Games & Higher-order Linear Dataflow

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Overview

- Towards Higher-Order Bigraphs
- We give a model of higher-order linear dataflow.
- This model is based upon fully complete models of linear logic by
  - Murawski & Ong (2003)
  - **Hyland & Ong (1993)**
  - Abramsky & Jagadeesan (1994)

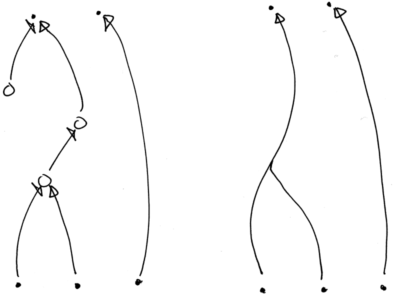
  ...and the Int-construction by Joyal, Street, and Verity.
- The model is reminiscent of:
  - Hughes (2006) MLL+unit proof nets
  - Hughes (2005) free *-autonomous category

Motivation

I wish that Robin Milner's bigraphs were symmetric monoidal closed.

Bigraphs are symmetric monoidal categories of graph contexts.
Dynamics of bigraphs are influenced by contexts.

1. Symmetric monoidal category $\sim$ multi-hole contexts.
2. Symmetric monoidal *closed* category $\sim$ higher-order contexts.



The problem

Have:

1. Category R_0 of finite sets and relations.
2. Category $T_0 \hookrightarrow R_0$ of finite sets and total functions.

Want:

1. Symmetric monoidal closed category
2. which embeds T_0
3. and which in some sense contains only total functions

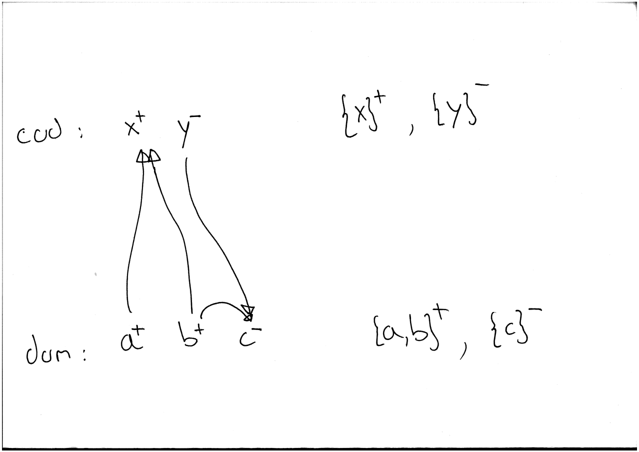
Idea

1. Do Int-construction on R_0, getting $\mathrm{Int}(R_0)$.
2. Then find a subcategory of $\mathrm{Int}(R_0)$ of total functions.

What is $\mathrm{Int}(R_0)$?

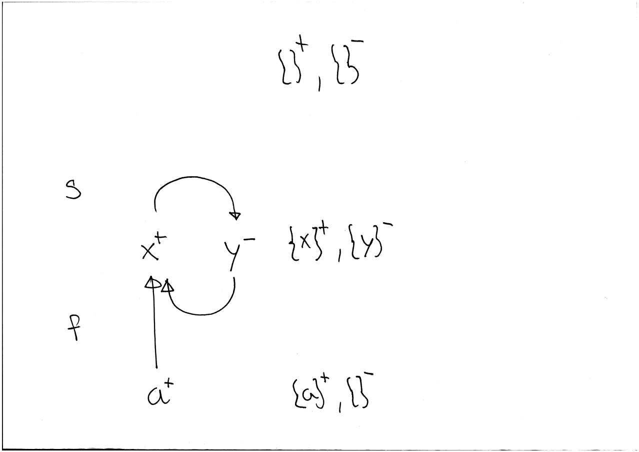
Objects pairs of finite sets (A^+, A^-).

Morphisms $f : (A^+, A^-) \to (B^+, B^-)$ relations
$f \subseteq A^+ + B^- \times B^+ + A^-$.



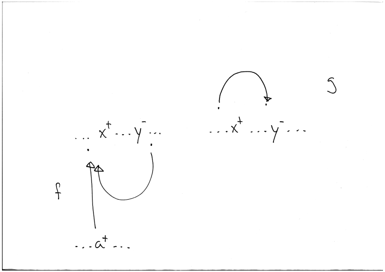
Composition is path-composition.

Alas, $\mathrm{Int}(R_0)$ has no (interesting) subcategory of total functions.



- Problem: Total functions of $\mathrm{Int}(R_0)$ are not closed under composition.
- Solution: Find a category H and faithful functor $F : H \to \mathrm{Int}(R_0)$, with image exactly the total functions.

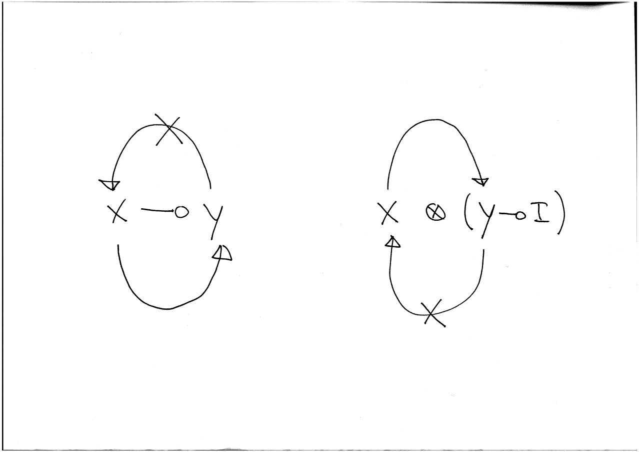
(Such refinements are known as *sortings* in the bigraph community.)



Intuition behind definition of H and F:

Objects	types A (over $I, \otimes, \multimap$) $F(A) = (A^+, A^-)$
Morphisms	$f : A \to B$ total functions $f : A^+ + B^- \to B^+ + A^-$ s.t. "f is a valid dataflow for $A \multimap B$". $F(f) = f$

Valid dataflow?



Formalisation?

Variation on Fair games of Hyland and Ong.

Games

Fair game: triple (M, λ, F) of

1. *moves* M (finite, contains at least two such);
2. *labelling function* $\lambda : M \to \{\mathrm{P}, \mathrm{O}\}$;
3. *maximal plays* F; a non-empty anti-chain of even-length sequences of alternately labelled moves, all beginning with an O-move.

The plays are the prefixes of the elements of F.

- The *tensor game* $A \otimes B$ has
  1. moves $M_A + M_B$;
  2. labelling function $[\lambda_A, \lambda_B]$; and
  3. maximal plays finite alternately-labelled sequences s over $M_A + M_B$ beginning with an O-move such that

$$s \upharpoonright A \in F_A \text{ and } s \upharpoonright B \in F_B .$$

- The *linear implication game* $A \multimap B$ has
  1. moves $M_A + M_B$;
  2. labelling function $[\overline{\lambda}_A, \lambda_B]$, and
  3. maximal plays finite alternately-labelled sequences over $M_A + M_B$ beginning with an O-move such that

$$s \upharpoonright A \in F_A \text{ and } s \upharpoonright B \in F_B .$$

Fair games are apparently unique in satisfying:

Proposition

Let σ be a total P-strategy for a game $A \multimap B$. Then $\sigma \restriction A$ is a total O-strategy for A and $\sigma \restriction B$ is a total P-strategy for B.

- The *atomic game*:

?	O
!	P

  Intuition: '?' requests data, '!' provides data.
- The *unit game* is simply the atomic game.

We now have games for each type. E.g., $a \multimap b$:

a	$\multimap$	b	
		?	O
?			P
!			O
		!	P

Games & total functions

Write $|A|$ for the atoms of A; A^+, A^- for the positive/negative atoms of A.

For a game $A \multimap B$:

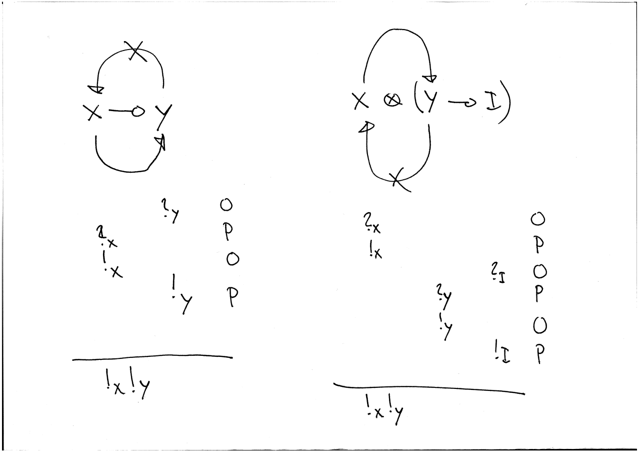
- A maximal play of $A \multimap B$ is a linear order of $M_A + M_B$.
- By restriction to !-moves, a maximum play of $A \multimap B$ is a *linear order on* $|A| + |B|$.
- A total strategy for $A \multimap B$ defines a *set of such linear orders*.

For a total function $f : A^+ + B^- \to B^+ + A^-$:

- The reflexive closure f^0 of f is a *partial order* on $|A| + |B|$.

A strategy $\sigma : A \multimap B$ *respects* f written $f \sqsubseteq \sigma$ iff for each linear order s of σ, the inclusion $f^0 \hookrightarrow s$ is order-respecting.

Example, revisited.



Objects linear types A (over $\otimes, \multimap, I$).

Morphisms $f : A \to B$ is a total function $f : A^+ + B^- \to B^+ + A^-$ of $\mathrm{Int}(R_0)$ s.t. there exists a strategy $\sigma : A \multimap B$ which respects f.

Theorem

1. *H is symmetric monoidal closed.*
2. *H embeds T_0.*
3. *If $f : A \multimap B \in H$ then $f : A^+ + B^- \to B^+ + A^-$ is a total function.*

Conclusion

1. Found a symmetric monoidal closed category H and a functor $F : H \to \mathrm{Int}(R_0)$ with image total functions.
2. From this we get (didn't say how) symmetric closed bigraphs.

Questions:

1. Did we really need Hyland-Ong fair games?

Thank you.