Decidability

We investigate the power of algorithms to solve problems. We demonstrate that certain problems can be solved algorithmically and others cannot. Our objective is to explore the limits of algorithmic solvability.

Encoding Turing Machines as strings

We can encode Turing Machines as strings over any non-empty alphabet.

E.g. if the string begins with the prefix

$$0^n\,1\,0^m\,1\,0^k\,1\,0^s\,1\,0^t\,1\,0^r\,1\,0^u\,1\,0^v$$

this might indicate that the machine has n states $(0, 1, \cdots, n-1)$; it has m tape symbols $(0, 1, \cdots, m-1)$, of which the first k are input symbols; the start, accept and reject states are s, t and r respectively, and the endmarker and blank symbols are u and v respectively. The remainder of the string can consist of a sequence of substrings specifying the transitions in δ.

E.g. $0^p\,1\,0^a\,1\,0^q\,1\,0^b 1\,0$ might indicate that δ contains the transition

$$((p, a), (q, b, L)).$$

The exact details are unimportant.

There are languages that are not RE

There are only countably many TMs.

Observe that Σ^* is countable, for any alphabet Σ. Since $\langle M \rangle \in \Sigma^*$ for each TM M, we can enumerate the set of TMs by simply omitting those strings that are not encodings of TMs.

The set of languages over Σ, $\mathcal{P}(\Sigma^*)$, is uncountable.

Suppose there were an enumeration of $\mathcal{P}(\Sigma^*)$, namely, $L_1, L_2, L_3, \cdots$. Let $x_1, x_2, x_3, \cdots$ be an enumeration of Σ^*. Define a new language L by: for $i \geq 1$

$$x_i \in L \iff x_i \notin L_i$$

Now $L \in \mathcal{P}(\Sigma^*)$, but L is not equal to any of the L_is.

Thus $\mathcal{P}(\Sigma^*)$ is not in 1-1 correspondence with the set of Turing machines (there are more languages than TMs). It follows that there is some language that is not accepted by any TM.

DFA Acceptance Problem

DFA Acceptance Problem: Given a DFA B and an input w, does B accept w?

The corresponding language is

$$A_{\mathrm{DFA}} = \{ \langle B, w \rangle : B \text{ is a DFA that accepts input string } w \}.$$

Lemma. A_{DFA} is a decidable language.

Proof. We construct a TM M that decides A_{DFA}: On input $\langle B, w \rangle$ where B is a DFA and w an input

1. Simulate B on input w
2. If the simulation ends in an accept state, *accept*. If it ends in a non-accepting state, *reject*.

Convince yourself that it follows that A_{NFA} and A_{regexp} (acceptance problems for NFA and regular expressions respectively) are also decidable.

Emptiness Problem for DFA: Given a DFA A, is $L(A)$ empty?

The corresponding language is

$$E_{\mathrm{DFA}} = \{ \langle A \rangle : A \text{ is a DFA such that } L(A) = \emptyset \}$$

Lemma. E_{DFA} is a decidable language.

Proof. We design a TM T that uses a marking algorithm. On input $\langle A \rangle$ where A is a DFA:

1. Mark the start state of A.
2. Repeat until no new states get marked:

   Mark any state that has a transition coming into it from any marked state.
3. If no accept state is marked, *accept*; otherwise *reject*.

Equivalence Problem for DFA: Given two DFAs, are they equivalent?

The corresponding language:

$EQ_{\mathrm{DFA}} = \{ \langle A, B \rangle$: A and B are DFA s.t. $L(A) = L(B) \}$.

Lemma. EQ_{DFA} is a decidable language.

Proof. Key idea: For sets P and Q, $P \subseteq Q$ iff $P \cap \overline{Q} = \emptyset$. We use T, the algorithm for the Emptiness Problem E_{DFA}.

On input $\langle A, B \rangle$, where A and B are DFAs:

1. Construct $C = (A \cap \overline{B}) \cup (B \cap \overline{A})$. (DFAs are closed under union, intersection and complementation.)
2. Run T on input $\langle C \rangle$.
3. If T accepts, accept; if T rejects, reject.

Acceptance problem for CFG

Consider the language:

$$A_{\mathrm{CFG}} = \{ \langle G, w \rangle : G \text{ is a CFG that generates } w \}$$

Lemma. A_{CFG} is a decidable language.

We use a fact: If G is in Chomsky normal form, any derivation of w has $2n-1$ steps, where n is the length of w.

Proof. The TM S for A_{CFG} is: On input $\langle G, w \rangle$, where G is a CFG and w is a string:

1. Convert G to an equivalent one in Chomsky normal form.
2. List all derivations with $2n-1$ steps, where n is the length of w.
3. If any of these derivations generate w, accept; otherwise, reject.

This is of course extremely inefficient, and will not be used in practice.

Emptiness Problem for CFG: Given a CFG G, is $L(G)$ empty?

Lemma. E_{CFG} is a decidable language.

Proof. The TM R for E_{CFG} is: On input $\langle G \rangle$, where $G = (V, \Sigma, R, S)$ is a CFG:

1. Mark all terminal symbols in G.
2. Repeat until no new variables get marked:

   Mark any variable A where G contains a rule $A \to U_1 \cdots U_k$ and each symbol $U_i \in \Sigma \cup V$ has already been marked.

3. If the start symbol is not marked, accept; otherwise, reject.

Equiv. Problem for CFG: Given two CFGs, are they equivalent?

Obvious attempt: Use the strategy for deciding EQ_{DFA}.

Unfortunately CFGs are neither closed under intersection nor complementation!

In fact EQ_{CFG} is *undecidable*.

Universal Turing Machines

A *universal* TM is a TM U that can simulate the actions of any Turing machine. I.e. for any TM M and any input x of M, U accepts (respectively rejects or loops on) $\langle M, x\rangle$, if and only if, M accepts (respectively rejects or loops on) x.

This is equivalent to an interpreter of C written in C.

Note this is different (quantifier order) from results such as every NDTM can be simulated by a deterministic TM.

Universal Turing Machines

Simulation. The tape of U is partitioned into 3 tracks:

- The top track holds the transition function δ_M of the input TM M.
- The middle track will be used to hold the simulated contents of M's tape.
- The bottom track will hold the current state of M, and the current position of M's tape head.

U simulates M on input x one step at a time, shuttling back and forth between the three tracks. In each step, it updates M's state and simulated tape contents and head position as dictated by δ_M. If ever M halts and accepts or halts and rejects, then U does the same.

Note: The input alphabet of M and U may be different.

A_{TM}: Acceptance Problem for TMs

Acceptance Problem for TM

$$A_{\mathrm{TM}} \;=\; \{\, \langle M, w\rangle : M \text{ is a TM that accepts input } w \,\}$$

Theorem. A_{TM} is RE.

Proof. We define a TM U that accepts A_{TM}: On input $\langle M, w\rangle$ where M is a TM and w is a string

1. *Simulate M on input w*.
2. If M ever enters its accept state, accept; if M ever enters its reject state, reject.

Note that U loops on $\langle M, w\rangle$ if M loops on w. This is why U is not a decider.

If the algorithm had some way to determine that M was not halting on w, it could reject.

A_{TM}: Acceptance Problem for TMs

Theorem. A_{TM} is undecidable.

Proof. Suppose, for a contradiction, TM H is a decider for A_{TM}. I.e.

$$H(\langle M, w\rangle) = \begin{cases} \textit{accept} & \text{if } M \text{ accepts } w \\ \textit{reject} & \text{if } M \text{ does not accept } w \end{cases}$$

We construct a new TM D with H as a subroutine. D: On input $\langle M\rangle$, where M is a TM

1. Run H on input $\langle M, \langle M\rangle\rangle$
2. Output the opposite of H. I.e. if H accepts, *reject*, and if H rejects, *accept*.

A_{TM} is undecidable (proof cont'd)

Thus

$$D(\langle M\rangle) \;=\; \begin{cases} \textit{reject} & \text{if } M \text{ accepts } \langle M\rangle \\ \textit{accept} & \text{if } M \text{ does not accept } \langle M\rangle \end{cases}$$

Now we run D on input $\langle D\rangle$. We have

$$D(\langle D\rangle) \;=\; \begin{cases} \textit{reject} & \text{if } D \text{ accepts } \langle D\rangle \\ \textit{accept} & \text{if } D \text{ does not accept } \langle D\rangle \end{cases}$$

I.e. D accepts $\langle D\rangle$ iff D rejects $\langle D\rangle$, which is a contradiction. Thus neither D nor H can exist. $\square$

To summarize, A_{TM} is (RE but) undecidable.

A language that is not RE

We say that L is *co-RE* if its complement $\overline{L}$ is RE.

Theorem. A language is decidable iff it is both RE and co-RE.

Proof. Suppose L and $\overline{L}$ are acceptable by M and M' respectively. Define a TM N as follows: On input x

1. Write x on the input tapes of M and M'.
2. Run M and M' in parallel (e.g. by dove-tailing the two computations).
3. If M accepts, *accept*; if M' accepts, *reject*.

N is a decider for L because for any input x, either $x \in L$ or $x \in \overline{L}$: if the former then N must accept because M must halt and accept at some point, if the latter, then N must reject because M' must halt and accept at some point.

Corollary. $\overline{A_{\mathrm{TM}}}$ is not RE. $\square$

The Halting Problem: “Given M and x, does M halt on x?”

$$\mathit{HALT}_{\mathrm{TM}} = \{ \langle M, w\rangle : M \text{ is a TM and } M \text{ halts on input } w \}$$

Theorem. The Halting Problem, *HALT*$_{\mathrm{TM}}$, is undecidable.

Proof. Suppose, for a contradiction, H is a decider for *HALT*$_{\mathrm{TM}}$. We use H to construct a TM S to decide A_{TM} as follows: On input $\langle M, w\rangle$ where M is a TM and w is an input:

1. Run TM H on input $\langle M, w\rangle$
2. If H rejects, *reject*.
3. If H accepts, run M on w until it halts.
4. If M has accepted, *accept*; if M has rejected, *reject*.

Since A_{TM} is undecidable, H cannot be a decider for *HALT*$_{\mathrm{TM}}$. □

The Emptiness Problem for TM is undecidable

$E_{\mathrm{TM}} = \{ \langle M \rangle : M$ is a TM and $L(M) = \emptyset \}$

Theorem. The Emptiness Problem for TM is undecidable.

Proof. Suppose, for a contradiction, E_{TM} is decidable by a TM E. We shall use E to construct a TM S that decides A_{TM}.

Definition of S: On input $\langle M, x \rangle$ where M is a TM and x an input

1. Construct a TM M_x defined as: On input y
   (a) If $y \neq x$ then *reject*.
   (b) Run M (input is x) and *accept* if M does.
2. Run E on input $\langle M_x \rangle$. If E accepts, *reject*, if E rejects, *accept*.

The Emptiness Problem for TM is undecidable (proof cont'd)

Now we have

$$\begin{array}{rcll} M \text{ accepts } x & \Rightarrow & L(M_x) = \{\, x \,\} & \text{i.e. } S \text{ accepts } \langle M, x\rangle \\ M \text{ does not accept } x & \Rightarrow & L(M_x) = \emptyset & \text{i.e. } S \text{ rejects } \langle M, x\rangle \end{array}$$

Thus S decides A_{TM}, which is a contradiction. $\square$

The Equivalence Problem for TM is undecidable

$EQ_{\mathrm{TM}} = \{ \langle M_1, M_2 \rangle : M_1$ and M_2 are TMs and $L(M_1) = L(M_2) \}$

Theorem. The Equivalence Problem for TM is undecidable.

Proof. Suppose, for a contradiction, EQ_{TM} is decidable by Q. We define a TM that decides E_{TM} by: On input M

1. Run Q on input $\langle M, M_1 \rangle$ where M_1 is a TM that accepts nothing i.e. $L(M_1) = \emptyset$.
2. If Q accepts, *accept*; if Q rejects, *reject*.

This gives a contradiction. $\square$

Proving Undecidabilty

We have seen three main methods of proving undecidability results: *diagonalization*, *Turing reduction* and *many-one reduction*.

Diagonalization.

E.g. A_{TM}: Show the problem cannot be decidable by creating a TM which cannot exist without a contradiction.

Advantage: Do not need to know about undecidable problems

Disadvantage: Only works easily for a small number of problems: A_{TM}, *HALT*$_{\mathrm{TM}}$, $\ldots$

Proving Undecidability: Reductions

Reductions transform one problem to another in a computable way. If there is a reduction from a known undecidable problem to some problem P, then P is also undecidable.

Reductions also work also for proving that a problem is or is not RE.

Turing reduction.

E.g. E_{TM}: Create an algorithm for a known undecidable problem using the problem to be reduced to as a subroutine.

Advantage: Easy way to show undecidability with known undecidable problems

Disadvantage: Algorithmic approach – does not provide insight into any relationship between the problems

Proving Undecidability: Reductions

Many-one reduction.

E.g. EQ_{TM}: Let A be a decision problem over Σ_A^* and let B a decision problem over Σ_B^*. A many-one reduction is a computable function $f : \Sigma_A^* \to \Sigma_B^*$ such that for all $w \in \Sigma_A^*$

$$w \in A \iff f(w) \in B$$

Advantage: True problem transformation – gives relationships useful in other areas (e.g. complexity)

Disadvantage: Stronger than Turing reduction, so harder to find reductions between problems

Another method of proving that a problem is not RE uses the fact that a problem is decidable if and only if the problem and its complement are both RE.

TOTAL $= \{\langle M \rangle : M$ halts on all inputs$\}$

Theorem. TOTAL is neither RE nor co-RE.

Proof. *HALT*$_{\mathrm{TM}}$ is not decidable, but it is RE. Hence $\overline{\textit{HALT}_{\mathrm{TM}}}$ is not RE.

Therefore it suffices to reduce $\overline{\textit{HALT}_{\mathrm{TM}}}$ to (i) TOTAL and to (ii) $\overline{\text{TOTAL}}$.

(i) Given $\langle M, x \rangle$ where M is a TM and x is an input, we set $N_{M,x}$ to be the following TM: On input y,

1. Run M on input x for up to $|y|$ steps.
2. If M has halted, then loop; if M has not yet halted, *accept*.

We have

$$\langle M, x \rangle \in \overline{\textit{HALT}_{\mathrm{TM}}} \iff N_{M,x} \text{ halts on all input } y \iff N_{M,x} \in \text{TOTAL}.$$

TOTAL is neither RE nor co-RE. (proof cont'd)

(ii) Given $\langle M, x\rangle$ where M is a TM and x is an input, we construct $N_{M,x}$ to be the following TM: On input y,

1. Erase y.
2. Write x on input tape, and run M on x.
3. If M accepts, *accept*; if M rejects, *reject*.

Note that either $N_{M,x}$ loops on all inputs, or it halts on all inputs. Now

$$\langle M, x\rangle \in \overline{\mathit{HALT}_{\mathrm{TM}}} \iff N_{M,x} \text{ loops on all inputs} \iff N_{M,x} \notin \text{TOTAL}.$$

Rice's Theorem

Rice's Theorem. Every non-trivial property of the RE sets is undecidable.

Reformulation Let P be any decision problem about TMs (expressed as a language) that satisfies the following properties:

(1) For any TMs M_1 and M_2 where $L(M_1) = L(M_2)$, we have $\langle M_1 \rangle \in P$ iff $\langle M_2 \rangle \in P$. I.e. the membership of a TM M in P depends only on the language of M.

(2) There are TMs M_1 and M_2 where $\langle M_1 \rangle \in P$ and $\langle M_2 \rangle \notin P$. I.e. P is non-trivial.

Then P is undecidable.

Proof of Rice's Theorem

Take a property P that satisfies (1) and (2). Wlog, assume that for any TM E, if $L(E) = \emptyset$ then $\langle E \rangle \notin P$. Take a TM K such that $\langle K \rangle \in P$. Given $\langle M, x \rangle$ where M is a TM and x is an input, we construct $N_{M,x}$ as follows: On input y

1. Save y on a separate track somewhere.
2. Write x on the input tape, and run M on input x.
3. If M halts on x, run K on y.
4. If K accepts, *accept*.

Either M halts on x or not.

If M does not halt on x, the simulation in 3. will not halt, and the input y of $N_{M,x}$ will not be accepted, and so, $L(N_{M,x}) = \emptyset$ i.e. $\langle N_{M,x} \rangle \notin P$.

On the other hand, if M halts on x, y is accepted by $N_{M,x}$ iff y is accepted by K i.e. $L(N_{M,x}) = L(K)$ i.e. $\langle N_{M,x} \rangle \in P$.

Proof of Rice's Theorem (cont'd)

Thus

$$\begin{array}{rcccl} M \text{ halts on } x & \Rightarrow & L(N_{M,x}) = L(K) & \Rightarrow & \langle N_{M,x}\rangle \in P \\ M \text{ does not accept } x & \Rightarrow & L(N_{M,x}) = \emptyset & \Rightarrow & \langle N_{M,x}\rangle \notin P \end{array}$$

Thus we have reduced *HALT*$_{\mathrm{TM}}$ to P. Since *HALT*$_{\mathrm{TM}}$ is undecidable, so is P.

[Assume for a contradiction P was decidable, say by decider A, then we can construct a decider S for *HALT*$_{\mathrm{TM}}$ as follows: on input $\langle M, x\rangle$ construct $\langle N_{M,x}\rangle$ and run A on $\langle N_{M,x}\rangle$. If A accepts *accept*, if A rejects *reject*.]

$\square$

Application of Rice's Theorem

For an arbitrary TM M with $\Sigma = \{a, b\}$, the following problems are undecidable:

- whether M accepts $ababa$ (i.e. $ababa \in L(M)$)
- whether $L(M)$ contains two different strings of the same length;
- whether $L(M)$ contains any string of length five;
- whether $L(M)$ is a regular language;
- whether $L(M)$ is a context-free language;

Note The following properties of TMs are not properties of RE sets and Rice's Theorem is not applicable:

- M has at least 341 states;
- M halts on all inputs;
- M rejects $ababa$
- there exists a smaller machine equivalent to M.

PCP: Post Correspondence Problem

A puzzle: Given a collection of dominos such as e.g.

$$P = \left\{ \left[\frac{b}{ca}\right], \left[\frac{a}{ab}\right], \left[\frac{ca}{a}\right], \left[\frac{abc}{c}\right] \right\}$$

find a list of dominos from P (repetitions permitted) so that the string we get by reading off the symbols on the top is the same as the sting of symbols on the bottom. This list is called a *match*.

E.g. the following is a match taking dominos $2, 1, 3, 2$ and 4 in this order:

$$\left\langle \left[\frac{a}{ab}\right], \left[\frac{b}{ca}\right], \left[\frac{ca}{a}\right], \left[\frac{a}{ab}\right], \left[\frac{abc}{c}\right] \right\rangle$$

(Reading off the top string we get $abcaaabc$, the same as reading off the bottom one.)

A formal statement of the PCP

Instance: A collection P of *dominos*:

$$P = \left\{ \left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \cdots, \left[\frac{t_k}{b_k}\right] \right\}$$

where $k \geq 1$, and all t_i and b_j are strings over Σ.

Question: Is there a *match* for P? I.e. a non-empty sequence $i_1, \cdots, i_l$ where each $1 \leq i_j \leq k$ and

$$t_{i_1}\, t_{i_2} \cdots t_{i_l} = b_{i_1}\, b_{i_2} \cdots b_{i_l}$$

Theorem. PCP is undecidable.

Examples

1. Let $\Sigma = \{\, 0, 1 \,\}$, and

$$P = \left\{ \left[\frac{11}{111} \right], \left[\frac{111}{11} \right], \left[\frac{1100}{001} \right] \right\}.$$

Here is a match for P: $\langle 1, 1, 3, 2 \rangle$, i.e.

$$\left\langle \left[\frac{11}{111} \right], \left[\frac{11}{111} \right], \left[\frac{1100}{001} \right], \left[\frac{111}{11} \right] \right\rangle$$

2. For some collection of dominos, finding a match may not be possible. E.g.

$$P' = \left\{ \left[\frac{abc}{ab} \right], \left[\frac{ca}{a} \right], \left[\frac{acc}{bc} \right] \right\}$$

(because every top string is longer than the bottom string)

Undecidability of PCP: outline proof

We modify *PCP* to

$$\textit{MPCP} \;=\; \{P \text{ is an instance of PCP with a match that starts with the first domino}\}$$

1. Suppose, for a contradiction, there is a TM R that decides the PCP. We use R to construct a TM S that decide A_{TM}. Let

$$M \;=\; (Q, \Sigma, \Gamma, \vdash, \sqcup, \delta, q_0, q_{\mathrm{acc}}, q_{\mathrm{rej}})$$

Take an input w for M. We construct an instance P' of the *MPCP* (over the alphabet $Q \cup \Gamma \cup \{\#\}$) that simulates M on w i.e.

$$P' \in \textit{MPCP} \quad \Longleftrightarrow \quad \langle M, w\rangle \in A_{\mathrm{TM}}.$$

Idea. Writing a configuration (u_i, q_i, v_i) as $\#u_i\, q_i\, v_i$, a match of P' gives a string; the first part

$$\#q_0 \vdash w\#u_1\, q_1\, v_1\#u_2\, q_2\, v_2\# \cdots \#u_i\, q_i\, v_i\# \cdots \#u_n\, q_{\mathrm{acc}}\, v_n$$

corresponds exactly to an accepting configuration; the rest of the string

$$\# \cdots \# \cdots \cdots \#q_{\mathrm{acc}}\#\#$$

consists of progressively shorter segments of the form $\# \cdots$ that finally ends in $\#\#$.

2. We convert P' to P, an instance of *PCP* which still simulates M on w.

Definition of P'

Part 1. Put $\left[\dfrac{\#}{\#q_0{\vdash}w_1\cdots w_n\#}\right]$ into P' as the first domino $\left[\dfrac{t_1}{b_1}\right]$

Part 2. For each $a, b \in \Gamma$, $q, r \in Q$, if $\delta(q,a) = (r, b, R)$, put $\left[\dfrac{qa}{br}\right]$ into P'

Part 3. For each $a, b, c \in \Gamma$, $q, r \in Q$, if $\delta(q,a) = (r, b, L)$, put $\left[\dfrac{cqa}{rcb}\right]$ into P'

Part 4. For every $a \in \Gamma$, put $\left[\dfrac{a}{a}\right]$ into P'.

Part 5. Put $\left[\dfrac{\#}{\#}\right]$ and $\left[\dfrac{\#}{\sqcup\#}\right]$ into P'.

Part 6. For every $a \in \Gamma$, put $\left[\dfrac{aq_{\mathrm{acc}}}{q_{\mathrm{acc}}}\right]$ and $\left[\dfrac{q_{\mathrm{acc}}a}{q_{\mathrm{acc}}}\right]$ into P'.

Part 7. Add $\left[\dfrac{q_{\mathrm{acc}}\#\#}{\#}\right]$ to P'.

An example

Consider the following TM M on input $w = ab$

$$M = (\underbrace{\{\, q_0, q_3, q_4, q_{\text{acc}}, q_{\text{rej}} \,\}}_{Q}, \underbrace{\{\, a, b, c \,\}}_{\Sigma}, \underbrace{\Sigma \cup \{\, \vdash, \sqcup \,\}}_{\Gamma}, \vdash, \sqcup, \delta, q_0, q_{\text{acc}}, q_{\text{rej}})$$

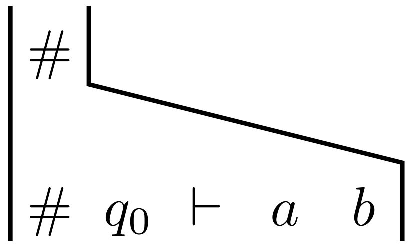
where δ is partly given by

$$\begin{aligned}
\delta(q_0, \vdash) &= (q_0, \vdash, R) \\
\delta(q_0, a) &= (q_3, c, R) \\
\delta(q_3, b) &= (q_4, a, R) \\
\delta(q_4, \sqcup) &= (q_{\text{acc}}, a, L)
\end{aligned}$$

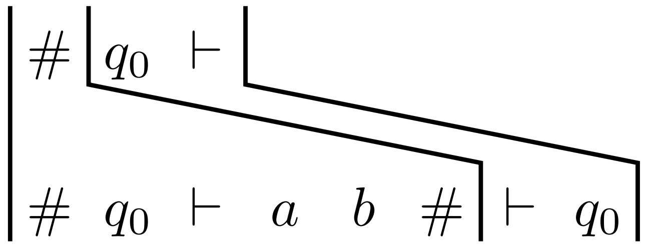
An accepting computation:

$$\begin{aligned}
& (\epsilon, q_0, \vdash ab) \\
\rightarrow\ & (\vdash, q_0, ab) \\
\rightarrow\ & (\vdash c, q_3, b) \\
\rightarrow\ & (\vdash ca, q_4, \sqcup) \\
\rightarrow\ & (\vdash c, q_{\text{acc}}, ab)
\end{aligned}$$

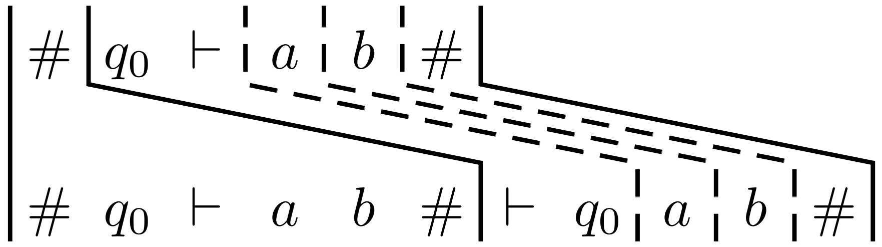
An example (cont'd)



$(\epsilon, q_0, \vdash ab)$



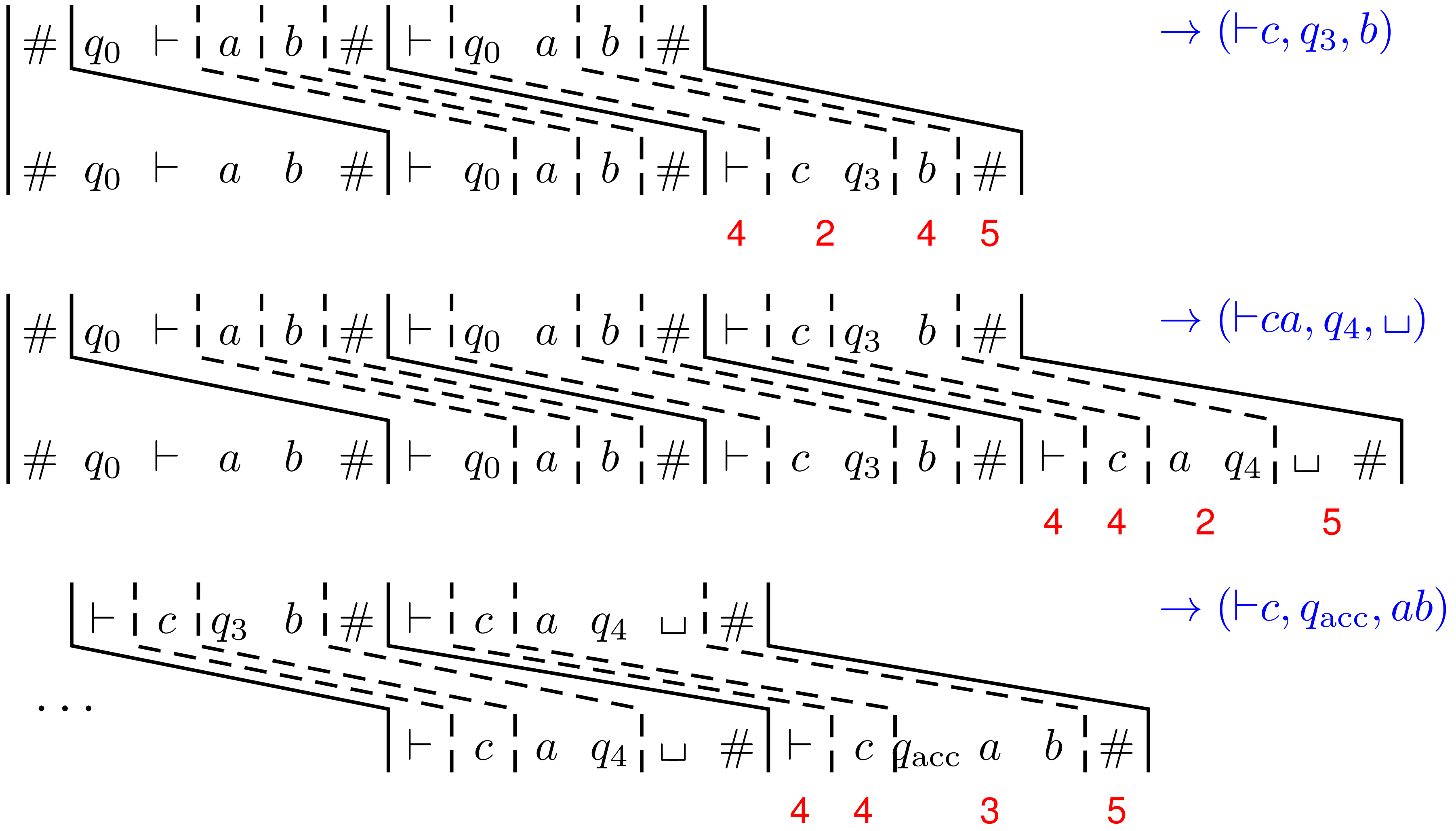
$\delta(q_0, \vdash) = (q_0, \vdash, R)$



$\#$ q_0 $\vdash$ a b $\#$
$\#$ q_0 $\vdash$ a b $\#$ $\vdash$ q_0 a b $\#$
parts 4 4 5

$\rightarrow (\vdash, q_0, ab)$

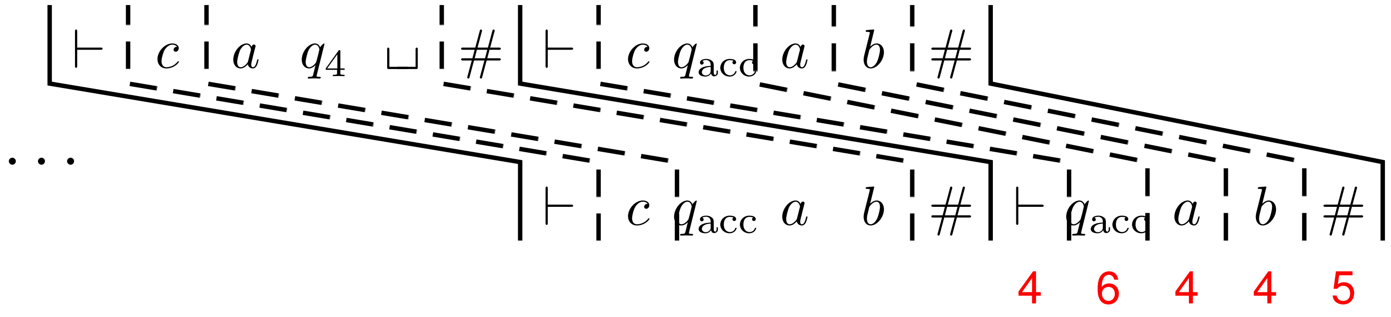
An example (cont'd)



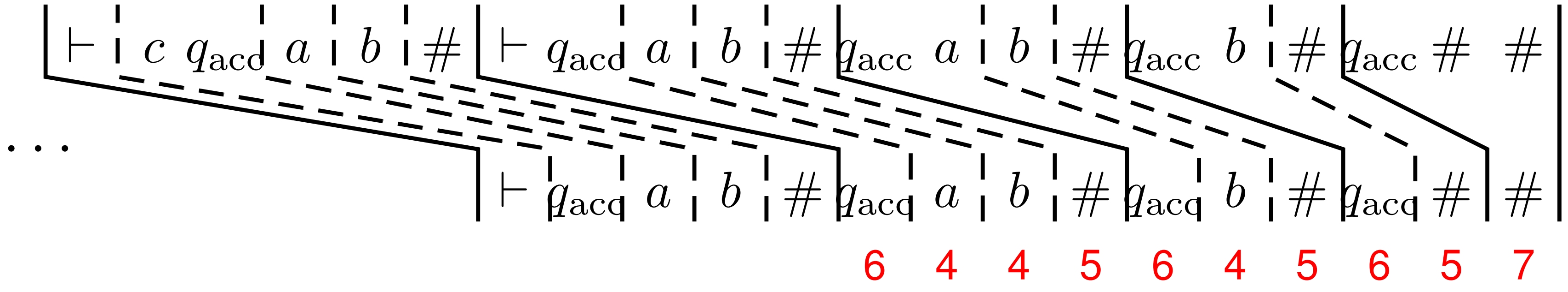
$\# \; q_0 \vdash a \; b \; \# \vdash q_0 \; a \; b \; \#$
$\# \; q_0 \vdash a \; b \; \# \vdash q_0 \; a \; b \; \# \vdash c \; q_3 \; b \; \#$
4 2 4 5
$\rightarrow (\vdash c, q_3, b)$
$\# \; q_0 \vdash a \; b \; \# \vdash q_0 \; a \; b \; \# \vdash c \; q_3 \; b \; \#$
$\# \; q_0 \vdash a \; b \; \# \vdash q_0 \; a \; b \; \# \vdash c \; q_3 \; b \; \# \vdash c \; a \; q_4 \; \sqcup \; \#$
4 4 2 5
$\rightarrow (\vdash ca, q_4, \sqcup)$
$\vdash c \; q_3 \; b \; \# \vdash c \; a \; q_4 \; \sqcup \; \#$
$\dots$
$\vdash c \; a \; q_4 \; \sqcup \; \# \vdash c \; q_{acc} \; a \; b \; \#$
4 4 3 5
$\rightarrow (\vdash c, q_{acc}, ab)$

An example (cont'd)

Tidying up.



$\vdash$ c a q_4 $\sqcup$ $\#$ $\vdash$ c $q_{\rm acc}$ a b $\#$
$\dots$
$\vdash$ c $q_{\rm acc}$ a b $\#$ $\vdash$ $q_{\rm acc}$ a b $\#$
4 6 4 4 5



$\vdash$ c $q_{\rm acc}$ a b $\#$ $\vdash$ $q_{\rm acc}$ a b $\#$ $q_{\rm acc}$ a b $\#$ $q_{\rm acc}$ b $\#$ $q_{\rm acc}$ $\#$ $\#$
$\dots$
$\vdash$ $q_{\rm acc}$ a b $\#$ $q_{\rm acc}$ a b $\#$ $q_{\rm acc}$ b $\#$ $q_{\rm acc}$ $\#$ $\#$
6 4 4 5 6 4 5 6 5 7

Converting a MPCP instance to a PCP instance

Finally we show how to convert P' to an instance of PCP which still simulates M on w. The idea is to build the requirement of starting with the first domino directly into the problem.

Notation. Take a string $u = u_1 \cdots u_n$. Define

$$\begin{aligned} \star u &= * u_1 * u_2 * \cdots * u_n \\ u \star &= u_1 * u_2 * \cdots * u_n * \\ \star u \star &= * u_1 * u_2 * \cdots * u_n * \end{aligned}$$

Converting a MPCP instance to a PCP instance (cont'd)

Suppose P' is the MPCP instance (over alphabet Θ)

$$\left\{ \left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \left[\frac{t_3}{b_3}\right], \ldots, \left[\frac{t_k}{b_k}\right] \right\}$$

Now define P to be the PCP instance (over alphabet $\Theta \cup \{*, \diamond\}$)

$$\left\{ \left[\frac{\star\, t_1}{\star\, b_1 \star}\right], \left[\frac{\star\, t_2}{b_2 \star}\right], \left[\frac{\star\, t_3}{b_3 \star}\right], \ldots, \left[\frac{\star\, t_k}{b_k \star}\right], \left[\frac{\star \diamond}{\diamond}\right] \right\}$$

Note that the only domino that can start a match is the first one, because it is the only one where both the top and bottom start with the same symbol, namely, $*$.

The original symbols in Θ now occur in even positions of the match.

The domino $\left[\frac{\star \diamond}{\diamond}\right]$ is to allow the top to add the extra $*$ at the end of the match.

Language	**regular**	**context-free**	**semi-decidable/RE**
Grammar	right-linear	context-free	unrestricted
rules	$A \to wB$, $A \to w$	$A \to \alpha$	$\alpha \to \beta$
Machine	DFA or NFA	NPDA	Turing machine
memory	finite	finite + one stack	unrestricted
Other Descr$^{\mathrm{n}}$	Regular expression Myhill-Nerode-Thm		λ-calculus μ-recursive funct$^{\mathrm{s}}$ RM, Hugs, Oberon, . . .
Example	$\{\, w : w$ contains $10 \,\}$ $\{\, 0^n 1^m : n, m \geq 0 \,\}$	$\{\, w\, w^R \,\}$ $\{\, 0^n 1^n : n \geq 0 \,\}$	$\{\, 0^n 1^m 0^{n*m} : n \geq 0 \,\}$ $\{\, Mx : M$ halts on $x \,\}$ HP
Counterexample	$\{\, 0^n 1^n : n \geq 0 \,\}$	$\{\, 0^n 1^n 0^n : n \geq 0 \,\}$ $\{\, w\, w \,\}$	$\{\, M : M$ halts on every $x \,\}$ TOTAL
Application	Tokenizer Model checker	Parser	General computing