditions [A3]-[A4] holds:*

*(A3) There are $r_k, E_k$ such that $r_k = r'$ and $\mathcal{T} \models E_k \sqsubseteq D'$.*

*(A4) There is $B \in sig_C(\mathcal{T})$ such that $\mathcal{T} \models B \sqsubseteq \exists r'.D'$ and $\mathcal{T} \models C \sqsubseteq B$ and for $C \sqsubseteq B$ at least one of the conditions [A1]-[A2] holds.*

*Proof.* We apply induction on the length of the proof. We start with the last applied rule and show for each possibility that the lemma holds. Rules AXTOP, AX and the case $C \bowtie D \in \mathcal{T}$ are the basis of induction, since each proof begins with one of them.

$(C \bowtie D \in \mathcal{T})$ In the case that $C \sqsubseteq D \in \mathcal{T}$ or $C \equiv D \in \mathcal{T}$, the lemma holds due to the flattening. Axioms within $\mathcal{T}$ can have the following form:

- $C, D \in \text{sig}_C(\mathcal{T})$. In this case, $\{C\} \in \texttt{Pre}(D)$. Therefore, condition [A1] holds.
- $C \in \text{sig}_C(\mathcal{T}), D = D_1 \sqcap ... \sqcap D_m$ with $D_1, ..., D_m \in \text{sig}_C(\mathcal{T})$. In this case, for each $D_i$ with $1 \leq i \leq m$ holds $\{C\} \in \texttt{Pre}(D_i)$. Therefore, condition [A1] holds for each $D_i$.
- $C \in \text{sig}_C(\mathcal{T}), D = \exists r'.D'$ with $D' \in \text{sig}_C(\mathcal{T})$. This case corresponds to the condition [A4].

(AXTOP) Since the conjunction is empty in case $D = \top$, the lemma holds.

(AX) Since $C = D$, for each $D_i$ there is a conjunct $C_i$ of $C$ with $C_i = D_i$. If $D_i$ is a concept symbol, condition [A1] of the lemma holds. Otherwise, [A3].

(EX) If EX was the last applied rule, then $D_i = \exists r_k.D'$ and $\mathcal{T} \vdash D_k \sqsubseteq D'$. Therefore, [A3] of the lemma holds.

(ANDL) Assume that $C' \sqcap C'' = C$ such that $C' \sqsubseteq D$ is the antecedent. By induction hypothesis, the lemma holds for $C' \sqsubseteq D$. Since all conjuncts of $C'$ are also conjuncts of $C$, the lemma holds also for $C \sqsubseteq D$.

(ANDR) Assume that $D = D_1 \sqcap D_2$, therefore, $C \sqsubseteq D_1$ and $C \sqsubseteq D_2$ is the antecedent. By induction hypothesis, the lemma holds for both, $C \sqsubseteq D_1$ and $C \sqsubseteq D_2$. Since all conjuncts of $D$ are from either $D_1$ or $D_2$, the lemma also holds for $C \sqsubseteq D$.

(CUT) By induction hypothesis, the lemma holds for both elements of the antecedent, $C \sqsubseteq C_1$ and $C_1 \sqsubseteq D$. Let $C_1 = \bigsqcap_{1 \leq p \leq r} A_p \sqcap \bigsqcap_{1 \leq s \leq t} \exists r'_s.E'_s$.

1. Assume that $D_i$ is a concept symbol. Then, there is $M_1 \in \mathtt{Pre}(D_i)$ such that [A1] or [A2] holds for each $B_u \in M_1$. We now consider each $C \sqsubseteq B_u$ and distinguish three cases, in one of which [A2] holds. In the remaining two cases, we can obtain $M_{\text{new}}$ by replacing $B_u$ within $M_1$ by the elements of some $M'_u \in \mathtt{Pre}(B_u)$ such that [A1] or [A2] holds for each $B' \in M_{\text{new}}$ and $C \sqsubseteq B'$:

   A1 Assume that there is a conjunct $A_p$ of $C_1$ with $A_p = B_1$. Then, by induction hypothesis, for $C \sqsubseteq A_p$, there is $M'_u \in \mathtt{Pre}(A_p)$ such that [A1] or [A2] holds for each $B' \in M'_u$. We can replace $B_u$ within $M_1$ by the elements of $M'_u$.

   A2 Assume that for $B_u$ there are $r'_s, E'_s$ and there exists $B' \in \text{sig}_C(\mathcal{T})$ such that $\mathcal{T} \models E'_s \sqsubseteq B'$ and $B \equiv \exists r'_s.B' \in \mathcal{T}$. Then, for $C \sqsubseteq \exists r'_s.E'_s$ either [A3] or [A4] can hold:

   -(A3) There are $r_k, E_k$ such that $r_k = r'_s$ and $\mathcal{T} \models E_k \sqsubseteq E'_s$. Then [A2] holds for $C \sqsubseteq B_u$, since $\mathcal{T} \models E_k \sqsubseteq B'$ and $B \equiv \exists r_k.B' \in \mathcal{T}$.

   -(A4) There is $B'' \in \text{sig}_C(\mathcal{T})$ such that $\mathcal{T} \models B'' \sqsubseteq \exists r'_s.E'_s$, $\mathcal{T} \models C \sqsubseteq B''$ and there is a set $M'_u \in \mathtt{Pre}(B'')$ such that for each element $B'$ of $M'_u$ at least one of the conditions [A1]-[A2] holds with respect to $C \sqsubseteq B'$.

   Let $M_{A1}$ be the set of all such $B_u \in M_1$ for which [A1] holds and let $M_{A4}$ be the set of all such $B_u \in M_1$ for which [A2] holds and for $C \sqsubseteq \exists r'_s.E'_s$ [A4] holds. Now we replace each $B_u$ within $M_1$ by the elements of the corresponding set $M'_u \in \mathtt{Pre}(B_u)$ that we have specified above and obtain $M_{\text{new}} = M_1 \setminus (M_{A1} \cup M_{A4}) \cup \bigcup \{M'_u \mid B_u \in M_{A1} \cup M_{A4}\}$. Clearly, $M_{\text{new}} \in \mathtt{Pre}(D_i)$ and [A1] or [A2] holds for each $B' \in M_{\text{new}}$ with respect to $C \sqsubseteq B'$, i.e., the lemma holds for $C \sqsubseteq D_i$.

2. Assume that $D_i = \exists r'.D'$. Then, [A3] or [A4] hold.

   A3 There are $r'_s, E'_s$ such that $r' = r'_s$ and $\mathcal{T} \models E'_s \sqsubseteq D'$. Then, for $C \sqsubseteq \exists r'_s.E'_s$ one of [A3], [A4] holds:

   -(A3) There are $r_k, E_k$ such that $r_k = r'_s$ and $\mathcal{T} \models E_k \sqsubseteq E'_s$. Then [A3] holds for $C \sqsubseteq D_i$, since $\mathcal{T} \models E_k \sqsubseteq D'$ and $r_k = r'$.

   -(A4) There is a concept symbol $B''$ such that $\mathcal{T} \models B'' \sqsubseteq \exists r'_s.E'_s$, $\mathcal{T} \models C \sqsubseteq B''$ and there is a set $M'' \in \mathtt{Pre}(B'')$ of concept symbols such that at least one of the conditions [A1]-[A2] holds for each element $B'$ of $M''$ and $C \sqsubseteq B'$. Since $\mathcal{T} \models B'' \sqsubseteq D_i$, [A4] holds for $\mathcal{T} \models C \sqsubseteq D_i$.

A4 There is a concept symbol $B$ such that $\mathcal{T} \models B \sqsubseteq \exists r'.D'$, $\mathcal{T} \models C_1 \sqsubseteq B$ and there is a set $M_1 \in \texttt{Pre}(B)$ such that at least one of the conditions [A1]-[A2] holds for each element $B_u$ of $M_1$ and for $C_1 \sqsubseteq B_u$. The argumentation is the same as for 1 ($D_i$ is a concept symbol). We consider each $C \sqsubseteq B_u$ and distinguish three cases, in one of which [A2] holds. In the remaining two cases, we can obtain $M_{\text{new}}$ by replacing $B_u$ within $M_1$ by the elements of some $M'_u \in \texttt{Pre}(B_u)$ such that [A1] or [A2] holds for each $B' \in M_{\text{new}}$ and $C \sqsubseteq B'$. Therefore, there is $M_1 \in \texttt{Pre}(B)$ such that either [A1] or [A2] holds for each $B_u \in M_1$. Then, [A4] holds for $C \sqsubseteq D_i$. □

The above lemma is focused on the derivation of subsumees. For the computation of uniform interpolants, we additionally need to show that, in flattened terminologies, every subsumption relation with an concept symbol and its subsumer being an existential restriction is derived from an equivalence axiom of the form $B_1 \equiv \exists r.B_2 \in \mathcal{T}$.

**Lemma 8.** *Let $\mathcal{T}$ be a flattened $\mathcal{EL}$ TBox, $A \in sig_C(\mathcal{T})$ and $r \in sig_R(\mathcal{T})$. Let $C$ be an $\mathcal{EL}$ concept such that $\mathcal{T} \models A \sqsubseteq \exists r.C$. Then, there are $B_1, B_2 \in sig_C(\mathcal{T})$ with $B_1 \equiv \exists r.B_2 \in \mathcal{T}$ such that $\mathcal{T} \models A \sqsubseteq B_1$, $\mathcal{T} \models B_2 \sqsubseteq C$.*

*Proof.* Lemma 16 [27] states that for a general $\mathcal{EL}$ TBox $\mathcal{T}$ with $\mathcal{T} \models C_1 \sqsubseteq \exists r.C_2$, where $C_1, C_2$ are $\mathcal{EL}$-concepts one of the following holds:

- there is a conjunct $\exists r.C'$ of $C_1$ such that $\mathcal{T} \models C' \sqsubseteq C_2$;
- there is a subconcept $\exists r.C'$ of $\mathcal{T}$ such that $\mathcal{T} \models C_1 \sqsubseteq \exists r.C'$ and $\mathcal{T} \models C' \sqsubseteq C_2$;

The first condition does not hold in this lemma, since $A$ is a concept symbol. Moreover, since in our case $\mathcal{T}$ is flattened, for each subconcept $\exists r.C'$ of $\mathcal{T}$ containing an existential restriction holds: there is an concept symbol $B_2 \in \text{sig}_C(\mathcal{T})$ such that $B_2 = C'$ and there is an axiom of the form $B_1 \equiv \exists r.B_2 \in \mathcal{T}$ with $B_1 \in \text{sig}_C(\mathcal{T})$. Additionally, from the above Lemma 16 follows $\mathcal{T} \models A \sqsubseteq \exists r.B_2$ and $\mathcal{T} \models B_2 \sqsubseteq C$. Since $\mathcal{T} \models B_1 \equiv \exists r.B_2$, it follows that also $\mathcal{T} \models A \sqsubseteq B_1$. □

### *6.2. Grammar Representation of Subsumees and Subsumers*

In this section, we show how, for a signature $\Sigma$, the sets of $\Sigma$-subsumees and $\Sigma$-subsumers of each concept symbol in a flattened $\mathcal{EL}$ TBox $\mathcal{T}$ can be described as languages generated by regular tree grammars on ranked ordered trees. In our definition of grammars, we uniquely represent each concept symbol $A \in \text{sig}_C(\mathcal{T})$ by a non-terminal $\mathfrak{n}_A$ (and denote the set of all non-terminals by $\mathcal{N}^{\mathcal{T}} = \{\mathfrak{n}_x | x \in \text{sig}_C(\mathcal{T}) \cup \{\top\}\}$). In what follows, we use the ranked alphabet $\mathcal{F} = (\text{sig}_C(\mathcal{T}) \cap \Sigma) \cup \{\top\} \cup \{\exists r^1 \mid r \in \text{sig}_R(\mathcal{T}) \cap \Sigma\} \cup \{\sqcap^i \mid 2 \leq i \leq |\text{sig}_C(\mathcal{T})|\}$, where $\top$ and concept symbols in $\text{sig}_C(\mathcal{T}) \cap \Sigma$ are constants, $\exists r^1$ for $r \in \text{sig}_R(\mathcal{T}) \cap \Sigma$ are unary functions and $\sqcap^i$ are functions of the arity $2 \leq i \leq |\text{sig}_C(\mathcal{T})|$. Due to flattening, $|\text{sig}_C(\mathcal{T})|$ is the highest arity of conjunctions that can occur in our TBox. In the following, it will be convenient to simply write $\sqcap$ and $\exists r$ if the arity of the corresponding function is clear from the context. Clearly, every $\mathcal{EL}$ concept $C$ with $\text{sig}(C) \subseteq \Sigma$ and at most $|\text{sig}_C(\mathcal{T})|$ conjuncts in each subconcept has a unique representation by the means of the above functions. We denote such a term representation of $C$ using $\mathcal{F}$ by $t_C$. For a term $t$, we denote its concept representation by $C_t$. Additionally, we use a substitution function $\sigma_{\mathcal{T},\mathcal{F}} : \{C \mid \text{sig}(C) \subseteq \text{sig}(\mathcal{T})\} \to T(\mathcal{F}, \mathcal{N}^{\mathcal{T}})$ with $\sigma_{\mathcal{T},\mathcal{F}}(C) = t_C\{\mathfrak{n}_\top/\top, \mathfrak{n}_{B_1}/B_1, ..., \mathfrak{n}_{B_n}/B_n\}$, where $B_1, ..., B_n$ are all concept symbols occurring in $C$. If the TBox and the set of non-terminals are clear from the context, we will denote such a representation of a concept $C$ simply by $\sigma(C)$.

As mentioned above, weak subsumees and subsumers are not required in order to obtain a uniform $\mathcal{EL}$ interpolant. In fact, including weak subsumees into our definition of the grammars would lead to a non-elementary upper bound on the generated language despite the bounded role depth. Also weak subsumers lead to an exponential blow-up in the size of the corresponding grammar. Thus, we avoid generating weak subsumees and subsumers by the corresponding grammars.

**Definition 7.** *Let $\mathcal{T}$ be a flattened $\mathcal{EL}$ TBox, $\Sigma$ a signature. Further, for each $B \in sig_C(\mathcal{T}) \cup \{\top\}$, let $R^{\sqsupseteq}$ be given by*

**(GL1)** $\mathfrak{n}_B \to B$ *if* $B \in \Sigma \cup \{\top\}$,

**(GL2)** $\mathfrak{n}_B \to \mathfrak{n}_{B'}$ *for all* $B' \in sig_C(\mathcal{T}) \cup \{\top\}$ *with* $\mathcal{T} \models B' \sqsubseteq B$, $B \neq B'$

**(GL3)** $\mathfrak{n}_B \to \sqcap(\mathfrak{n}_{B_1}, ..., \mathfrak{n}_{B_n})$ *for all* $B \equiv B_1 \sqcap \ldots \sqcap B_n \in \mathcal{T}$,

**(GL4)** $\mathfrak{n}_B \to \exists r(\mathfrak{n}_{B'})$ *for all* $B \equiv \exists r.B' \in \mathcal{T}$ *with* $r \in sig_R(\mathcal{T}) \cap \Sigma$.

*Let $R^{\sqsubseteq}$ be given for all $B \in sig_C(\mathcal{T}) \cup \{\top\}$ by*

**(GR1)** $\mathfrak{n}_B \to B$ *if* $B \in \Sigma \cup \{\top\}$,

**(GR2)** $\mathfrak{n}_B \to \mathfrak{n}_{B'}$ *if* $B \neq B'$ *and either* $B' = \top$ *or* $B'$ *is the only concept symbol such that* $\mathcal{T} \models B \sqsubseteq B'$,

**(GR3)** $\mathfrak{n}_B \to \sqcap(\mathfrak{n}_{B_1}, ..., \mathfrak{n}_{B_n})$ *if* $\{B_1, \ldots, B_n\} = \{B' \in sig_C(\mathcal{T}) \mid \mathcal{T} \models B' \sqsupseteq B\}$ *and* $n \geq 2$,

**(GR4)** $\mathfrak{n}_B \to \exists r(\mathfrak{n}_{B'})$ *for all* $B \equiv \exists r.B' \in \mathcal{T}$ *with* $r \in sig_R(\mathcal{T}) \cap \Sigma$.

*For every $A \in sig_C(\mathcal{T})$, the regular tree grammar $G^{\sqsupseteq}(\mathcal{T}, \Sigma, A)$ is given by $(\mathfrak{n}_A, \mathcal{N}^{\mathcal{T}}, \mathcal{F}, R^{\sqsupseteq})$. Likewise, the regular tree grammar $G^{\sqsubseteq}(\mathcal{T}, \Sigma, A)$ is given by $(\mathfrak{n}_A, \mathcal{N}^{\mathcal{T}}, \mathcal{F}, R^{\sqsubseteq})$.*

We denote the set of tree grammars $\{G^{\sqsupseteq}(\mathcal{T}, \Sigma, A) \mid A \in \text{sig}_C(\mathcal{T})\}$ by $\mathbb{G}^{\sqsupseteq}(\mathcal{T}, \Sigma)$ and the set $\{G^{\sqsubseteq}(\mathcal{T}, \Sigma, A) \mid A \in \text{sig}_C(\mathcal{T})\}$ by $\mathbb{G}^{\sqsubseteq}(\mathcal{T}, \Sigma)$. For the construction of grammars the following result holds.

**Theorem 2.** *Let $\mathcal{T}$ be a flattened $\mathcal{EL}$ TBox and let $\Sigma$ be a signature. $\mathbb{G}^{\sqsupseteq}(\mathcal{T}, \Sigma)$ and $\mathbb{G}^{\sqsubseteq}(\mathcal{T}, \Sigma)$ can be computed from $\mathcal{T}$ in polynomial time and are at most polynomial in the size of $\mathcal{T}$.*

*Proof.* Flattening and classification can be done all together in polynomial time [11] and yield an at most polynomial result. From this result, the grammars are constructed in polynomial time. □

The following example demonstrates the grammar construction.

**Example 5.** *Let $\mathcal{T} = \{A_1 \sqsubseteq \exists r A_2, \exists r B_1 \sqcap B_3 \sqsubseteq B_2, A_2 \sqsubseteq B_1\}$. In order to flatten the given TBox, we introduce fresh concept names for $\exists r A_2, \exists r B_1$ and $B_1' \sqcap B_3$ to obtain $\mathcal{T}'$:*

$$\begin{array}{rl} A_1 \sqsubseteq A_2' & \quad A_2 \sqsubseteq B_1 \\ B_2' \sqsubseteq B_2 & \quad B_1' \sqcap B_3 \equiv B_2' \\ \exists r B_1 \equiv B_1' & \quad \exists r A_2 \equiv A_2' \end{array}$$

*Let $\Sigma = sig(\mathcal{T}) \setminus \{B_1\}$. Then, we introduce terminals for each concept symbol from $\Sigma$ and the $\top$ concept according to (GL1) and (GR1):*

$$\mathfrak{n}_{A_1} \to A_1 \qquad \mathfrak{n}_{A_2} \to A_2 \qquad \mathfrak{n}_{B_2} \to B_2 \qquad \mathfrak{n}_{\top} \to \top \tag{10}$$

*If we only use subsumees given before the classification of $\mathcal{T}'$, we obtain the following set of transitions $R^{\sqsupseteq}$ for generating subsumees of concept symbols:*

$$\mathfrak{n}_{A_2'} \rightarrow \mathfrak{n}_{A_1} \qquad \mathfrak{n}_{B_1} \rightarrow \mathfrak{n}_{A_2} \tag{11}$$

$$\mathfrak{n}_{B_2} \rightarrow \mathfrak{n}_{B_2'} \qquad \mathfrak{n}_{B_2'} \rightarrow \sqcap(\mathfrak{n}_{B_1'}, \mathfrak{n}_{B_3}) \tag{12}$$

$$\mathfrak{n}_{B_1'} \rightarrow \exists r(\mathfrak{n}_{B_1}) \qquad \mathfrak{n}_{A_2'} \rightarrow \exists r(\mathfrak{n}_{A_2}) \tag{13}$$

*We see that the subsumee $\exists r.A_2 \sqcap B_3$ of $B_2$ is not generated by the above set of transitions. If we classify $\mathcal{T}'$ before constructing the grammar, we obtain additionally*

$$\mathfrak{n}_{B_1'} \rightarrow \mathfrak{n}_{A_2'} \qquad \mathfrak{n}_{B_1'} \rightarrow \mathfrak{n}_{B_2'} \qquad \mathfrak{n}_{B_3} \rightarrow \mathfrak{n}_{B_2'} \qquad \mathfrak{n}_{B_1'} \rightarrow \mathfrak{n}_{A_1} \tag{14}$$

*Accordingly, $R^{\sqsubseteq}$ is given by Rules 10,13 and, additionally*

$$\mathfrak{n}_{A_1} \rightarrow \mathfrak{n}_{\top} \qquad \mathfrak{n}_{A_2} \rightarrow \mathfrak{n}_{\top} \qquad \mathfrak{n}_{B_1} \rightarrow \mathfrak{n}_{\top} \qquad \mathfrak{n}_{B_2} \rightarrow \mathfrak{n}_{\top} \tag{15}$$

$$\mathfrak{n}_{B_3} \rightarrow \mathfrak{n}_{\top} \qquad \mathfrak{n}_{A_2'} \rightarrow \mathfrak{n}_{\top} \qquad \mathfrak{n}_{B_1'} \rightarrow \mathfrak{n}_{\top} \qquad \mathfrak{n}_{B_2'} \rightarrow \mathfrak{n}_{\top} \tag{16}$$

$$\mathfrak{n}_{A_1} \rightarrow \mathfrak{n}_{A_2'} \qquad \mathfrak{n}_{A_2} \rightarrow \mathfrak{n}_{B_1} \qquad \mathfrak{n}_{A_2'} \rightarrow \mathfrak{n}_{B_1'} \tag{17}$$

$$\mathfrak{n}_{B_2'} \rightarrow \sqcap(\mathfrak{n}_{B_1'}, \mathfrak{n}_{B_3}, \mathfrak{n}_{B_2}) \tag{18}$$

In the above example, we can generate all non-weak subsumees using the complete grammar construction, i.e., after including the results of classification in addition to transitions representing explicitly given subsumptions. For instance, the subsumee $\exists r.A_2 \sqcap B_3$ of $B_2$ can be generated using the first additional rule in 14 as follows: $\mathfrak{n}_{B_2} \rightarrow \mathfrak{n}_{B_2'} \rightarrow \sqcap(\mathfrak{n}_{B_1'}, \mathfrak{n}_{B_3}) \rightarrow \sqcap(\mathfrak{n}_{A_2'}, \mathfrak{n}_{B_3}) \rightarrow \sqcap(\exists r(\mathfrak{n}_{A_1}), \mathfrak{n}_{B_3}) \rightarrow \sqcap(\exists r(A_1), B_3)$.

We now consider various properties of the above grammars that are of interest for the computation of uniform interpolants. The following theorem states that the grammars derive only terms representing $\Sigma$-subsumees and $\Sigma$-subsumers of the corresponding concept symbol.

**Theorem 3.** *Let $\mathcal{T}$ be a flattened $\mathcal{EL}$ TBox, $\Sigma$ a signature and $A \in sig_C(\mathcal{T})$.*

1. *For each $t \in L(G^{\sqsupseteq}(\mathcal{T}, \Sigma, A)) \cup L(G^{\sqsubseteq}(\mathcal{T}, \Sigma, A))$ it holds that $sig(C_t) \subseteq \Sigma$.*
2. *For each $t \in L(G^{\sqsupseteq}(\mathcal{T}, \Sigma, A))$ it holds that $\mathcal{T} \models C_t \sqsubseteq A$.*
3. *For each $t \in L(G^{\sqsubseteq}(\mathcal{T}, \Sigma, A))$ it holds that $\mathcal{T} \models A \sqsubseteq C_t$.*

*Proof.* 1. It is easy to check given Definition 7 that the grammars derive only terms containing concept symbols and roles from $\Sigma$, since $\mathfrak{n}_B \to B$ only if $B \in \Sigma \cup \{\top\}$ and $\mathfrak{n}_B \to \exists r(t')$ only if $r \in \Sigma$. Therefore, for any $A \in \mathrm{sig}_C(\mathcal{T})$ and any $t \in L(G^{\sqsubseteq}(\mathcal{T}, \Sigma, A)) \cup L(G^{\sqsupseteq}(\mathcal{T}, \Sigma, A))$ holds $\mathrm{sig}(C_t) \subseteq \Sigma$.

2. We use an easy induction on the maximal nesting depth of functions in $t$ using the rules given in Definition 7:

   - Assume that $C_t$ is a concept symbol $B$ or $\top$. The term $B$ can only be derived from $\mathfrak{n}_A$ by $n$ empty transitions (GL2), and, once $\mathfrak{n}_B$ is derived, the rule (GL1). Let $B_1, ..., B_n$ be such that $\mathfrak{n}_A \to \mathfrak{n}_{B_1} \to ... \to \mathfrak{n}_{B_n} \to \mathfrak{n}_B$. Then, by Definition 7, for each pair $B_i, B_{i+1}$ holds $\mathcal{T} \models B_i \sqsupseteq B_{i+1}$, for $B_n, B$ holds $\mathcal{T} \models B_n \sqsupseteq B$ and for $A, B_1$ holds $\mathcal{T} \models A \sqsupseteq B_1$. It follows that also $\mathcal{T} \models A \sqsupseteq B$.
   - Assume that $t = \exists r(t')$ for some term $t'$. Then, the derivation of $t$ from $\mathfrak{n}_A$ starts with $n$ empty transitions (GL2) such that $\mathfrak{n}_{B'}$ for some $B' \in \mathrm{sig}_C(\mathcal{T}) \cup \{\top\}$ is derived, and a subsequent application of (GL4) such that $\mathfrak{n}_B$ for some $B \in \mathrm{sig}_C(\mathcal{T}) \cup \{\top\}$ is derived. As argued above about the applications of empty transitions, $\mathcal{T} \models A \sqsupseteq B'$ holds. Moreover, By Definition 7 (GL4) holds $B' \equiv \exists r.B \in \mathcal{T}$, and, therefore, $\mathcal{T} \models A \sqsupseteq \exists r.B$. Let $C' = C_{t'}$. Then, by induction hypothesis, $\mathcal{T} \models B \sqsupseteq C'$. Therefore, $\mathcal{T} \models A \sqsupseteq \exists r.C'$, while $\exists r.C' = C_t$.
   - Assume that $t = \sqcap(t_1, ..., t_n)$ for a set of terms $t_1, ..., t_n$. Then, the derivation of $t$ from $\mathfrak{n}_A$ starts with $m$ empty transitions (GL2) such that $\mathfrak{n}_{B'}$ for some $B' \in \mathrm{sig}_C(\mathcal{T}) \cup \{\top\}$ is derived, and a subsequent application of (GL3) such that we derive $\sqcap(\mathfrak{n}_{B_1}, ..., \mathfrak{n}_{B_n})$, where $t_i \in L(G^{\sqsupseteq}(\mathcal{T}, \Sigma, \mathfrak{n}_{B_i}))$ for $1 \leq i \leq n$. As argued above about the applications of empty transitions, $\mathcal{T} \models A \sqsupseteq B'$ holds. Let $C_i = C_{t_i}$. By induction hypothesis, $\mathcal{T} \models B_i \sqsupseteq C_i$. By Definition 7, $B' \equiv B_1 \sqcap ... \sqcap B_n \in \mathcal{T}$. Therefore, $\mathcal{T} \models B' \sqsupseteq C_1 \sqcap ... \sqcap C_n$ and $\mathcal{T} \models A \sqsupseteq C_1 \sqcap ... \sqcap C_n$ with $C_1 \sqcap ... \sqcap C_n = C_t$.

3. The proof of soundness of $\mathbb{G}^{\sqsubseteq}(\mathcal{T}, \Sigma))$ can be done in the same manner, i.e., by induction on the maximal nesting depth of functions in $t$:

   - Assume that $C_t$ is a concept symbol $B$ or $\top$. The term $B$ can only be derived from $\mathfrak{n}_A$ by $n$ empty transitions (GR2), and, once $\mathfrak{n}_B$ is derived, the rule (GR1). Let $B_1, ..., B_n$ be such that $\mathfrak{n}_A \to \mathfrak{n}_{B_1} \to ... \to \mathfrak{n}_{B_n} \to \mathfrak{n}_B$. Then, by Definition 7, for each pair $B_i, B_{i+1}$ holds

$\mathcal{T} \models B_i \sqsubseteq B_{i+1}$, for $B_n, B$ holds $\mathcal{T} \models B_n \sqsubseteq B$ and for $A, B_1$ holds $\mathcal{T} \models A \sqsubseteq B_1$. It follows that also $\mathcal{T} \models A \sqsubseteq B$ with $C_t = B$.

- Assume that $t = \exists r(t')$ for some term $t'$. Then, the derivation of $t$ from $\mathfrak{n}_A$ starts with $n$ empty transitions (GR2) such that $\mathfrak{n}_{B'}$ for some $B' \in \mathrm{sig}_C(\mathcal{T}) \cup \{\top\}$ is derived, and a subsequent application of a non-empty transition (GR4) such that $\exists r.\mathfrak{n}_B$ for some $B \in \mathrm{sig}_C(\mathcal{T}) \cup \{\top\}$ is derived. As argued above about the applications of empty transitions, $\mathcal{T} \models A \sqsubseteq B'$ holds. Moreover, By Definition 7, it holds that $\mathcal{T} \models B' \equiv \exists r.B$, and, therefore, $\mathcal{T} \models A \sqsubseteq \exists r.B$. Let $C' = C_{t'}$. By induction hypothesis, $\mathcal{T} \models B \sqsubseteq C'$. Therefore, $\mathcal{T} \models A \sqsubseteq \exists r.C'$ with $C_t = \exists r.C'$.

- Assume that $t = \sqcap(t_1, ..., t_n)$ for a set of terms $t_1, ..., t_n$. Then, the derivation of $t$ from $\mathfrak{n}_A$ starts with $m$ empty transitions (GR2) such that $\mathfrak{n}_{B'}$ for some $B' \in \mathrm{sig}_C(\mathcal{T}) \cup \{\top\}$ is derived, and a subsequent application of (GR3) such that we derive $\sqcap(\mathfrak{n}_{B_1}, ..., \mathfrak{n}_{B_n})$, where $t_i \in L(G^{\sqsupseteq}(\mathcal{T}, \Sigma, \mathfrak{n}_{B_i}))$ for $1 \leq i \leq n$ and $n \geq 2$. As argued above about the applications of empty transitions, $\mathcal{T} \models A \sqsubseteq B'$ holds. Let $C_i = C_{t_i}$. By induction hypothesis, $\mathcal{T} \models B_i \sqsubseteq C_i$. By Definition 7, $\mathcal{T} \models B' \sqsubseteq B_1 \sqcap ... \sqcap B_n$. Therefore, $\mathcal{T} \models B' \sqsubseteq C_1 \sqcap ... \sqcap C_n$ and $\mathcal{T} \models A \sqsubseteq C_1 \sqcap ... \sqcap C_n$ with $C_1 \sqcap ... \sqcap C_n = C_t$.

To be able to show completeness of the grammars, we first show that the commutative associative closure of the generated $G^{\sqsupseteq}$ language contains all elements of `Pre`.

**Lemma 9.** *Let $\mathcal{T}$ be flattened $\mathcal{EL}$ TBox and $\Sigma$ a signature. Let $G = G^{\sqsupseteq}(\mathcal{T}, \Sigma, A))$ and, for a concept symbol $A$, let $K \in$ `Pre`$(A)$. Then, $\sigma(\prod_{B \in K} B) \in L_u^*(G^{\sqsupseteq}(\mathcal{T}, \Sigma, A)))$.*

*Proof.* The lemma can be shown by an easy induction on the depth of derivation of $K$ from $A$. We distinguish three cases for the last derivation step.

- If $K = \{A\}$, then the lemma is a direct consequence of Definition 7 (GL1).

- Assume that $K$ has been obtained from $K' \in$ `Pre`$(A)$ by replacing some $B$ by some $B'$ such that $\mathcal{T} \models B' \sqsubseteq B$. By induction hypothesis, $\sigma(\prod_{B'' \in K'} B'') \in L_u^*(G^{\sqsupseteq}(\mathcal{T}, \Sigma, A)))$. By Definition 7 (GL2), we have $\mathfrak{n}_B \rightarrow \mathfrak{n}_{B'} \in R^{\sqsupseteq}$. Thus, also $\sigma(\prod_{B \in K} B) \in L_u^*(G^{\sqsupseteq}(\mathcal{T}, \Sigma, A)))$.

- Assume that $K$ has been obtained from $K' \in \texttt{Pre}(A)$ by replacing some $B$ by some $B_1, \ldots, B_n$ such that $B \equiv B_1 \sqcap ... \sqcap B_n \in \mathcal{T}$. By induction hypothesis, $\sigma(\bigsqcap_{B'' \in K'} B'') \in L_u^*(G^{\sqsupseteq}(\mathcal{T}, \Sigma, A)))$. By Definition 7 (GL3), we have $\mathfrak{n}_B \to \sqcap(\mathfrak{n}_{B_1}, ..., \mathfrak{n}_{B_n}) \in R^{\sqsupseteq}$. Thus, also $\sigma(\bigsqcap_{B \in K} B) \in L_u^*(G^{\sqsupseteq}(\mathcal{T}, \Sigma, A)))$. □

As discussed above, grammars do not guarantee to capture weak subsumees and subsumers. Therefore, we obtain the following result for the completeness of the grammars.

**Theorem 4.** *Let $\mathcal{T}$ be a flattened $\mathcal{EL}$ TBox, $\Sigma$ a signature and $A$ a concept symbol.*

1. *For each $C$ with $sig(C) \subseteq \Sigma$ such that $\mathcal{T} \models C \sqsubseteq A$ there is a concept $C'$ with $t_{C'} \in L^*(G^{\sqsupseteq}(\mathcal{T}, \Sigma, A))$ such that $C$ can be obtained from $C'$ by adding arbitrary conjuncts to arbitrary subconcepts.*
2. *For each $C$ with $sig(C) \subseteq \Sigma$ such that $\mathcal{T} \models A \sqsubseteq C$ there is a concept $C'$ with $t_{C'} \in L^*(G^{\sqsubseteq}(\mathcal{T}, \Sigma, A))$ such that $C$ can be obtained from $C'$ by removing $\top$ conjuncts from arbitrary subconcepts.*

*Proof.* The theorem is proved by induction on the role depth of C using the properties of the flattening, for instance, stated in Lemmas 7, in addition to Definition 7 and Lemma 9. Let

$$C = \bigsqcap_{1 \leq j \leq n} A_j \sqcap \bigsqcap_{1 \leq k \leq m} \exists r_k.E_k$$

where $A_j$ are concept symbols, $r_k$ are role symbols and $E_k$ are arbitrary $\mathcal{EL}$ concepts. W.l.o.g., we can assume that all $A_j$ are pairwise different.

1. We prove the first claim as follows:
   - Assume role depth = 0. Then, $C$ is a conjunction of concept symbols, i.e., $C = \bigsqcap_{1 \leq j \leq n} A_j$. By Lemma 7, there is a set $M' \in \texttt{Pre}(A)$ of concept symbols such that, for each $B \in M'$, there is an $A_j$ with $A_j = B$. By Lemma 9, $\sigma(\bigsqcap_{B \in M'} B) \in L_u^*(G^{\sqsupseteq}(\mathcal{T}, \Sigma, A)))$. Since each $B \in M'$ is in $\Sigma$, by Definition 7 (GL1), $\mathfrak{n}_B \to B \in R^{\sqsupseteq}$. It follows that $t_C \in L^*(G^{\sqsupseteq}(\mathcal{T}, \Sigma, A))$.
   - Assume that the role depth is greater than 0. As in the case above, there is a set $M' \in \texttt{Pre}(A)$ of concept symbols such that, for each $B \in M'$, [A1] or [A2] holds. Let $M'_1$ be the subset of $M'$ where [A1] holds, i.e., $M'_1 = M' \cap \{A_1, ...A_n\}$, and let $M'_2 = M' \setminus M'_1$. In accordance with this separation of $M'$ into $M'_1$ and $M'_2$, we can also identify the

two corresponding sub-conjunctions of $C$: Let $C'_1 = \bigsqcap_{B \in M'_1} B$, and $C'_2 = \bigsqcap_{1 \leq f \leq p} \exists r'_f.E'_f$ such that for each $f$ there is a corresponding $B_f \in M'_2$.

For each $f$ it holds that there exists a concept symbol $B'_f$ with $\mathcal{T} \models E'_f \sqsubseteq B'_f$ and $B_f \equiv \exists r.B'_f \in \mathcal{T}$. By induction hypothesis, for each $f$ there exists a concept $E''_f$ such that $t_{E''_f} \in L^*(G^{\sqsupseteq}(\mathcal{T}, \Sigma, B'_f))$ and $E'_f$ can be obtained from $E''_f$ by adding arbitrary conjuncts to arbitrary subconcepts. By Definition 7 (GL4), $\mathfrak{n}_{B_f} \to \exists r'_f(\mathfrak{n}_{B'_f}) \in R^{\sqsupseteq}$. Therefore, $\exists r'_f(t_{E''_f}) \in L^*(G^{\sqsupseteq}(\mathcal{T}, \Sigma, B_f))$ and $\exists r'_f.E'_f$ can be obtained from $\exists r'_f.E''_f$ by adding arbitrary conjuncts to arbitrary subconcepts.

Since each $B \in M'_1$ is in $\Sigma$, we have $\mathfrak{n}_B \to B \in R^{\sqsupseteq}$ by Definition 7 (GL1). By Lemma 9, $\sigma(\bigsqcap_{B \in M'} B) \in L^*_u(G^{\sqsupseteq}(\mathcal{T}, \Sigma, A)))$. Thus, we obtain a concept expression $C''' = \bigsqcap_{B \in M'_1} B \sqcap \bigsqcap_{B_f \in M'_2} \exists r'_f.E''_f$ with $t_{C''} \in L^*(G^{\sqsupseteq}(\mathcal{T}, \Sigma, A))$ such that $C$ can be obtained from it by adding arbitrary conjuncts to arbitrary subconcepts.

2. We proceed with showing that for each such general $C$ with $\mathrm{sig}(C) \subseteq \Sigma$ such that $\mathcal{T} \models A \sqsubseteq C$ there is a concept $C'$ such that $t_{C'} \in L^*(G^{\sqsubseteq}(\mathcal{T}, \Sigma, A))$ and $C$ can be obtained from $C'$ by removing $\top$ conjuncts from arbitrary subconcepts. For each $A_j$, we know that $\mathcal{T} \models A \sqsubseteq A_j$ and $A_j \in \Sigma \cup \{\top\}$. By Definition 7 (GR1) $\mathfrak{n}_{A_j} \to A_j \in R^{\sqsubseteq}$ for all $A_j$. Assume a role depth $0$.

   - Assume that $n = 1$, i.e., $C = A_1$, and assume that $A_1$ is the only concept symbol such that $\mathcal{T} \models A \sqsubseteq A_1$. By Definition 7 (GR2) $\mathfrak{n}_A \to \mathfrak{n}_{A_1} \in R^{\sqsubseteq}$. Thus, $t_C \in L^*(G^{\sqsubseteq}(\mathcal{T}, \Sigma, A))$.

   - Assume that there are more than one concept symbol $A_i$ such that $\mathcal{T} \models A \sqsubseteq A_i$. By Definition 7 (GR3), $\mathfrak{n}_A \to \sqcap(\mathfrak{n}_{A_1}, ..., \mathfrak{n}_{A_x}) \in R^{\sqsubseteq}$ for some $x \geq n$. By Definition 7 (GR2), there is $\mathfrak{n}_{A_i} \to \mathfrak{n}_\top \in R^{\sqsubseteq}$ for all $A_i$. By applying (GR1) for all $A_j$ and $\mathfrak{n}_{A_i} \to \mathfrak{n}_\top$, $\mathfrak{n}_\top \to \top$ for all $i > n$, we obtain a term $t_{C \sqcap C'}$, where $C'$ is a conjunction of $x - n$ concepts $\top$. Thus, the theorem holds for role depth 0.

   Assume a role depth $> 0$. For each $\exists r_k.E_k$, it follows from Lemma 8 that there are $B_k, B''_k \in \mathrm{sig}_C(\mathcal{T})$ with $B_k \equiv \exists r_k.B''_k \in \mathcal{T}$ such that $\mathcal{T} \models A \sqsubseteq B_k$, $\mathcal{T} \models B''_k \sqsubseteq E_k$. By Definition 7 (GR4), $\mathfrak{n}_{B_k} \to \exists r_k(\mathfrak{n}_{B''_k}) \in R^{\sqsubseteq}$. By induction hypothesis, there is a concept $E'_k$ such that $t_{E'_k} \in L^*(G^{\sqsubseteq}(\mathcal{T}, \Sigma, B''_k))$ and $E_k$ can be obtained from $E'_k$ by removing $\top$ conjuncts from arbitrary subconcepts.

- Assume that there is the only one concept symbol $B'$ such that $\mathcal{T} \models A \sqsubseteq B'$. Then, $C = \exists r_1.E_1$ and $B_1 = B'$. By Definition 7 (GR2) $\mathfrak{n}_A \to \mathfrak{n}_{B'} \in R^{\sqsubseteq}$. Thus, $t_{\exists r_1.E'_1} \in L(G^{\sqsubseteq}(\mathcal{T}, \Sigma, A))$ and $\exists r_1.E_1$ can be obtained from $\exists r_1.E'_1$ by removing $\top$ conjuncts from arbitrary subconcepts.
- Assume that there are more than one concept symbol $B'$ such that $\mathcal{T} \models A \sqsubseteq B'$. By Definition 7 (GR3), $\mathfrak{n}_A \to \sqcap(\mathfrak{n}_{B'_1}, ..., \mathfrak{n}_{B'_x}) \in R^{\sqsubseteq}$ for some $x \geq n+m$ such that $B'_j = A_j$ for $1 \leq j \leq n$ and $B'_{n+k} = B_k$ for $1 \leq k \leq m$. By Definition 7 (GR2), there is $\mathfrak{n}_{B'_i} \to \mathfrak{n}_\top \in R^{\sqsubseteq}$ for all $B'_i$. Now, we derive the term $t_{C'' \sqcap C'}$ from $\mathfrak{n}_A$ by first applying $\mathfrak{n}_A \to \sqcap(\mathfrak{n}_{B'_1}, ..., \mathfrak{n}_{B'_x})$ and then proceeding as follows:
  - from each $B'_i$ with $i > n+m$, we derive $\top$ by applying $\mathfrak{n}_{B'_i} \to \mathfrak{n}_\top$, $\mathfrak{n}_\top \to \top$;
  - from each $B'_j = A_j$ with $1 \leq j \leq n$, we derive $A_j$ by applying $\mathfrak{n}_{B'_j} \to A_j$;
  - from each $B'_{n+k} = B_k$ with $1 \leq k \leq m$, we derive $t_{\exists r_k.E'_k}$.

  We obtain a term $t_{C'' \sqcap C'} \in L^*(G^{\sqsubseteq}(\mathcal{T}, \Sigma, A))$, where $C'$ is a conjunction of concepts $\top$ and $C'' = \bigsqcap_{1 \leq j \leq n} A_j \sqcap \bigsqcap_{1 \leq k \leq m} \exists r_k.E'_k$. Clearly, $C$ can be obtained from $C''$ by removing $\top$ conjuncts from arbitrary subconcepts. Thus, $C$ can be obtained from $C'' \sqcap C'$ by removing $\top$ conjuncts from arbitrary subconcepts. □

### *6.3. From Grammars to Uniform Interpolants*

Now we show that, as a consequence of Lemma 4 and Theorem 4, in case a finite uniform interpolant exists, we can construct it from the subsumees and subsumers of maximal depth $N = 2^{4 \cdot |\mathrm{sub}(\mathcal{T})|} + 1$ generated by the grammars $\mathbb{G}^{\sqsupseteq}(\mathcal{T}, \Sigma), \mathbb{G}^{\sqsubseteq}(\mathcal{T}, \Sigma)$. Given the grammars, the corresponding subsumee/subsumer relation pair $\langle L_{\sqsupseteq}, L_{\sqsubseteq} \rangle$ is given by $L_{\bowtie} = \{(A, C) \mid t_C \in L(G^{\bowtie}(\mathcal{T}, \Sigma, A)), d(C) \leq N\}$ for $\bowtie \in \{\sqsupseteq, \sqsubseteq\}$ and $A \in \mathrm{sig}_C(\mathcal{T})$. Note that, if all subsumees and subsumers are using only concepts and roles from $\Sigma$ (follows from Theorem 3), then $\mathrm{sig}(\mathtt{M}(L_{\sqsupseteq}, L_{\sqsubseteq}, \Sigma)) \subseteq \Sigma$. We obtain the following result concerning the size of uniform $\mathcal{EL}$ $\Sigma$-interpolants:

**Theorem 5.** *Let $\mathcal{T}$ be a flattened version of an $\mathcal{EL}$ TBox $\mathcal{T}_{nf}$ and $\Sigma$ a signature with $\Sigma \cap sig(\mathcal{T}) \subseteq sig(\mathcal{T}_{nf})$. For $N = 2^{4 \cdot |sub(\mathcal{T}_{nf})|} + 1$, $\bowtie \in \{\sqsupseteq, \sqsubseteq\}$ and $A \in sig_C(\mathcal{T})$, let $L_{\bowtie}(A) = \{C \mid t_C \in L(G^{\bowtie}(\mathcal{T}, \Sigma, A)), d(C) \leq N\}$. The following statements are equivalent:*

1. *There exists a uniform $\mathcal{EL}$ $\Sigma$-interpolant of $\mathcal{T}_{nf}$.*

2. $\mathtt{M}(L_{\sqsupseteq}, L_{\sqsubseteq}, \Sigma) \equiv_{\Sigma}^{\mathcal{EL}} \mathcal{T}_{nf}$

3. *There exists a uniform $\mathcal{EL}$ $\Sigma$-interpolant $\mathcal{T}'$ of $\mathcal{T}_{nf}$ with $|\mathcal{T}'| \in O(2^{2^{2^{|\mathcal{T}_{nf}|}}})$.*

*Proof.* We prove the implications $1 \Rightarrow 2$ and $2 \Rightarrow 3$. All other implications are either trivial or follow from the others. For convenience, let $\mathcal{T}_\Sigma$ denote the TBox $\mathtt{M}(L_{\sqsupseteq}, L_{\sqsubseteq}, \Sigma)$.

$1 \Rightarrow 2$**:** First, note that the statement $\mathcal{T}_\Sigma \equiv_{\Sigma}^{\mathcal{EL}} \mathcal{T}_{nf}$ follows from Lemma 5 and the fact that $\Sigma \cap \text{sig}(\mathcal{T}) \subseteq \text{sig}(\mathcal{T}_{nf})$. Thus, it is sufficient to prove $\mathcal{T}_\Sigma \equiv_{\Sigma}^{\mathcal{EL}} \mathcal{T}$. By Definition 2, the statement $\mathcal{T}_\Sigma \equiv_{\Sigma}^{\mathcal{EL}} \mathcal{T}$ consists of two directions: (1) for all $\mathcal{EL}$ concepts $C, D$ with $\text{sig}(C) \cup \text{sig}(D) \subseteq \Sigma$ holds $\mathcal{T}_\Sigma \models C \sqsubseteq D \Rightarrow \mathcal{T} \models C \sqsubseteq D$ and (2) for all $\mathcal{EL}$ concepts $C, D$ with $\text{sig}(C) \cup \text{sig}(D) \subseteq \Sigma$ holds $\mathcal{T}_\Sigma \models C \sqsubseteq D \Leftarrow \mathcal{T} \models C \sqsubseteq D$.

**(1)** The first direction follows from Theorem 3 and Definition 5. Theorem 3 ensures that the subsumee/subsumer relation pair $\langle L_{\sqsupseteq}, L_{\sqsubseteq} \rangle$ does not contain any subsumees or subsumers not being entailed by $\mathcal{T}$ and that it consists only of symbols from $\Sigma \cup \{\top\}$. Definition 5 ensures that $\mathcal{T}_\Sigma$ does not contain any concepts that do not occur in $\langle L_{\sqsupseteq}, L_{\sqsubseteq} \rangle$.

**(2)** For the second direction, assume that there exists a uniform $\mathcal{EL}$ $\Sigma$-interpolant of $\mathcal{T}_{nf}$ and, subsequently, $\mathcal{T}$. Then, by Lemma 4, there exists a uniform $\mathcal{EL}$ $\Sigma$-interpolant $\mathcal{T}'$ of $\mathcal{T}_{nf}$ and $\mathcal{T}$ with $d(\mathcal{T}') \leq N$. It is sufficient to show that for each $C \sqsubseteq D \in \mathcal{T}'$ holds $\mathcal{T}_\Sigma \models C \sqsubseteq D$. Assume that $C \sqsubseteq D \in \mathcal{T}'$. We prove by induction on maximal role depth of $C, D$ that also $\mathcal{T}_\Sigma \models C \sqsubseteq D$. Let $D = \bigsqcap_{1 \leq i \leq l} D_i$ and

$$C = \bigsqcap_{1 \leq j \leq n} A_j \sqcap \bigsqcap_{1 \leq k \leq m} \exists r_k . E_k$$

where $A_j$ are concept symbols, $r_k$ are role symbols and $E_k$ are arbitrary $\mathcal{EL}$ concepts. Clearly, $\mathcal{T} \models C \sqsubseteq D$, iff $\mathcal{T} \models C \sqsubseteq D_i$ for all $i$ with $1 \leq i \leq l$.

- If $D_i$ is a concept symbol, then, it follows from Theorem 4 that there is a concept $C'$ such that $t_{C'} \in L^*(G^{\sqsupseteq}(\mathcal{T}, \Sigma, A))$ and $C$ can be obtained from $C'$ by adding arbitrary conjuncts to arbitrary subconcepts. Since $d(C) \leq N$, also $d(C') \leq N$. Therefore, $\mathcal{T}_\Sigma \models C \sqsubseteq D_i$.

- If $D_i = \exists r.D'$ for some $r, D'$, then, by Lemma 7, one of the following is true:

  (A3) There are $r_k, E_k$ in $C$ such that $r_k = r$ and $\mathcal{T} \models E_k \sqsubseteq D'$. Since $d(E_k) < N$ and $d(D') < N$, by induction hypothesis holds $\mathcal{T}_\Sigma \models E_k \sqsubseteq D'$. It follows that $\mathcal{T}_\Sigma \models \exists r_k.E_k \sqsubseteq D_i$ and $\mathcal{T}_\Sigma \models C \sqsubseteq D_i$.

  (A4) There is a concept symbol $B \in \mathrm{sig}_C(\mathcal{T})$ such that $\mathcal{T} \models B \sqsubseteq \exists r.D'$ and $\mathcal{T} \models C \sqsubseteq B$. Then,

  - it follows from Theorem 4 that there is a concept $C_1'$ such that $t_{C_1'} \in L^*(G^{\sqsupseteq}(\mathcal{T}, \Sigma, A))$ and $C$ can be obtained from $C_1'$ by and adding arbitrary conjuncts to arbitrary subconcepts. Since $d(C) \leq N$, also $d(C_1') \leq N$. Therefore, $(B, C_1'') \in L_{\sqsupseteq}$ for some associative commutative variant $C_1''$ of $C_1'$.
  - it follows from Theorem 4 that there is a concept $C_2'$ such that $t_{C_2'} \in L^*(G^{\sqsubseteq}(\mathcal{T}, \Sigma, B))$ and $\exists r.D'$ can be obtained from $C_2'$ by removing $\top$ conjuncts from arbitrary subconcepts. Since $d(\exists r.D') \leq N$, also $d(C_2') \leq N$ and it follows that $(B, C_2'') \in L_{\sqsubseteq}$ for some associative commutative variant $C_2''$ of $C_2'$.

  By Definition 5, $C_1'' \sqsubseteq C_2'' \in \mathcal{T}_\Sigma$, and, therefore, $\mathcal{T}_\Sigma \models C \sqsubseteq D_i$.

$2 \Rightarrow 3$**:** Observe that $\mathbb{G}_1, \mathbb{G}_2$ have $n = |\mathrm{sig}_C(\mathcal{T})|$ non-terminals and $n$ is also the maximal arity of $\sqcap$. Now we consider the stepwise generation of terms in $L(G^{\sqsupseteq}(\mathcal{T}, \Sigma, A))$ and $L(G^{\sqsubseteq}(\mathcal{T}, \Sigma, A))$. Initially, terms are given by transitions. Assume that $m$ is the maximal number of transitions in $\mathbb{G}_1, \mathbb{G}_2$, where is polynomial in $n$. Each of these outgoing transitions has at most $n$ occurring non-terminals. For a term $t$ of role depth $x$, we can obtain a term of the role depth $x + 1$ by first applying transition rules of type GL1-GL3 ( GR1-GR3 in case of subsumer terms) to replace non-terminals $\mathfrak{n}$ by terms $t'$ and then applying transitions of type GL4 (GR4). In case of subsumees, we can assume that it is sufficient to consider terms $t'$ with a maximal function depth $m$ (maximal number of transitions), since a repeated application of the same transition of type GL3 generates a weak subsumee that is not required for the construction of the uniform interpolant. The total maximal depth of function nestings in subsumee terms is then $N \cdot m$. In case of subsumers, the term of the role depth $x+1$ is obtained by applying at most one rule of type GR3 for each non-terminal, since the corresponding conjunctions in GR3 contain all non-terminals that can be obtained by infinitely many successive applications of GR1-GR3. The total maximal depth of function nestings in subsumer terms is then $N \cdot 2$. Given the maximal function depth $N \cdot m$, the maximal arity $n$ of functions and the number $n$ of different non-terminals, we obtain at most $n^{n^{N \cdot m}}$ different terms. Since in $N \in O(2^n)$, the size of terms is in $O(2^{2^n})$ while the number of terms is in $O(2^{2^{2^n}})$.

These complexity results correspond to the size and number of axioms in Example 4 used to demonstrate the triple-exponential lower bound. □

## 7. Related Work

In addition to the already discussed results on uniform interpolation in description logics [19, 18, 20, 28, 29, 16, 17], in this section we discuss the work on inseparability and conservative extensions. The latter two notions form the foundation for module extraction, e.g., [30, 17, 26], and decomposition of ontologies into modules, e.g., [31, 32, 33]. The notion of a conservative extension is defined using inseparability: A TBox $\mathcal{T}_1$ is called a $\Sigma$-conservative extension of a TBox $\mathcal{T}_2$ if $\mathcal{T}_1$ is $\Sigma$-inseparable from $\mathcal{T}_2$ and $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

Ghilardi, Lutz and Wolter [34] investigate modularity of ontologies based on concept-inseparability. They show that deciding if a subontology is a module in the description logic $\mathcal{ALC}$ is 2ExpTime-complete. In a subsequent work, Lutz, Walter and Wolter [35] show that the same problem is 2ExpTime-complete for $\mathcal{ALCQI}$, but undecidable for $\mathcal{ALCQIO}$. The authors also investigate a stronger notion of inseparability and conservative extensions defined directly on models instead of entailed consequences: given two TBoxes $\mathcal{T}_1$ and $\mathcal{T}_2$, $\mathcal{T}_1$ is a *model-conservative extension* of $\mathcal{T}_2$ iff for every model $\mathcal{I}$ of $\mathcal{T}_2$, there exists a model of $\mathcal{T}_1$ which can be obtained from $\mathcal{I}$ by modifying the interpretation of symbols in $\text{sig}(\mathcal{T}_1) \setminus \text{sig}(\mathcal{T}_2)$ while leaving the interpretation of symbols in $\text{sig}(\mathcal{T}_2)$ fixed. The authors show that the corresponding problem based on the latter notion is undecidable for $\mathcal{ALC}$.

In a more recent work, Konev, Lutz, Walter and Wolter [26] consider the decidability of the above problem based on model-conservative extensions for $\mathcal{ALC}$ under different additional restrictions, e.g., restriction of the relevant signature to concept names, and obtain complexity results ranging from $\Pi_2^p$ to undecidable. Further, the authors consider the problem for acyclic $\mathcal{EL}$ terminologies. It is interesting that, in contrast to acyclic $\mathcal{ALC}$ terminologies, for which the problem remains undecidable, for acyclic $\mathcal{EL}$ terminologies the complexity goes down to PTime. In a later work [36], the above authors present a full complexity picture for $\mathcal{ALC}$ and its common extensions. They investigate a broad range of query languages (languages in which the relevant consequences are expressed), starting with the language allowing for expressing inconsistency only and ending with Second Order Logic. More recently, Lutz and Wolter [27] show that the above notion of model-conservative extensions is undecidable also for such a lightweight logic as $\mathcal{EL}$.

Kontchakov, Wolter and Zakharyaschev [37] investigate the above decision problem for two representatives of the DL-Lite family of description logics as ontology languages and existential $\Sigma$-queries as a query language. They show that, for DL-Lite$_{\texttt{horn}}$, the problem is coNP-complete, and for DL-Lite$_{\texttt{bool}}$ $\Pi_2^p$-complete.

The high complexity results for already rather simple logics have lead to a development of alternative ways to extract modules not requiring checking inseparability. For instance, Cuenca Grau, Horocks, Kazakov and Sattler [30], propose a tractable algorithm for computing modules from OWL DL ontologies based on the notion of *syntactic locality* [38] that defines the locality of an axiom on the syntactic level, i.e., states syntactic conditions for the potential logical relevance of axioms. It is guaranteed that the extracted module preserves all relevant consequences, but the obtained modules are not necessarily minimal.

## 8. Summary and Outlook

In this article, we have discussed the task of uniform interpolation, which guarantees a preservation of the relevant subset of the deductive closure while eliminating all references to irrelevant entities.

We provided an approach to computing uniform interpolants of general $\mathcal{EL}$ terminologies based on proof theory and regular tree languages. Moreover, we showed that, if a finite uniform $\mathcal{EL}$ interpolant exists, then there exists one of at most triple exponential size in terms of the original TBox, and that, in the worst-case, no shorter interpolant exists, thereby establishing a tight triple exponential bound. This is an important foundational insight, since it reveals the effect of structure sharing in the basic logic $\mathcal{EL}$.

The result brings about some insights when it comes to the practical applicability of uniform interpolation for module extraction and related tasks. In order to prevent a triple exponential blowup in the worst-case, we need to impose restrictions on rewriting, in that certain signature elements are kept even if not considered relevant. For instance, in [39], we obtain first, preliminary results in this direction. We show that, despite the worst-case triple exponential blowup, uniform interpolation can be very useful as a basis for rewriting aiming at an elimination of irrelevant information from ontologies.

On the other hand, the results of this article reveal the potential of structure sharing for improving the conciseness of ontologies. By introducing a reverse operation to uniform interpolation, namely the elimination of structural redundancy from ontologies via vocabulary extension, we maybe able to "compress" ontologies in a semantics-preserving way, obtaining up to triple-exponentially more concise representations of $\mathcal{EL}$ ontologies in the best case. This raises a new practically relevant research question, which is particularly interesting for improving reasoning efficiency.


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## Appendix A. Model-Theoretic Properties of $\mathcal{EL}$ Concepts

In Section 2, we characterize $\mathcal{EL}$ concept membership and $\mathcal{EL}$ concept subsumption in the absence of terminological background knowledge. In this section, we include the according proofs.

**Lemma 1.** For any $\mathcal{EL}$ concept expression $C$ and any interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ and $x \in \Delta^{\mathcal{I}}$ it holds that $x \in C^{\mathcal{I}}$ if and only if there is a homomorphism from $(\mathcal{I}_C, x_C)$ to $(\mathcal{I}, x)$.

*Proof.* We prove both directions by structural induction over $C$.

We start with the if-direction, letting $\varphi$ be the homomorphism from $(\mathcal{I}_C, x_C)$ to $(\mathcal{I}, x)$:

- For $C = \top$, the case is trivial.
- For $C = A \in N_C$, we find $x_A \in A^{\mathcal{I}_A}$, therefore the existence of the homomorphism ensures that $x = \varphi(x_A) \in A^{\mathcal{I}}$.
- For $C = C_1 \sqcap C_2$, we find that $\varphi_\iota : \Delta^{\mathcal{I}_{C_\iota}} \to \Delta^{\mathcal{I}}$ defined by

$$\varphi_\iota(y) = \begin{cases} x & \text{if } y = x_{C_\iota} \\ \varphi(y') & \text{if } y = (y', \iota) \end{cases}$$

for $\iota \in \{1,2\}$ are homomorphisms from $(\mathcal{I}_{C_1}, x_{C_1})$ to $(\mathcal{I}, x)$ and $(\mathcal{I}_{C_2}, x_{C_2})$ to $(\mathcal{I}, x)$, respectively. Invoking the induction hypothesis, we conclude that $x \in C_1^{\mathcal{I}}$ as well as $x \in C_2^{\mathcal{I}}$ and thus $x \in C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}} = (C_1 \sqcap C_2)^{\mathcal{I}}$.

- Considering $C = \exists r.C_1$, we find that $\varphi' = \varphi|_{\Delta^{\mathcal{I}_{C_1}}}$ is a homomorphism from $(\mathcal{I}_{C_1}, x_{C_1})$ to $(\mathcal{I}, \varphi(x_{C_1}))$. Invoking the induction hypothesis, we conclude $\varphi'(x_{C_1}) = \varphi(x_{C_1}) \in C_1^{\mathcal{I}}$. On the other hand, by construction of $\mathcal{I}_C$ we find $(x_C, x_{C_1}) \in r^{\mathcal{I}_C}$ and thus, since $\varphi$ is a homomorphism $(x, \varphi(x_{C_1}) = (\varphi(x_C), \varphi(x_{C_1}) \in r^{\mathcal{I}}$. Together, this allows to conclude $x \in (\exists r.C_1)^{\mathcal{I}}$.

We proceed with the only-if direction.

- For $C = \top$, the case is trivial.

- For $C = A \in N_C$, the mapping $\varphi = \{x_A \mapsto x\}$ is the required homomorphism since by assumption it holds that $x \in A^{\mathcal{I}}$.

- For $C = C_1 \sqcap C_2$, we have by assumption $x \in C^{\mathcal{I}} = C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}}$ therefore $x \in C_1^{\mathcal{I}}$ and $x \in C_2^{\mathcal{I}}$. Invoking the induction hypothesis we find homomorphisms $\varphi_1$ from $(\mathcal{I}_{C_1}, x_{C_1})$ to $(\mathcal{I}, x)$ and $\varphi_2$ from $(\mathcal{I}_{C_2}, x_{C_2})$ to $(\mathcal{I}, x)$. Consequently, by construction of $\mathcal{I}_C$, the mapping $\varphi : \Delta^{\mathcal{I}_C} to \Delta^{\mathcal{I}}$ defined by

$$\varphi(y) = \begin{cases} x & \text{if } y = x_C \\ \varphi_1(y') & \text{if } y = (y', 1) \\ \varphi_2(y') & \text{if } y = (y', 2) \end{cases}$$

  is a homomorphism from $(\mathcal{I}_C, x_C)$ to $(\mathcal{I}, x)$.

- For $C = \exists r.C_1$, we find by assumption $x \in (\exists r.C_1)^{\mathcal{I}}$ thus there exists an $x' \in \Delta^{\mathcal{I}}$ with $(x, x') \in r^{\mathcal{I}}$ and $x' \in C_1^{\mathcal{I}}$. Invoking the induction hypothesis, we find a homomorphism $\varphi'$ from $(\mathcal{I}_{C_1}, x_{C_1})$ to $(\mathcal{I}, x')$. Consequently the mapping $\varphi : \Delta^{\mathcal{I}_C} \to \Delta^{\mathcal{I}}$ with $\varphi = \varphi' \cup \{x_C \mapsto x\}$ is a homomorphism from $(\mathcal{I}_C, x_C)$ to $(\mathcal{I}, x)$.

□

**Lemma 2.** *Let $C$ and $C'$ be two $\mathcal{EL}$ concept expressions. Then $\emptyset \models C \sqsubseteq C'$ if and only if there is a homomorphism from $(\mathcal{I}'_C, x'_C)$ to $(\mathcal{I}_C, x_C)$.*

*Proof.* For the if-direction, let $\varphi$ be the homomorphism from $(\mathcal{I}'_C, x'_C)$ to $(\mathcal{I}_C, x_C)$. Now let $\mathcal{I}$ be an interpretation and pick an arbitrary $x \in \Delta^{\mathcal{I}}$ with $x \in C^{\mathcal{I}}$. By

Lemma 1, there exists a homomorphism $\varphi'$ from $(\mathcal{I}_C, x_C)$ to $(\mathcal{I}, x)$. Then $\varphi' \circ \varphi$ is a homomorphism from $(\mathcal{I}_{C'}, x_{C'})$ to $(\mathcal{I}, x)$ and by the other direction of Lemma 1, we can conclude $x \in C'$. Thus $C^{\mathcal{I}} \subseteq C'^{\mathcal{I}}$ for all interpretations $\mathcal{I}$ and therefore $\emptyset \models C \sqsubseteq C'$.

For the only-if-direction, assume $\emptyset \models C \sqsubseteq C'$. Now consider the pointed interpretation $(\mathcal{I}_C, x_C)$. As the identity on $\Delta^{\mathcal{I}_C}$ is a homomorphism from $(\mathcal{I}_C, x_C)$ to itself, we use Lemma 1 to conclude $x_C \in C^{\mathcal{I}_C}$. By $\emptyset \models C \sqsubseteq C'$ we can infer that $x_C \in C'^{\mathcal{I}_C}$. Invoking the if-direction of Lemma 1, we find that there must be a homomorphism from $(\mathcal{I}'_C, x'_C)$ to $(\mathcal{I}_C, x_C)$. □

## Appendix B. $\mathcal{EL}$ Automata

In this appendix section, we recall core notions on $\mathcal{EL}$ automata [20] before giving the proof of Lemma 4.

**Definition 11 [20].** *An $\mathcal{EL}$* automaton (EA) *is a tuple $\mathcal{A} = (Q, P, \Sigma_N, \Sigma_E, \delta)$, where $Q$ is a finite set of* bottom up states, *$P$ is a finite set of* top down states, *$\Sigma_N \subseteq N_C$ is the finite* node alphabet, *$\Sigma_E \subseteq N_R$ is the finite* edge alphabet, *and $\delta$ is a set of transitions of the following form:*

$$
\begin{array}{rclcrcl}
\mathit{true} & \rightarrow & q & \quad & p & \rightarrow & p_1 \qquad\qquad (B.1)\\
A & \rightarrow & q & & p & \rightarrow & \langle r \rangle p1 \qquad\qquad (B.2)\\
q_1 \wedge \ldots \wedge q_n & \rightarrow & q & & p & \rightarrow & A \qquad\qquad (B.3)\\
\langle r \rangle q1 & \rightarrow & q & & p & \rightarrow & \mathit{false} \qquad\qquad (B.4)\\
q & \rightarrow & p & & & & \qquad\qquad (B.5)
\end{array}
$$

*where $q, q_1, ..., q_n$ range over $Q$, $p, p_1$ range over $P$, $A$ ranges over $\Sigma_N$, and $r$ ranges over $\Sigma_E$.*

**Definition 12 [20].** *Let $\mathcal{I}$ be an interpretation and $\mathcal{A} = (Q, P, \Sigma_N, \Sigma_E, \delta)$ an EA. A* run *of $\mathcal{A}$ on $\mathcal{I}$ is a map $\rho : \delta \rightarrow 2^{Q \cup P}$ such that for all $d \in \Delta^{\mathcal{I}}$, we have:*

1. *if $\mathit{true} \rightarrow q \in \delta$, then $q \in \rho(d)$;*
2. *if $A \rightarrow q \in \delta$, and $d \in A^{\mathcal{I}}$, then $q \in \rho(d)$;*
3. *if $q_1, ..., q_n \in \rho(d)$ and $q_1 \wedge \ldots \wedge q_n \rightarrow q \in \delta$, then $q \in \rho(d)$;*
4. *if $(d, e) \in r^{\mathcal{I}}$, $q_1 \in \rho(e)$ and $\langle r \rangle q_1 \rightarrow q \in \delta$, then $q \in \rho(d)$;*
5. *if $q \in \rho(d)$ and $q \rightarrow p \in \delta$, then $p \in \rho(d)$;*

6. *if* $p \in \rho(d)$ *and* $p \to p_1 \in \delta$*, then* $p_1 \in \rho(d)$*;*
7. *if* $p \in \rho(d)$ *and* $p \to \langle r \rangle p_1 \in \delta$*, then there is an* $(d, e) \in r^{\mathcal{I}}$ *with* $p_1 \in \rho(e)$*;*
8. *if* $p \in \rho(d)$ *and* $p \to A \in \delta$*, then* $d \in A^{\mathcal{I}}$*;*
9. *if* $p \to \mathit{false} \in \delta$*, then* $p \notin \rho(d)$*.*

The following Proposition specifies how the corresponding EA $\mathcal{A}$ for any TBox $\mathcal{T}$ can be constructed such that $\mathcal{T}_\Sigma(\mathcal{A}) \equiv_\Sigma^{\mathcal{EL}} \mathcal{T}$ for any $\Sigma$.

**Construction from Proposition 13 [20]** *Let* $\mathcal{T}$ *be a TBox,* $s(\mathcal{T})$ *subconcepts of* $\mathcal{T}$ *and* $\mathcal{A} = (Q, P, \mathit{sig}_C(\mathcal{T}), \mathit{sig}_R(\mathcal{T}), \delta)$ *with* $Q = \{q_C | C \in s(\mathcal{T})\}, P = \{p_C | C \in s(\mathcal{T})\}$ *and* $\delta$ *given by*

- $\mathit{true} \to q_\top$ *if* $\top \in s(\mathcal{T})$*;*
- $A \to q_A$ *and* $q_A \to p_A$ *for all* $A \in \mathit{sig}_C(\mathcal{T})$*;*
- $q_C \wedge q_D \to q_{C \sqcap D}$*;*
- $\langle r \rangle q_C \to q_{\exists r.C}$ *and* $q_{\exists r.C} \to \langle r \rangle p_C$ *for all* $\exists r.C \in s(\mathcal{T})$*;*
- $q_C \to q_D$ *for all* $C, D \in s(\mathcal{T})$ *with* $\mathcal{T} \models C \sqsubseteq D$*;*
- $p_A \to A$ *for all* $A \in \mathit{sig}_C(\mathcal{T})$*;*
- $p_{\exists r.C} \to \langle r \rangle p_C$ *for all* $\exists r.C \in s(\mathcal{T})$*;*
- $p_C \to p_D$ *for all* $C, D \in s(\mathcal{T})$ *with* $\mathcal{T} \models C \sqsubseteq D$*;*
- $p_\bot \to \mathit{false}$ *if* $\bot \in s(\mathcal{T})$*.*

An EA $\mathcal{A}$ is said to entail a subsumption $C \sqsubseteq D$ if every model accepted by $\mathcal{A}$ satisfies $C \sqsubseteq D$. Subsequently, an EA $\mathcal{A}$ and a TBox $\mathcal{T}$ are $\mathcal{EL}$ $\Sigma$-inseparable, in symbols $\mathcal{A} \equiv_\Sigma^{\mathcal{EL}} \mathcal{T}$, if $\mathcal{A} \models C \sqsubseteq D$ iff $\mathcal{T} \models C \sqsubseteq D$ for all $\mathcal{EL}$ $\Sigma$-inclusions $C \sqsubseteq D$. Further, for a signature $\Sigma$, $\mathcal{T}_\Sigma(\mathcal{A}) = \{C \sqsubseteq D \mid \mathcal{A} \models C \sqsubseteq D, \mathrm{sig}(C) \cup \mathrm{sig}(D) \subseteq \Sigma\}$. For a natural number $m$, $\mathcal{T}_\Sigma^m(\mathcal{A}) = \{C \sqsubseteq D \mid C \sqsubseteq D \in \mathcal{T}_\Sigma(\mathcal{A}), d(C) \leq m$ and $d(D) \leq m\}$.

**Excerpt from Lemma 55 [20].** *Let* $\mathcal{A}$ *be an EA and* $M_\mathcal{A} = 2^{|P \cup Q|}$*. The following conditions are equivalent:*

1. *There exists* $k > M_\mathcal{A}^2 + 1$ *such that* $\mathcal{T}_\Sigma^{M_\mathcal{A}^2+1} \not\models \mathcal{T}_\Sigma^k$*;*
4. *There does not exists an* $\mathcal{EL}$ *TBox* $\mathcal{T}$ *with* $\mathcal{A} \equiv_\Sigma^{\mathcal{EL}} \mathcal{T}$*.*

**Lemma 4.** *Let $\mathcal{T}$ be an $\mathcal{EL}$ TBox, $\Sigma$ a signature. The following statements are equivalent:*

1. *There exists a uniform $\mathcal{EL}$ $\Sigma$-interpolant of $\mathcal{T}$.*
2. *There exists a uniform $\mathcal{EL}$ $\Sigma$-interpolant $\mathcal{T}'$ of $\mathcal{T}$ for which holds $d(\mathcal{T}') \leq 2^{4\cdot(|sub(\mathcal{T})|)} + 1$.*

*Proof.* Assume that a uniform $\mathcal{EL}$ $\Sigma$-interpolant of $\mathcal{T}$ exists and let $M = 2^{(2\cdot|\mathrm{sub}(\mathcal{T})|)}$. Then, by Lemma 55 [20], there is no $k > M^2+1$ such that $\mathcal{T}_\Sigma^{M^2+1}(\mathcal{A}) \not\models \mathcal{T}_\Sigma^{k}(\mathcal{A})$, where $\mathcal{A}$ is the corresponding $\mathcal{EL}$ automaton for $\mathcal{T}$. Then $\mathcal{T}_\Sigma^{M^2+1}(\mathcal{A}) \models \mathcal{T}_\Sigma(\mathcal{A})$. Therefore, $\mathcal{T}_\Sigma^{M^2+1}(\mathcal{A}) \equiv_\Sigma^{\mathcal{EL}} \mathcal{T}$, i.e., $\mathcal{T}_\Sigma^{M^2+1}(\mathcal{A})$ is a uniform $\mathcal{EL}$ $\Sigma$-interpolant $\mathcal{T}'$ of $\mathcal{T}$ with $d(\mathcal{T}') \leq M^2+1$. We can replace $M^2+1$ by $2^{4\cdot(|\mathrm{sub}(\mathcal{T})|)}+1$ and obtain $d(\mathcal{T}') \leq 2^{4\cdot(|\mathrm{sub}(\mathcal{T})|)} + 1$. □

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