UNIQUE FIXED POINTS.

Let P be a predicate
Let e be an integer valued ~~variable~~ escpression
Let n be a fresh integer variable.

$$P\restriction_e n \triangleq (e < n \Rightarrow P)$$

in future we will omit the subscript e

Theorem L0 $P\restriction 0 = \text{true}$ ✓

L1 $P \Rightarrow P\restriction n{+}1 \Rightarrow P\restriction n$ (not used)

L2 $(\forall n.\ P\restriction n) = P$ ✓

L3 $(\forall m.\ P_m)\restriction n = \forall m.\ (P_m\restriction n)$ ✓

L4 If P_n is a descending chain $\forall n.\ P_n = \forall n (P_n \restriction n)$ ✓

A function F is <u>constructive</u> if for all P

$$F(P)\restriction n{+}1 = F(P\restriction n)\restriction n{+}1$$

Theorem 1 If F is constructive, then

$$F^n(P)\restriction n = F^n(\text{true})\restriction n$$

Proof by induction on n.

$F^0(P)\restriction 0 = \text{true} = F^0(\text{true})\restriction 0$ by L0

$n>0$ $\quad F^n(P)\restriction n = F(F^{n-1}(P)\restriction n{-}1)\restriction n$ F constructive

$= F(F^{n-1}(\text{true})\restriction n{-}1)\restriction n$ induction hyp

$= F^n(\text{true})\restriction n$

corollary $\quad F^{n+m}(P)\restriction n = F^n(\text{true})\restriction n$

$LHS = F^n(F^m(P))\restriction n$

Theorem 2 If F is constructive, the equation

$$X = F(X)$$

has at most one solution

Proof let X be such a solution.

by induction, $X = F^n(X)$ for all n

$$\therefore\ X = \forall n. X\upharpoonright n = \forall n. F^n(X)\upharpoonright n = \qquad \text{L2}$$

$$= \forall n\ F^n(\mathrm{true})\upharpoonright n \qquad \text{just proved}$$

which is independent of X

F is said to be a condensing function if

$$\forall n\ F^{n+1}(\mathrm{true}) \Rightarrow F^n(\mathrm{true})$$

Theorem 3. If F is a condensing and constructive function, $X = F(X)$ ~~X~~ has one solu

$$F(\forall m\ F^m(\mathrm{true})) = \forall n. (F(\forall m. F^m(\mathrm{true})))\upharpoonright n \qquad \text{L2}$$

$$= \forall n>0\ F((\forall m\ F^m(\mathrm{true}))\upharpoonright n-1)\upharpoonright n \qquad \text{F constructive}$$

$$= \forall n>0\ F(\forall m(F^m(\mathrm{true})\upharpoonright n-1))\upharpoonright n \qquad \text{L3}$$

$$= \forall n>0\ F((\forall m<n. F^m(\mathrm{true}))\upharpoonright n-1)\upharpoonright n \qquad \text{Theorem 1 corollary}$$

$$= \forall n>0. F(\cancel{\forall} F^{n-1}(\mathrm{true})\upharpoonright n-1)\upharpoonright n \qquad \text{F condensing}$$

$$= \forall n>0\ F(F^{n-1}(\mathrm{true}))\upharpoonright n \qquad \text{F constructive}$$

$$= \forall n>0\ F^n(\mathrm{true})\upharpoonright n \qquad \text{''}$$

$$= \forall n>0\ F^n(\mathrm{true}) \qquad \text{L4 F condensing}$$

Clearly all monotonic functions are condensing.