A Concrete Representation of Observational Equivalence for PCF

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Overview

- Observational equivalence for PCF terms
- This talk describes some work to give a concrete representation of (a superset of) the equivalence classes
- This goes via the game semantics model of the mid-nineties by Hyland, Ong, Abramsky et al
- We define a mapping obs into sets of finite sets which equates equivalent PCF terms.

Types and Terms of PCF

- Prototypical functional programming language introduced by Plotkin
- Based on Scott's LCF.

Types are of the form:

$$T = \mathtt{nat} \mid T_1 \to T_2$$

Terms are of the form:

$$M := x \mid \lambda x : A.M \mid M_1 M_2 \mid \mathtt{succ} M \mid \mathtt{pred} M \mid \\ n \mid \mathtt{ifzero}\ M_1\ \mathtt{then}\ M_2\ \mathtt{else}\ M_3 \mid \mathtt{Y}_A\ M$$

Observational Equivalence 1

- We define a relation $\Downarrow$ between closed terms and values.
- S is *refined by* T if replacing S by T in any terminating program gives a terminating program.
- A *context* is a PCF term possibly with a placeholder/hole $-$.
- Given closed terms M and N of the same type, $M \leq_{obs} N$ iff for all valid contexts $C[-]$, $C[M] \Downarrow$ implies $C[N] \Downarrow$.
- Write $S =_{obs} T$ if $S \leq_{obs} T$ and $T \leq_{obs} S$

Observational Equivalence 2

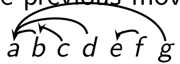
- $=_{obs}$ involves a large quantification over all contexts.
- Undecidable for finite types (Loader).

Denotational (games) models:

- In the mid-nineties, Hyland/Ong, Abramsky/Jagadeesan/Malacaria, Nickau provided a model of PCF based on game semantics.
- Gives an intrinsic account of PCF terms as *innocent strategies* + definability/quotienting.

Games and Plays

- A *play* is a sequence of moves where most moves are equipped with a pointer to some previous move.



- A *game* is a set of moves + further data defining which plays are *legal*.
- Moves are divided into *player* moves and *opponent* moves; plays must be *alternating* [1]

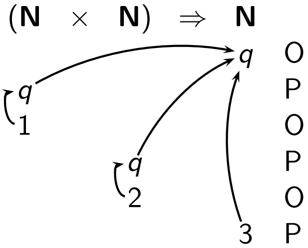
Example : Game **N** — O-move q + P-response for each $n \in \mathbb{N}$.
Example legal play: q 5 q 6 q 7 q 42.

[1] We also require visibility and well-bracketing.

Function Space

- If A and B are games, we can define $A \Rightarrow B$ and $A \times B$
- Plays in these games consists of a play in A interleaved with a play in B
- In the case of $A \Rightarrow B$, the roles of P and O are reversed in the subgame A

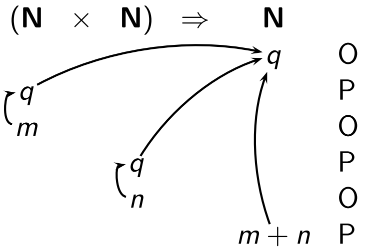
Example of a play in $(\mathbf{N} \times \mathbf{N}) \Rightarrow \mathbf{N}$:



Strategies

- A P-strategy on a game is a set of even-length plays that are even-prefixed closed and even-branching.
- Represents a partial function from odd-lengthed plays to the next P-move.

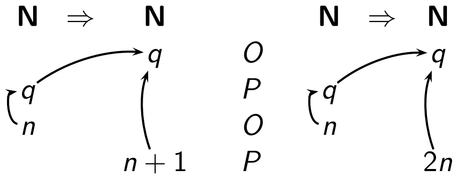
Example of a strategy on $(\mathbf{N} \times \mathbf{N}) \Rightarrow \mathbf{N}$:



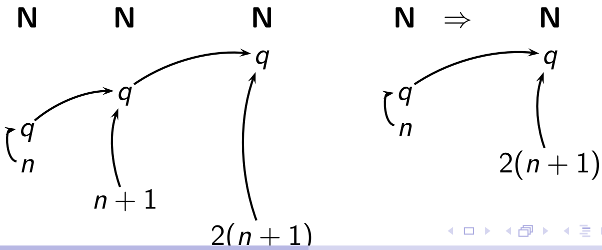
We can compose strategies.

Composition

Let $\sigma : \mathbf{N} \to \mathbf{N}$ and $\tau : \mathbf{N} \to \mathbf{N}$ have maximal plays



For $\sigma;\tau$ we use "parallel composition plus hiding"



Views

The P-view of a play s is the subsequence of s removing moves between an opponent move and its justifier.

- $\ulcorner \epsilon \urcorner = \epsilon$
- $\ulcorner sp \urcorner = \ulcorner s \urcorner p$ where p is a P-move
- $\ulcorner si \urcorner = i$ where i is an initial move
- $\ulcorner sp t o \urcorner = \ulcorner s \urcorner p o$, where P-move p is the justifier of O-move o

Can also define *O-view* of s:

- $\llcorner \epsilon \lrcorner = \epsilon$
- $\llcorner so \lrcorner = \llcorner s \lrcorner o$ where o is an O-move
- $\llcorner s o t p \lrcorner = \llcorner s \lrcorner o p$, where O-move o justifies P-move p

Innocent Strategies

- An *innocent strategy* σ over a game is strategy where the next P-move depends only on the P-view.
- We can give the denotation of each PCF term as an innocent strategy.
- Soundness + *definability* — all compact innocent strategies represent some PCF term.
- This allows us to give a semantic definition of observational equivalence; and via quotienting a fully abstract model of PCF.

Innocent Equivalence

- We define $\leq_{ib}$ on innocent strategies giving a semantic definition of the $\leq_{obs}$
- Let Σ denote the game with one initial O-move q and it's P-response a enabled by q. Let $\top$ denote the strategy $\{\epsilon, qa\}$ on Σ.
- Let σ and τ be innocent strategies over a game A. $\sigma \leq_{ib} \tau$ if for any innocent strategy $\alpha : A \Rightarrow \Sigma$ if $\sigma;\alpha = \top$ then $\tau;\alpha = \top$.

Theorem

Given two PCF terms $M, N : A$ we have $M \leq_{obs} N$ iff $[\![M]\!] \leq_{ib} [\![N]\!]$

Innocent Tests

- Given a strategy $\sigma : A$ we consider innocent tests passed by σ, i.e. functions from P-views of plays in $A \Rightarrow \Sigma$ to the next move.
- But P-views of plays in $A \Rightarrow \Sigma \cong$ O-views of plays in A.
- Thus an innocent test on A corresponds to an O-view function on A. We can represent this as a *set* of O-views.

Definition

Let s be a play over some game. Define $\mathrm{ovw}(s) = \{\llcorner t \lrcorner : t \sqsubseteq s\}$.

The obs Construction

Definition

A play s is *O-innocent* if for $s_1 o_1, s_2 o_2 \sqsubseteq s$ with $\llcorner s_1 \lrcorner = \llcorner s_2 \lrcorner$ we must have $o_1 = o_2$.

Definition

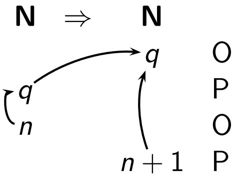
Let σ be an innocent strategy over some game. Define $\overline{\sigma}$ to be the subset of σ consisting of only the O-innocent, single-threaded, complete plays.

Definition

Let σ be an innocent strategy. Define $\mathrm{obs}(\sigma) = \{\mathrm{ovw}(s) : s \in \overline{\sigma}\}$

Example 1

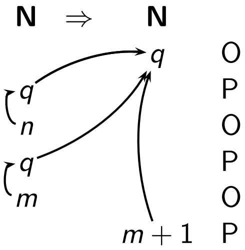
- We describe an innocent strategy succ



- Then $\text{obs}(\text{succ}) = \{\{\epsilon, q_2, q_2 q_1, q_2 q_1 n_1, q_2 (n+1)_2\} : n \in \mathbb{N}\}$
  (Maximal O-views: $\{\{q_2 q_1 n_1, q_2 (n+1)_2\} : n \in \mathbb{N}\}$.)

Example 2

- We also consider a strategy succ_2 with maximal plays



- O-innocence implies $m = n$. Thus $\mathsf{obs}(\mathsf{succ}_2) = \{\{q_2 q_1 m_1, q_2 (m+1)_2\} : m \in \mathbb{N}\}$ (maximal O-views only.)

Forgetfulness

- We see succ $=_{ib}$ succ_2 and $\mathrm{obs}(succ) = \mathrm{obs}(succ_2)$.
- obs forgets the order and number of times the arugments are interrogated (and O-innocence guarantees the same each time.)
- Similarly, strategies for left-strict and right-strict addition ($\neq$ but $=_{ib}$) both obs to $\{\{q_3 q_1 m_1, q_3 q_2 n_2, q_3(m+n)_3\} : m, n \in \mathbb{N}\}$.

Concrete Representation of PCF

We can show

Theorem

Let σ and τ be innocent strategies over a game A. Then $\sigma =_{ib} \tau$ iff $\mathrm{obs}(\sigma) = \mathrm{obs}(\tau)$.

Thus, combining this with the full abstraction results for PCF of the mid nineties, we have:

Corollary

If S and T are terms of PCF then $S =_{obs} T$ iff $\mathrm{obs}([\![S]\!]) = \mathrm{obs}([\![T]\!])$.

Observational Preorder

We can also give a characterisation of $\leq_{obs}$ in this setting.

Definition

Suppose σ and τ are sets of O-view sets over an arena A. Write $\sigma \leq_{os} \tau$ if $\forall S \in \sigma \; \exists T \in \tau$ with $T \subseteq S$.

- We can show that $\mathrm{obs}(\sigma) \leq_{os} \mathrm{obs}(\tau)$ iff $\sigma \leq_{ib} \tau$ (so corresponds to $\leq_{obs}$.)

Definability

- No concrete representation of the image of obs (not effectively presentable, Loader.)
- We could describe a category where objects are games and arrows are sets of the form obs(σ) for an innocent strategy σ; this would be a fully abstract model.
- Can we define composition in terms of the O-view sets directly?
- Loader's result places some restrictions on this.

Composition?

Possible definition of composition:

Definition

Given sets of O-view sets $\sigma : A \Rightarrow B$ and $\tau : B \Rightarrow C$ we define

$$\sigma;\tau = \{\mathrm{ovw}(s|_{A,C}) : \begin{array}{l} s \in \mathrm{int}(A,B,C) \wedge \\ \mathrm{singlethreaded}(s) \wedge \\ \mathrm{complete}(s) \wedge \\ \mathrm{Oinnocent}(s|_{A,C}) \wedge \\ \mathrm{ovw}(s|_{B,C}) \in \tau \wedge \\ (\forall q \in \mathrm{init}(s|_{A,B}))(\mathrm{ovw}(s|_{A,B}|_q) \in \sigma) \end{array} \}$$

But it is not yet clear which conditions on these sets are needed for associativity to work (and such that composition preserves such conditions.)

Questions?