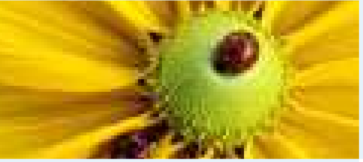
A categorical perspective on Entropy and a categorification of topological semimodules

Vaia Patta and Jamie Vicary
University Of Oxford

September 17, 2017

Where it all started: LY'-categories



Background

We generalise the work of Lieb and Yngvason[1] that models thermodynamical systems in order to show existence of Entropy based on almost self-evident axioms.

Lieb and Yngvason's framework was tantamount to a category:

- Objects: Thermodynamical states.
- Morphisms: Existence of adiabatic processes.
- Monoidal product: Compound thermodynamical systems.
- Any system can be scaled by some factor (monoidal endofunctor).
- Any system can be split into parts and recombined (natural transform).
- A small enough system does not affect the adiabaticity of a process (convergence functor).

[1] "The mathematical structure of the Second Law of Thermodynamics", 2003

Motivation

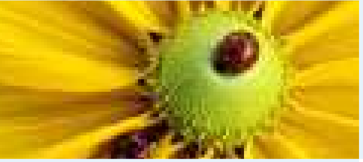
We introduce LY'-categories to model distinct classes of adiabatic processes (eg grouped by duration) rather than just their existence. This places more restrictions on the convergence functor:

- It has to make sense in terms of topological convergence. There are 3 axioms that a function has to satisfy in order for it to correspond to convergence of sequences in a topological space.
- It has to cooperate with the rest of the categorical structure (composition, identity, monoidal structure, scaling endofunctors, splitting and recombination transform) in a physically plausible way.

An LY'-category that is a strict symmetric strict monoidal preorder over $\mathbb{R}^{\geq 0}$ where the λ endofunctors are strict recovers Lieb and Yngvason's category, which we call LY.

Preliminaries

- All semirings are required to be commutative and cancellative with addition $a + b$ and multiplication ab (partial functions: subtraction $a - b$, division $\frac{a}{b}$, integer exponents $a^{\pm n}$)
- A semifield is a semiring with inverses.
- Semirings denoted $\Lambda(\text{additive unit}, \text{multiplicative unit})$
- *Nice* topologies: Sequential topologies where every convergent sequence has a unique limit.
- All topological semirings are required to be nice.
- Topological semirings denoted $(\text{semiring}, \text{topology})$
- $\mathbf{NiceTop}$ is the category of nice topologies.



Weakly linear categories

A *weakly linear category* on a topological semiring $(\Lambda(0,1),\tau)$ is a symmetric monoidal category $(\mathbf{C},\oplus)$:

- Equipped with a family of symmetric monoidal endofunctors λ (with associated natural transforms J_λ) indexed by the elements of Λ such that multiplication of elements of Λ coincides with composition of functors.
- Satisfying the *splitting and recombination* property: For every $X \in \mathrm{Ob}(\mathbf{C})$ and every $\lambda_1, \lambda_2 \in \Lambda$ there exists a natural isomorphism $c_{X,\lambda_1,\lambda_2} : (\lambda_1+\lambda_2)X \to \lambda_1 X \oplus \lambda_2 X$.

A *primary sequence* in a weakly linear category is a sequence of objects of the form $X \oplus \lambda_1 Y, X \oplus \lambda_2 Y, \ldots$ with $\lambda_i \to 0$.

A *sequence category* of a category $\mathbf{C}$ is a category whose objects are infinite sequences in $\mathrm{Ob}(\mathbf{C})$ and whose morphisms are infinite sequences in $\mathrm{Mor}(\mathbf{C})$, with composition defined pointwise.

Convergence in sequential spaces

Let S be the set of infinite sequences on a set X and let $f : S \to X$ be a partial function. Then f is the topological convergence of some space on X if and only if the following hold:

- $f(x, x, \ldots) = x$
- If $f(s) = x$ and s' is a subsequence of s then $f(s') = x$
- Either $f(s) = x$ or there exists a subsequence s' of s such that for all subsequences s'' of s' either $f(s'') \neq x$ or f is undefined on s''.

Subsequence endofunctors

The *primary sequence category* of a weakly linear category $\mathbf{C}$ is the sequence category of $\mathbf{C}$ whose objects are primary sequences and pointwise monoidal products thereof.

The primary sequence category of a weakly linear category is itself a weakly linear category over the same topological semiring, with every operation defined pointwise.

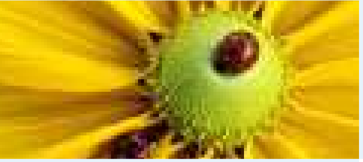
In a sequence category, a *subsequence endofunctor* F_I (where I is an infinite list of strictly increasing positive integers) maps an infinite sequence of morphisms to the infinite subsequence specified by indices I.

In the primary sequence category of a weakly linear category, F_I are defined for all I.

Convergence categories

Given a category $\mathbf{C}$, let $\mathbf{S}$ be a sequence category of $\mathbf{C}$ such that subsequence endofunctors F_I are defined for all I. A *convergence category* $(\mathbf{S}', \mathrm{conv})$ (with respect to $\mathbf{S}$ and $\mathbf{C}$) is a subcategory $\mathbf{S}'$ of $\mathbf{S}$ (along with an injection $i : \mathbf{S}' \to \mathbf{S}$) that is closed under all F_I and is equipped with a functor $\mathrm{conv} : \mathbf{S}' \to \mathbf{C}$ satisfying the following properties:

- conv maps any constant sequence of morphisms to the sole distinct element of the sequence.
- The following triangle commutes:
$$\begin{tikzcd} \mathbf{S}' \arrow[r, "F"] \arrow[dr, "\mathrm{conv}"'] & \mathbf{S}' \arrow[d, "\mathrm{conv}"] \\ & \mathbf{C} \end{tikzcd}$$
- Let s be a morphism in $\mathbf{S}$. If for every subsequence endofunctor F_k there exists a subsequence endofunctor F_j such that $(F_j \circ F_k)(s)$ is in the image of i, then s is also in the image of i and $\mathrm{conv}((i^{-1} \circ F_j \circ F_k)(s)) = \mathrm{conv}(i^{-1}(s))$.

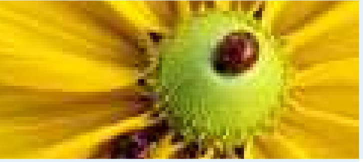


LY'-categories

An LY'-category is a weakly linear category where the following hold.
Let $\mathbf{S}$ be the primary sequence category of $\mathbf{C}$. There exists a monoidal subcategory $\mathbf{S}'$ of $\mathbf{S}$, called the *stability category* of $\mathbf{C}$, which is surjective on objects and closed under the λ endofunctors, such that $\mathbf{S}'$ has at least one morphism in each homset that is nonempty in $\mathbf{S}$ and such that there exists a strict symmetric monoidal functor $\mathrm{conv} : \mathbf{S}' \to \mathbf{C}$, called the *stability functor* of $\mathbf{C}$, making the following square commute for each λ:

$$
\begin{tikzcd}
\mathbf{S}' \arrow[r, "\mathrm{conv}"] \arrow[d, "\lambda"] & \mathbf{C} \arrow[d, "\lambda"] \\
\mathbf{S}' \arrow[r, "\mathrm{conv}"] & \mathbf{C}
\end{tikzcd}
$$

Furthermore, $(\mathbf{S}', \mathrm{conv})$ is a convergence category with respect to $\mathbf{S}$ and $\mathbf{C}$.



Kan extension

Extended primary sequence category $\mathbf{Q}'$: Objects (morphisms) are concatenations QS of sequences where S is an object (morphism) of the primary sequence category.

Extended stability category $\mathbf{S}'$: Objects (morphisms) are concatenations QS of sequences where S is an object (morphism) of the stability category.

Kan extension
$$\begin{tikzcd} \mathbf{S}' \arrow[r, "\mathrm{conv}"] \arrow[d, hook] & \mathbf{C} \\ \mathbf{Q}' \arrow[ur, "\mathrm{conv}"'] & \end{tikzcd}$$

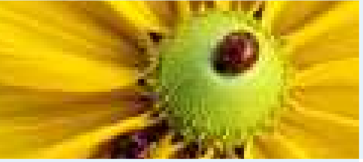
Categories of LY'-categories

Let $i : (\Lambda, \tau) \to (\Lambda'(0,1), \tau')$ be an inclusion of topological semirings. Define an LY'-*functor* $f : \mathbf{C} \to \mathbf{C}'$ between LY'-categories as a strict monoidal functor such that the following squares commute in $\mathbf{Cat}$ for all $\lambda \in \Lambda$

$$\begin{tikzcd} \mathbf{C} \arrow[r, "\lambda"] \arrow[d, "f"] & \mathbf{C} \arrow[d, "f"] \\ \mathbf{C}' \arrow[r, "i\lambda"] & \mathbf{C}' \end{tikzcd} \qquad \begin{tikzcd} \mathbf{S} \arrow[r, "\mathrm{conv}"] \arrow[d, "f\times f\times\cdots"'] & \mathbf{C} \arrow[d, "f"] \\ \mathbf{S}' \arrow[r, "\mathrm{conv}'"] & \mathbf{C}' \end{tikzcd}$$

where $\mathbf{S}$ and $\mathbf{S}'$ are the respective stability categories of $\mathbf{C}$ and $\mathbf{C}'$, and conv and conv' are the respective convergence functors.

Define $\mathbf{LY}'$ as the category of LY'-categories and LY'-functors. Define $\mathbf{LY}'_{\boldsymbol{\Lambda}}$ as its subcategory of LY'-categories over the topological semiring Λ where $i = \mathrm{id}_\Lambda$.



Topological properties

Let $\mathbf{C}$ be any category admitting a convergence category $(\mathbf{G}, \mathrm{conv})$. Then it is always possible to define appropriate T1 topologies on $\mathrm{Ob} := \mathrm{Ob}(\mathbf{C})$ and $\mathrm{Mor} := \mathrm{Mor}(\mathbf{C})$ such that in each the function conv maps each convergent sequence to its limit and is undefined on divergent sequences.

In particular, given an LY'-category
$(\mathbf{C}, \oplus, 0X, a, l, r, \lambda, \Lambda(0,1), \tau, \{J\}_\lambda, s, c, \mathrm{conv})$
(for some $X \in \mathrm{Ob}(\mathbf{C})$), these are the finest topologies such that the topology on Ob agrees with τ on the λ functors.

These topologies are nice.

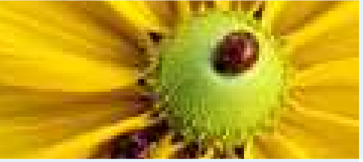
Where it all went: Topological weak semimodules

Preliminaries

Introducing a categorification of semimodules constituting a more elegant categorical object that generalises LY'-categories.

Preliminary definitions follow.

Semiring category: Given a semiring $\Lambda(0,1)$ we define the category $\mathbf{\Lambda}$ to be the discrete category whose objects are the semiring elements, equipped with a (primary) monoidal product $\oplus$ corresponding to the addition in Λ. We also define a (secondary) monoidal structure $\otimes$ corresponding to multiplication.



Let ($\mathbf{C}$ be a symmetric monoidal category and let $\mathbf{\Lambda}$ be a semiring category. A *weak semimodule* over Λ is (the codomain of) a functor $\cdot : \mathbf{\Lambda} \times \mathbf{C} \to \mathbf{C}$ such that for every $f \in \mathrm{Mor}(\mathbf{C})$ we have $\cdot(1, f) \mapsto f$, for every $\lambda \in \mathrm{Ob}(\Lambda)$ and every $f \in \mathrm{Mor}(\mathbf{C})$ the functors $\cdot(\lambda, \_) : \mathbf{C} \to \mathbf{C}$ and $\cdot(\_, f) : \mathbf{\Lambda} \to \mathbf{C}$ are symmetric monoidal (wrt the primary monoidal product) and the following diagram commutes in $\mathbf{Cat}$:

$$
\begin{tikzcd}
\mathbf{\Lambda} \times \mathbf{\Lambda} \times \mathbf{C} \arrow[r, "\otimes \times \mathrm{id}_{\mathbf{C}}"] \arrow[d, "\mathrm{id}_{\mathbf{\Lambda}} \times \cdot"] & \mathbf{\Lambda} \times \mathbf{C} \arrow[d, "\cdot"] \\
\mathbf{\Lambda} \times \mathbf{C} \arrow[r, "\cdot"] & \mathbf{C}
\end{tikzcd}
$$

If $\mathbf{\Lambda}$ is a ring then $\cdot$ is called a *weak module*. If $\mathbf{\Lambda}$ is a semifield then $\cdot$ is called a *weak semivector space*. If $\mathbf{\Lambda}$ is a field then $\cdot$ is called a *weak vector space*.

A weak semimodule over SMC

Given any symmetric monoidal category $\mathbf{C}$, one may construct a weak semimodule of $\mathbf{C}$ as the semimodule over $\mathbb{N}$ where $\cdot(n, f) = \underbrace{f \otimes f \otimes \cdots \otimes f}_{n}$, with 0 mapping every morphism to the identity of the monoidal unit.

Example: Let $(\mathbf{M_b}, \otimes, J_n)$ denote the following category:

- The objects are positive integers k.
- The morphisms are $b^k \times b^k$ matrices.
- Composition is matrix multiplication.
- The monoidal product $\otimes$ is the Kronecker product.
- There is a family of monoidal endofunctors n with natural transforms J_n, indexed by the natural numbers, acting as Kronecker powers.

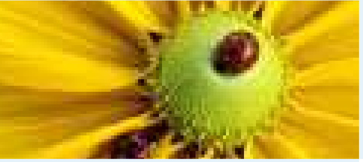
Linear maps and linear extensions

A *linear map* $f : (\cdot_1 : \mathbf{\Lambda} \times \mathbf{C_1} \to \mathbf{C_1}) \to (\cdot_2 : \mathbf{\Lambda} \times \mathbf{C_2} \to \mathbf{C_2})$ between weak semimodules over Λ is defined to be a strict monoidal functor $f : \mathbf{C_1} \to \mathbf{C_2}$ that makes the following diagram commute in $\mathbf{Cat}$:

$$\begin{tikzcd} \mathbf{\Lambda} \times \mathbf{C_1} \arrow[r, "\cdot_1"] \arrow[d, "\mathrm{id}_{\mathbf{\Lambda}} \times f"] & \mathbf{C_1} \arrow[d, "f"] \\ \mathbf{\Lambda} \times \mathbf{C_2} \arrow[r, "\cdot_2"] & \mathbf{C_2} \end{tikzcd}$$

Let $i : \Lambda(0,1) \to \Lambda'(0,1)$ be a semiring inclusion. A *linear extension* $f : (\cdot_1 : \mathbf{\Lambda} \times \mathbf{C_1} \to \mathbf{C_1}) \to (\cdot_2 : \mathbf{\Lambda'} \times \mathbf{C_2} \to \mathbf{C_2})$ between weak semimodules over Λ and Λ' respectively is defined to be a strict monoidal functor $f : \mathbf{C_1} \to \mathbf{C_2}$ that makes the following diagram commute in $\mathbf{Cat}$:

$$\begin{tikzcd} \mathbf{\Lambda} \times \mathbf{C_1} \arrow[r, "\cdot_1"] \arrow[d, "i \times f"] & \mathbf{C_1} \arrow[d, "f"] \\ \mathbf{\Lambda'} \times \mathbf{C_2} \arrow[r, "\cdot_2"] & \mathbf{C_2} \end{tikzcd}$$



Topological weak semimodules

Let $\mathbf{\Lambda}$ be a topological semiring category. A *topological weak semimodule* over Λ is a weak semimodule on a topological category, where the monoidal product, the symmetry and the structural functors defining the weak semimodule are continuous with respect to the topologies involved (presuming the product topology on the product).

Example: Let $(\mathbf{M}'_{\mathbf{b}}, \otimes, J_n, |\cdot|)$ denote the category defined as $\mathbf{M}_{\mathbf{b}}$ but where each matrix M is assigned a norm $|M|$. The category $\mathbf{M}'_{\mathbf{b}}$ is a topological weak semimodule over $(\mathbb{N}, \mathrm{discrete})$ where τ'_{Ob} is the discrete topology and τ_{Mor} is defined by the norm (*i.e.* within each homset the norm defines a distance $d(M_1, M_2) = |M_1 - M_2|$ that in turn yields a metric space on the homset).

- (Topological) weak semimodules over Λ and (topological) linear maps
- (Topological) weak semimodules and (topological) linear extensions
- (Topological) 2-semimodules over Λ and (topological) linear maps
- (Topological) 2-semimodules and (topological) linear extensions

Conjecture: These are all complete and cocomplete.

Topological weak semimodules and LY'-categories

A weak semimodule is precisely a weakly linear category.

A topological category is precisely a category with a convergence category.

The relation between topological weak semimodules and LY'-categories is more nuanced: For a semifield Λ satisfying certain conditions (call Λ *well-behaved*), the category of LY'-categories over (Λ, τ) is a coreflective subcategory of the category of topological weak semivector spaces over (Λ, τ) (where τ can be any topology).

All fields are well-behaved. The semifields $\mathbb{Q}^{\geq 0}$ and $\mathbb{R}^{\geq 0}$ are also well-behaved. This makes the above statement relevant for many physical applications.

Physical application (1)

Consider a homogeneous thermal system with only two degrees of freedom, its mass M and its temperature T, which can only undergo adiabatic processes (of duration t); specifically, a substance in a rigid adiabatic container fitted with a stirrer, on which we can only do dissipative work. We can describe this system by a traced LY'-category over $\mathbb{R}^{\geq 0}$.

- Objects are pairs (M, T) of positive real numbers.
- $\mathrm{Hom}((M, T, E), (M', T', E')) \neq \emptyset$ if and only if $M = M'$ $T \leq T'$. Each morphism f_t is characterised by an index $t \in \mathbb{R}^{\geq 0}$ and the index 0 is reserved for identities.
- Composition: Let $\mathrm{Hom}(A, B) = \{f\}_t$, $\mathrm{Hom}(B, C) = \{g\}_t$ and $\mathrm{Hom}(A, C) = \{h\}_t$. Then $g_{t_2} \circ f_{t_1} = h_{t_1+t_2}$. Composition physically corresponds to performing one process after the other.

Physical application (2)

- Strict symmetric strict monoidal product $\oplus$:
  - For nonempty objects, $A \oplus B = (M_A + M_B, T_{A\oplus B})$, where $T_{A\oplus B}$ is defined to be the solution to a specific equation. This monoidal product physically corresponds to a merge of systems A and B.
  - For nonempty objects, for morphisms $f_{t_A} : A \to A'$ and $g_{t_B} : B \to B'$, set $f_{t_A} \oplus g_{t_B} = h_{t_A+t_B} : A \oplus B \to A' \oplus B'$.
  - Empty systems $(0, T)$ with arbitrary T are all units; furthermore, we identify them all to be the same object O.
- Strict monoidal endofunctors $\lambda \in \mathbb{R}^{\geq 0} : (M, T) \mapsto (\lambda M, T), f_t \mapsto g_{\lambda t}$. Physically, these functors correspond to scaling the systems by some factor without changing the average power of the heating process.

Physical application (3)

- Let $\{f_{t_i} : X \oplus \lambda_i X' \to Y \oplus \lambda_i Y'\}_i$ be a sequence of morphisms where $\lambda_i \to 0$. If the corresponding sequence t_i converges to t, then let $\mathrm{conv}(\{f_{t_i}\}) = f_t : X \to Y$.