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Continuous Analogs of Axiomatized Digital Surfaces

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In image processing, workers sometimes wish to display three dimensional objects on a CRT screen, and so tools for detecting surfaces by computer need to be developed. In recent papers [D. G. Morgenthaler and A. Rosenfeld, *Information and Control* **51**, 1981, 227–247; G. M. Reed and A. Rosenfeld, submitted for publication; G. M. Reed, *Computer Vision, Graphics, and Image Processing* **25**, 1984, 226–235] Morgenthaler, Reed, and Rosenfeld have introduced the concept of a simple surface point of a subset of $\mathbb{Z}^3$. But simple surface points are defined by means of axioms, and the axioms do not reveal what simple surface points "look like." In this paper eight of the nine varieties of simple surface points are shown to have natural "continuous analogs," and the one remaining variety is shown to be very different from the other types. This work yields substantial generalizations of the main theorems on simple surface points that were proved by Morgenthaler, Reed, and Rosenfeld.

1. INTRODUCTION

A cuboid **Q** can be divided into small nonoverlapping cuboids of equal shape and size. In image processing, the small cuboids in such a partition are called "voxels," the word voxel being a contraction of "volume element." A subset **S** of **Q** can be well approximated by the union of a collection of voxels, provided these are very small relative to **Q**; this approximation allows the subset **S** to be represented in computer memory as the set of 1's in a certain $n_1 \times n_2 \times n_3$ array of 1's and 0's, where n_1, n_2, and n_3 are the numbers of voxels along three perpendicular edges of **Q**. Alternatively, **S** can be represented in a suitable $n_1 \times n_2 \times n_3$ array A of "real numbers" as the set of those array elements whose values lie between specified upper and lower "thresholds." (In fact the representation of **S** as the set of 1's in a binary array can *in theory* be regarded as a special case of this in which all elements of A are 0 or 1 and the upper and lower "thresholds" are both equal to 1.)

Three-dimensional "image arrays" of this second type are generated (by the method of "reconstruction from projections"—see [3]) in electron microscopy and diagnostic radiography. In the former application the subset **S** might typically be the tail of a bacteriophage, and in the latter **S** might be a hospital patient's heart. In both cases it is convenient to be able to display different views of **S** on a CRT screen, and for this purpose we need to "find" and "store" the outer surface of **S**. (While recognizing their extreme importance, we regard "removal of hidden points," "smoothing," and "shading" as "cosmetic" operations that are logically separate from the process of surface finding.)

When designing an "outer-surface detection algorithm" in this context we must of course specify what that algorithm is intended to detect, because **S** is essentially stored as a discrete set of points in $\mathbb{Z}^3$ and if we call this set of points $\rho(\mathbf{S})$ then it is not clear how the "outer surface" of $\rho(\mathbf{S})$ should best be defined. Perhaps the simplest approach (which is essentially due to Rosenfeld [8]) is to call the set $\{p \in \rho(\mathbf{S}) \mid p \text{ is } not \text{ surrounded by } \rho(\mathbf{S}) \setminus \{p\}\}$ the *outer border* of $\rho(\mathbf{S})$. (The term

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"surrounded by" is precisely defined in [8].) Readers interested in this definition are referred to [10]. Another natural definition of the "outer surface" of $\rho(\mathbf{S})$ is the following: Let $V(\mathbf{S})$ be the collection of voxels in $\mathbf{Q}$ corresponding to $\rho(\mathbf{S})$. Then $\cup V(\mathbf{S})$ approximates $\mathbf{S}$, so it is reasonable to *define* the outer surface of $\rho(\mathbf{S})$ to be the outer surface of $\cup V(\mathbf{S})$. The *surface* of $\cup V(\mathbf{S})$ is $\cup\{\mathbf{F} \mid \mathbf{F}$ is a face of exactly one voxel in $V(\mathbf{S})\}$, and the *outer surface* is a subset of the surface: so if we use this definition then it is easy to encode the outer surface of $\rho(\mathbf{S})$ in computer memory. In [2] and [4] Artzy, Frieder, Herman, and Webster propose a surface-tracking routine based on this representation of surfaces. Their algorithm accepts as input any one face on the outer surface of $\cup V(\mathbf{S})$ and produces from this the entire outer surface of $\cup V(\mathbf{S})$. The examples in [2] suggest that the algorithm works well.

But in a series of interesting papers [5, 7, 6] Morgenthaler, Reed, and Rosenfeld introduced the concept of a "simple surface point" of a subset of $\mathbb{Z}^3$, and it is natural to ask if their idea can be used to formulate an alternative definition of the "outer surface" of $\rho(\mathbf{S})$. However, Morgenthaler, Reed, and Rosenfeld defined their "simple surface points" axiomatically, and it is difficult to understand this concept just by reading the axioms. Accordingly, the main aim of our paper is to prove some "structure theorems" which reveal the "geometric meaning" of all but one of the nine different varieties of "simple surface point." (The remaining case appears to be quite dissimilar to the other eight.) These results (which are new) are established in Sections 3 and 4, and in Section 5 we use our "structure theorems" to derive powerful generalizations of the main theorems of [7] and [6]. We also show (in the corollary to Proposition 16) that one of the three axioms used by the earlier authors to define "simple surface points" can be deduced from the other two—provided we exclude the one anomalous variety of "simple surface point" referred to above.

Our treatment of this subject is original. We transform our (hard) digital topology problems into (fairly easy) problems of *continuous* topology, which we are able to solve. An important advantage of this approach to the theory of surface points is that readers should get an intuitive understanding of why the main results "must be true." In a future paper a similar technique will be used to prove a strong form of an important but fairly deep theorem that was stated without proof by Rosenfeld in [8], namely that the outer border (as defined above) of a "connected" subset of $\mathbb{Z}^3$ is itself "connected."

2. BASIC CONCEPTS

A. Definitions of Some Terms and Notations Used in Digital Topology

The following definitions are essentially equivalent to the corresponding definitions in Rosenfeld [8] except for Definitions 7 and 8 which are new. In Definition 4 we insist that a β-path should be a sequence of *distinct* points to avoid a clash with the terminology of graph theory.

Remark. $\mathbb{Z}^3$ denotes the set of all ordered triples of integers.

1. If p and q are *distinct* points in $\mathbb{Z}^3$ then we say that q is a **26-neighbor** of p if each coordinate of q differs from the same coordinate of p by *at most* 1. (Thus p has exactly twenty-six 26-neighbors.)

2. If p and q are *distinct* points in $\mathbb{Z}^3$ then we say that q is an **18-neighbor** of p if q is a 26-neighbor of p **and** at least one of the coordinates of q is equal to the same coordinate of p. (Thus p has exactly eighteen 18-neighbors.)

3. If p and q are *distinct* points in $\mathbb{Z}^3$ then we say that q is a **6-neighbor** of p if q is a 26-neighbor of p **and** q differs from p in *exactly one* coordinate. (Thus p has exactly six 6-neighbors.)

In all subsequent definitions α and β denote numbers which may be 6, 18, or 26.

4. We say p is **β-adjacent** to q if p is a β-neighbor of q. If $W \subseteq \mathbb{Z}^3$ then a **β-path in W from u to v** is a sequence of *distinct* points $(x_i | 0 \leq i \leq n)$ (where $n \geq 0$) such that $x_0 = u$, $x_n = v$, $x_i \in W$ $(0 < i < n)$, and x_i is β-adjacent to x_{i+1} $(0 \leq i < n)$.

5. If $W \subseteq \mathbb{Z}^3$ then W is **β-connected** if given any two points x and y in W there is a β-path in W from x to y. If $T \subseteq \mathbb{Z}^3$ then a **β-component** of T is a maximal β-connected subset of T (i.e., a β-connected subset of T that is not a proper subset of any other β-connected subset of T).

6. If $W \subseteq \mathbb{Z}^3$ then we say W is **β-adjacent** to u (or, equivalently, u is β-adjacent to W), if W contains some β-neighbor of u.

7. If $p \in \mathbb{Z}^3$ then $\mathrm{N}(p)$ denotes the set of all points x in $\mathbb{R}^3$ such that each coordinate of x differs from the corresponding coordinate of p by at most 1. (So $\mathrm{N}(p)$ is a cube with sides of length two.)

8. If W is an expression denoting a subset of $\mathbb{Z}^3$ then $W\langle p \rangle$ denotes the set $W \cap \mathrm{N}(p)$.

9. If $V \subseteq \mathbb{Z}^3$ then $\overline{V}$ denotes $\mathbb{Z}^3 \setminus V$.

B. Surface Points and Digital Surfaces

In [5], [7], and [6] Morgenthaler, Reed, and Rosenfeld define "simple surface points" and "simple closed surfaces." Their definitions can be restated as follows (see the remarks after Propositions 14 and 16).

Let each of α and β denote one of integers 6, 18, or 26. If $p \in W \subseteq \mathbb{Z}^3$ then we call p an **$(\alpha, \overline{\beta})$ surface point** of W if p is β-adjacent to exactly two β-components of $\overline{W}\langle p \rangle$ **and** each α-neighbor of p contained in W is β-adjacent to both of these β-components. We say W is an **$(\alpha, \overline{\beta})$ digital surface** if every point in W is an $(\alpha, \overline{\beta})$ surface point of W.

The basic goal of our paper is to present a visual interpretation of surface points (except in the case $\alpha = \beta = 6$), and to use this visual interpretation to give very natural proofs of several nontrivial theorems about digital surfaces. One of the results proved is that if α and β are not both equal to 6 then the complement of an $(\alpha, \overline{\beta})$ digital surface is **not** β-connected. We also exhibit a $(6, \overline{6})$ digital surface whose complement **is** 6-connected, thus showing that $(6, \overline{6})$ surface points are quite unlike the other kinds of surface point.

Remark. Observe that a $(26, \overline{\beta})$ surface point of a set S is always an $(18, \overline{\beta})$ surface point of S and an $(18, \overline{\beta})$ surface point of S is always a $(6, \overline{\beta})$ surface point of S.

C. Some Abbreviations

WLOG = "without loss of generality."
iff = "if and only if."
□ = "The result has been proved " (□ is similar in meaning to "Q.E.D.").
= "This is a contradiction."

D. Mathematical Terminology

The definitions in this section are standard in mathematics except for Definition 3. Although Definition 3 is not standard it is used by many authors.

1. $\mathbb{R}^3$ is the set of all ordered triples of real numbers. (Intuitively we think of $\mathbb{R}^3$ as the set of all points in space.)

2. If $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ are any two points in $\mathbb{R}^3$ then the **distance** from x to y is $\sqrt{((x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2)}$.

(Note that this agrees with our intuitive notion of "distance" in space.)

3. If $x \in \mathbb{R}^3$ and $r > 0$ then we define $B(x, r)$ to be the set of all points y in $\mathbb{R}^3$ such that the distance from x to y is strictly less than r. (Thus $B(x, r)$ is the set of points inside a sphere of radius r whose center is at x. In fact "B" is for "Ball.") $B(x, r)$ is undefined if $r \leq 0$.

4. A subset U of $\mathbb{R}^3$ is **open** if for each x in U there is a value of r such that $B(x, r) \subseteq U$. A subset of $\mathbb{R}^3$ is **closed** if its complement in $\mathbb{R}^3$ is open.

5. If $X \subseteq \mathbb{R}^3$ then the **interior** of X (written int(X)) is defined by $x \in \text{int}(X)$ **iff** there is a value of r such that $B(x, r) \subseteq X$.

6. If $X \subseteq \mathbb{R}^3$ then the **closure** of X (written cl(X)) is defined by $x \in \text{cl}(X)$ **iff** $B(x, r) \cap X \neq \varnothing$ for all $r > 0$.

7. If x and y are real numbers such that $x < y$ then (x, y) is sometimes used to denote the set $\{w | x < w < y\}$.

E. Definitions of Special Terms and Notations Used In This Paper

1. If u and v are distinct points in space (i.e., in $\mathbb{R}^3$) then uv denotes the straight line segment whose endpoints are u and v; if w is a point such that u, v, and w are not collinear then Δuvw denotes a closed triangle whose corners are u, v, and w. (A straight line segment is defined to include its endpoints.)

2. If $\mathbf{Q}$ is a cuboid then $\partial\mathbf{Q}$ denotes the surface of $\mathbf{Q}$. (Thus $\partial\mathbf{Q}$ is a union of six closed rectangles.)

3. A **closed unit cell** in $\mathbb{R}^3$ is a closed cube with sides of length 1 whose corners are all in $\mathbb{Z}^3$.

A **unit cell** in $\mathbb{Z}^3$ is the set of corners of a closed unit cell in $\mathbb{R}^3$.

4. Suppose $n \geq 1$ and $\{x_i | 0 \leq i \leq n\}$ is a set of points in space such that whenever $i \neq j$ $x_i x_{i+1} \cap x_j x_{j+1} = \{x_i, x_{i+1}\} \cap \{x_j, x_{j+1}\}$. Define $\gamma = \bigcup \{x_i x_{i+1} | 0 \leq i \leq n\}$. Then if the x_i are all distinct we shall call γ a **simple polygonal arc** joining x_0 to x_n. If $n \geq 2$, $x_0, \ldots, x_{n-1}$ are distinct **and** $x_n = x_0$ then we shall call γ a **simple closed polygonal curve**.

5. We shall say that a subset C of $\mathbb{R}^3$ is **polygonally connected** if any two points in C can be joined by a simple polygonal arc. If a polygonally connected subset D of a set X is *not* a subset of any other polygonally connected subset of X then we call D a **polygonally connected component** of X.

6. Suppose $n \geq 0$ and $\{T_i | 0 \leq i \leq n\}$ is a set of closed triangles in space such that the following both hold:

(i) Whenever $i \neq j$ $T_i \cap T_j$ is **either** a side of both T_i and T_j **or** a corner of both T_i and T_j **or** the empty set. (Strictly speaking the second alternative should of course be stated as "a set whose only member is a corner of both T_i and T_j.")
(ii) Each side of a triangle T_i is a side of *at most* one other T_i.

Then the set $\cup\{T_i | 0 \leq i \leq n\}$ will be called a **polyhedral surface.**

7. If T_i $(0 \leq i \leq n)$ are as in (6) and Σ is the union of the T_i, then the **boundary** of Σ (which we shall denote by $\partial\Sigma$) is $\cup\{s | s$ is a side of exactly one $T_i\}$.

This definition is a little tricky because if Σ is a polyhedral surface then there will be infinitely many ways of dissecting Σ into triangles satisfying 6i and ii: we must check that the definition of $\partial\Sigma$ produces the same set irrespective of the dissection we choose. While it is not hard to prove this we will not do so since the result should be "obvious" by geometric intuition.

We shall say that Σ is a polyhedral surface **without boundary** if Σ is a polyhedral surface such that $\partial\Sigma = \varnothing$.

8. A **strongly connected** polyhedral surface is a polyhedral surface Σ with the property that if F is any *finite* set of points then $\Sigma \setminus F$ is polygonally connected. (Thus a square and the surface of a cube are both strongly connected. But the union of two triangles that meet at just one point is not strongly connected.)

9. A polyhedral surface π will be called a **plate** if

either π is a face of some closed unit cell in $\mathbb{R}^3$

or π is strongly connected and satisfies each of the following three conditions:

(i) There is a unique closed unit cell in $\mathbb{R}^3$ which contains π: this cell will be denoted by $\mathbf{K}(\pi)$.
(ii) $\partial\pi$ is a simple closed polygonal curve, and each straight line segment contained in $\partial\pi$ is **either** an edge of $\mathbf{K}(\pi)$ **or** a diagonal of a face of $\mathbf{K}(\pi)$.
(iii) $\partial\pi = \pi \cap \partial\mathbf{K}(\pi)$.

10. If π is a plate then a **vertex** of π is any point in $\pi \cap \mathbb{Z}^3$; an **edge** of π is a straight line segment contained in $\partial\pi$ that joins two vertices of π.

11. If $p \in \mathbb{Z}^3$ then a **plate cycle** at p is a sequence $(\pi_i | 0 \leq i \leq n)$ (where $n \geq 1$) of *distinct* plates such that:

(i) There is a sequence $(e_i | 0 \leq i \leq n)$ in which e_i and e_{i+1} are distinct edges of π_i $(0 \leq i < n)$, e_n and e_0 are distinct edges of π_n, and p is an endpoint of each e_i.
(ii) If $i \neq j$ then $\pi_i \cap \pi_j$ is the union of a number (which may be zero) of straight line segments each of which is an edge of both plates and a set of points each of which is a vertex of both plates.
(iii) Any edge of a π_i is an edge of *at most* one other π_i.

The **plate set** of a plate cycle is the set of plates in that plate cycle.

12. If $\mathbb{P}$ is an expression denoting a set of plates and $p \in \mathbb{Z}^3$ then $\mathbb{P}\langle p\rangle$ denotes the set of those plates in $\mathbb{P}$ that have a vertex at p.

F. The (Plate) Cycle-Finding Algorithm

Let p be a point in $\mathbb{Z}^3$ and let $\mathbb{P}$ be a set of plates such that the following are all true:

(i) If $\pi \in \mathbb{P}\langle p \rangle$ and e is an edge of π that contains p then e is an edge of *at least* two plates in $\mathbb{P}\langle p \rangle$.

(ii) If three plates in $\mathbb{P}\langle p \rangle$ have an edge in common then p is an endpoint of one of the edges common to all three plates.

(iii) The intersection of any two distinct plates in $\mathbb{P}\langle p \rangle$ is the union of a number (which may be zero) of straight line segments each of which is an edge of both plates and a set of points each of which is a vertex of both plates.

Then if π' and π'' are plates in $\mathbb{P}$ that have an edge in common containing p the following simple algorithm will generate a plate cycle at p whose plate set is a subset of $\mathbb{P}$.

Cycle-Finding Algorithm

Arguments $\pi', \pi'' : \mathbb{P}$
where π' and π'' have an edge in common which contains p.
Let π_0 be π'.
Let π_1 be π''.
Let e_0 be an edge common to π_0 and π_1 that contains p.
For $i = 1, 2, 3, \ldots$
let e_i be the edge of π_i that contains p but is different from e_{i-1};
let π_{i+1} be a plate different from π_i such that e_i is an edge of π_{i+1}.
until there is $m < i$ such that $e_i = e_m$.
Let n be the largest i for which e_i is defined.
Let m be the (unique) integer less than n such that $e_m = e_n$.
Let **result** be the sequence $(\pi_i | m < i \leq n)$.

G. An Important Result

PROPOSITION 0. *Let* $\mathbf{X}$ *be a closed cuboid in* $\mathbb{R}^3$, *and let* Σ *be a polyhedral surface contained in* $\mathbf{X}$ *such that* $\Sigma \cap \operatorname{int}(\mathbf{X}) \neq \varnothing$ *and* $\partial\Sigma \subseteq \partial\mathbf{X}$. *Let* p *be any point in* $\Sigma \cap \operatorname{int}(\mathbf{X})$. *Then* p *is in the closure of two* different *polygonally connected components of* $\mathbf{X} \setminus \Sigma$. *Furthermore, if* Σ *is strongly connected and* $\partial\Sigma = \Sigma \cap \partial\mathbf{X}$ *then* $\mathbf{X} \setminus \Sigma$ *has exactly two polygonally connected components, and* Σ *is a subset of the closure of each of these two sets.*

Remark. This result should be very plausible by "geometric intuition." A proof is outlined in the Appendix, but this is intended for readers who are reasonably familiar with the concepts of general topology.

H. Three Important Remarks

(a) In a few of the proofs we use a string "*abc*" (where a, b, and c are digits) to denote the point with coordinates (a, b, c) (e.g., 201 denotes the point $(2, 0, 1)$.) Figure 1 should make it easy to follow the proofs which use this notation.

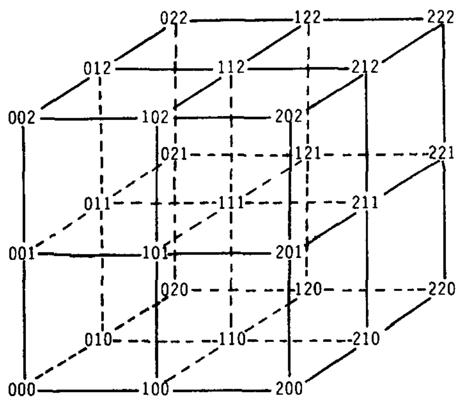

FIG. 1. $\mathbb{Z}^3\langle 111\rangle$.

(b) In the rest of this paper α and β denote any one of 6, 18, or 26 except where there is a statement to the contrary.

(c) We use "by $(SP(W), \mathbb{Z}^3\langle xyz\rangle, uvw)$" [as in "by $(SP(W), \mathbb{Z}^3\langle 201\rangle, 101)$"] as an abbreviation for the clause "since xyz is an $(\alpha, \bar{\beta})$ surface point of W, uvw is β-adjacent to each of the two (and only two) β-components of $\overline{W}\langle xyz\rangle$ that are β-adjacent to xyz."

3. A THEORY OF PLATES

Definitions

I. Let $\mathbf{Q}$ be a closed cuboid whose corners are in $\mathbb{Z}^3$ and whose edges are parallel to the coordinate axes. Let S be a subset of $\mathbb{Z}^3$ and let β be 6, 18, or 26. We shall say that a collection $\mathbb{P}$ of plates is **$\bar{\beta}$-natural** with respect to $(S, \mathbf{Q})$ **iff** the following conditions (a), (b), (c), (d) hold for every closed unit cell $\mathbf{K}$ in $\mathbf{Q}$:

(a) At most one plate in $\mathbb{P}$ meets int($\mathbf{K}$). If there is such a plate and V is its vertex set then $S \cap \mathbf{K} \cap \text{int}(\mathbf{Q}) \subseteq V \subseteq S \cap \mathbf{K}$.
(b) The straight line segment joining two β-adjacent points in $\bar{S} \cap \mathbf{K}$ does not meet any plate in $\mathbb{P}$ that is contained in $\mathbf{Q}$.
(c) If no plate in $\mathbb{P}$ meets int($\mathbf{K}$) then $\mathbf{K} \cap \bar{S}$ is nonempty and β-connected. If there is a plate π in $\mathbb{P}$ that meets int($\mathbf{K}$) and $\mathbf{C}$ is any polygonally connected component of int($\mathbf{K}$)$\setminus \pi$ then cl($\mathbf{C}$) $\cap \bar{S}$ is nonempty and β-connected.
(d) Let $\mathbf{F}$ be any face of $\mathbf{K}$ such that $\mathbf{F}$ is **not** a subset of $\partial\mathbf{Q}$. Then $\mathbf{F} \in \mathbb{P}$ **iff** all corners of $\mathbf{F}$ are in S. Also, if a diagonal of $\mathbf{F}$ is an edge of a plate in $\mathbb{P}$ then the corners of $\mathbf{F}$ which are not on this diagonal are in $\bar{S}$.

We shall say that $\mathbb{P}$ is **$\bar{\beta}$-natural** with respect to S **iff** (a), (b), (c), and (d) hold when $\mathbf{Q}$ is replaced by $\mathbb{R}^3$. (We define $\partial\mathbb{R}^3 = \varnothing$.)

Remarks. 1. If $\mathbb{P}$ is $\bar{\beta}$-natural wrt S then $\mathbb{P}$ is $\bar{\beta}$-natural wrt $(S, \mathbf{Q})$ for any choice of $\mathbf{Q}$.

2. If $\mathbb{P}$ is $\bar{\beta}$-natural wrt $(S, \mathbf{Q})$ and $p \in S \cap \text{int}(\mathbf{Q})$ then $\mathbb{P}\langle p\rangle$ is $\bar{\beta}$-natural wrt $(S, \mathbf{N}(p))$.

II. Suppose $p \in S \subseteq \mathbb{Z}^3$ and $\mathbb{P}$ is a collection of plates.

We shall say that $\mathbb{P}$ satisfies the **$\bar{\beta}$-form of condition 1** with respect to (S, p) **iff** the following subconditions both hold:

(a) If $\mathbb{P}\langle p\rangle$ is the plate set of a plate cycle at p then $\bar{S}\langle p\rangle$ has exactly two β-components.

(b) If $\mathbb{P}\langle p\rangle \supseteq \mathbb{P}' \cup \mathbb{P}''$ where $\mathbb{P}'$ and $\mathbb{P}''$ are the plate sets of two *different* plate cycles at p then $\bar{S}\langle p\rangle$ has at least three β-components.

We shall say that $\mathbb{P}$ satisfies the **$(\alpha, \bar{\beta})$-form of condition 2** with respect to (S, p) **iff** the following three subconditions all hold:

(a) If v is a vertex of two plates π_1, π_2 in $\mathbb{P}\langle p\rangle$ and vp is an edge of π_1 then vp is an edge of π_2.

(b) If $x \in S\langle p\rangle$ and y is a β-neighbor of x that is contained in $\bar{S}\langle p\rangle$ then the straight line segment xy **either** does not meet $\cup\mathbb{P}\langle p\rangle$ **or** meets $\cup\mathbb{P}\langle p\rangle$ only at x.

(c) If $\pi \in \mathbb{P}\langle p\rangle$ and $e = vp$ is an edge of π then v is α-adjacent to p.

The first proposition in this chapter explains why we chose to use the word "natural" in I above.

PROPOSITION 1. *Let S be a subset of $\mathbb{Z}^3$ and let $\mathbb{P}$ be a collection of plates which is $\bar{\beta}$-natural with respect to $(S, \mathbf{Q})$, where $\mathbf{Q}$ is a closed cuboid whose corners are in $\mathbb{Z}^3$ and whose edges are parallel to the coordinate axes. Let $\mathbb{P}(\mathbf{Q})$ denote the set of plates in $\mathbb{P}$ which are contained in some closed unit cell in $\mathbf{Q}$. Then two points in $\bar{S}$ are in the same β-component of $\bar{S} \cap \mathbf{Q}$* **iff** *they are in the same polygonally connected component of $\mathbf{Q} \setminus \cup\, \mathbb{P}(\mathbf{Q})$.*

Proof. "Only if" follows trivially from part (b) of the definition of $\bar{\beta}$-natural.

To prove "if," let Γ be a simple polygonal arc in $\mathbf{Q} \setminus \cup\, \mathbb{P}(\mathbf{Q})$ from x to y where x, y are in $\bar{S} \cap \mathbf{Q}$. Now there is $\epsilon > 0$ such that the distance from any point on Γ to any point on $\cup\mathbb{P}(\mathbf{Q})$ exceeds ϵ; hence by modifying Γ if necessary we may assume that $\Gamma \setminus \{x, y\}$ meets no edge of any closed unit cell in $\mathbf{Q}$.

Let l be the length of Γ, and for $0 \le s \le l$ let $\gamma(s)$ denote the point on Γ whose distance from x measured along Γ, is s. (Thus $\gamma(0) = x$, and $\gamma(l) = y$.) Let $\mathbf{K}_1, \mathbf{K}_2 \dots, \mathbf{K}_n$ be all the closed unit cells in $\mathbf{Q}$ (in any order, provided no cell appears twice).

Define $\mathbf{N}_i = \{s \in (0, l) | \gamma(s + \epsilon) \in \mathbf{K}_i$ and $\gamma(s - \epsilon) \notin \mathbf{K}_i$ for all sufficiently small $\epsilon\}$. Let $\mathbf{M} = \cup\{\mathbf{N}_i | i = 1, 2 \dots, n\} \cup \{0, l\}$. Since Γ is polygonal, $\mathbf{M}$ is a finite set. By perturbing Γ if necessary we may assume that every s such that $\gamma(s)$ is on a face of a $\mathbf{K}_i$ is in $\mathbf{M}$. Let the members of $\mathbf{M}$ be $0 = s_0 < s_1 \cdots < s_r = l$. Then for each $0 \le i < r$ there is a cell $\mathbf{K}_{m(i)}$ containing the set of points $\{\gamma(s) | s_i \le s \le s_{i+1}\}$. For all i such that $0 < i < r$ define a face $\mathbf{F}_i$ by $\mathbf{F}_i = \mathbf{K}_{m(i-1)} \cap \mathbf{K}_{m(i)}$. Then $\gamma(s_i) \in \mathbf{F}_i$. For every $0 < i < r$ pick a point u_i in $\bar{S} \cap \mathbf{F}_i$ such that the straight line segment from $\gamma(s_i)$ to u_i meets no plate in $\mathbb{P}(\mathbf{Q})$. (Such a point exists since by part (d) of the definition of "natural" there is a point q in $\mathbf{F} \cap \bar{S}$: now if the straight line segment from $\gamma(s_i)$ to q meets an edge e of a plate in $\mathbb{P}(\mathbf{Q})$ then e is a diagonal of $\mathbf{F}$ so (by d) the corner of $\mathbf{F}$ opposite q is also in $\bar{S}$, and the straight line segment from $\gamma(s_i)$ to this corner meets no plate in $\mathbb{P}(\mathbf{Q})$.) Define $u_0 = \gamma(0) = x$, $u_r = \gamma(l) = y$.

By our definitions of $m(i)$, u_i and u_{i+1} there is a polygonal arc in $\mathbf{K}_{m(i)}$ from u_i to u_{i+1} which meets no plate in $\mathbb{P}(\mathbf{Q})$. We may assume WLOG that this arc meets $\partial\mathbf{K}_{m(i)}$ only at its endpoints (for we can always perturb the arc so as to make this true). Hence by part c of our definition of "natural" there is a β-path in $\bar{S} \cap \mathbf{K}_{m(i)}$ from u_i to u_{i+1} $(0 \leq i < r)$. It follows that $x = u_0$ and $y = u_r$ are in the same β-component of $\bar{S} \cap \mathbf{Q}$, as required. □

COROLLARY. *If $\mathbb{P}$ is $\bar{\beta}$-natural wrt $(S, \mathbf{Q})$ then $\bar{S} \cap \mathbf{Q}$ has exactly as many β-components as $\mathbf{Q} \setminus \cup \mathbb{P}(\mathbf{Q})$ has polygonally connected components.*

Proof. Suppose $x \in \mathbf{W}$ where $\mathbf{W}$ is a polygonally connected component of $\mathbf{Q} \setminus \cup \mathbb{P}(\mathbf{Q})$. Let $\mathbf{K}$ be a closed unit cell in $\mathbf{Q}$ that contains x. There is $\epsilon > 0$ such that $B(x, \epsilon)$ meets no plate in $\mathbb{P}(\mathbf{Q})$ so there is a point y in $\text{int}(\mathbf{K}) \cap \mathbf{W}$. By part c of the definition of "natural" there is q in $\bar{S}$ such that $q \in \text{cl}(\mathbf{C})$ where $\mathbf{C}$ is the polygonally connected component of $\text{int}(\mathbf{K}) \setminus \cup \mathbb{P}$ containing y. So $q \in \mathbf{W}$. This argument shows that every polygonally connected component of $Q \setminus \cup \mathbb{P}(\mathbf{Q})$ meets $\bar{S}$. The corollary now follows from Proposition 1. □

Remark. Although its proof is fairly simple, Proposition 1 is one of the most important results in this paper.

PROPOSITION 2. *Suppose $p \in S \subseteq \mathbb{Z}^3$ and $\mathbb{P}$ is a collection of plates satisfying the following conditions* i, ii, iii:

(i) *If $\pi \in \mathbb{P}\langle p \rangle$ then $(\pi \cap \partial\mathrm{N}(p))$ is precisely the union of the edges of π that do not contain p.*

(ii) *If π_1, π_2 are in $\mathbb{P}\langle p \rangle$ then $\pi_1 \cap \pi_2$ is* **either** *$\{p\}$* **or** *an edge of π_1 and π_2 that contains p* **or** *the union of the two edges of π_1 and π_2 that contain p.*

(iii) *$\mathbb{P}\langle p \rangle$ is $\bar{\beta}$-natural wrt $(S, \mathrm{N}(p))$.*

Then $\mathbb{P}$ satisfies the $\bar{\beta}$-form of condition 1 wrt (S, p). Furthermore, if $\mathbb{P}\langle p \rangle$ is the plate set of a plate cycle at p then $\mathrm{N}(p) \setminus \cup \mathbb{P}\langle p \rangle$ has exactly two polygonally connected components, and if $x \in \pi \in \mathbb{P}\langle p \rangle$ then x is in the closure of both polygonally connected components.

Proof. We shall assume throughout this proof that $\mathbb{P}$ satisfies i, ii, and iii.

Case I. $\mathbb{P}\langle p \rangle$ is the plate set of a plate cycle at p.

It is easy to see from the definition of "plate cycle" that $\cup \mathbb{P}\langle p \rangle$ is a strongly connected polyhedral surface. If $\pi \in \mathbb{P}\langle p \rangle$ and e is an edge of π that does not contain p then by ii e is an edge of no other plate in $\mathbb{P}\langle p \rangle$, so $e \subseteq \partial(\cup \mathbb{P}\langle p \rangle)$. Conversely, if e is an edge of a plate and $e \subseteq \partial(\cup \mathbb{P}\langle p \rangle)$ then by the definition of a plate cycle $p \notin e$. Hence i implies that $\cup \mathbb{P}\langle p \rangle \cap \partial\mathrm{N}(p) = \partial(\cup \mathbb{P}\langle p \rangle)$. The result now follows from Propositions 0 and 1 (Corollary).

Case II. $\mathbb{P}\langle p \rangle \supseteq \mathbb{P}' \cup \mathbb{P}''$ where $\mathbb{P}'$ and $\mathbb{P}''$ are the plate sets of two different plate cycles at p.

By the argument given in the previous paragraph, $\mathrm{N}(p) \setminus \cup \mathbb{P}'$ has two polygonally connected components $\mathbf{K}_1', \mathbf{K}_2'$, and $\mathrm{N}(p) \setminus \cup \mathbb{P}''$ has two polygonally connected components $\mathbf{K}_1'', \mathbf{K}_2''$. Let π be a plate in $\mathbb{P}''/\mathbb{P}'$, and let x be a point on π which is not on an edge of π. Then x is on no other plate in $\mathbb{P}\langle p \rangle$, and x is not on $\partial\mathrm{N}(p)$. So WLOG $x \in \text{int}(\mathbf{K}_1')$. But (by Proposition 0) $x \in \cup \mathbb{P}''$ implies $x \in \text{cl}(\mathbf{K}_1'')$ and $x \in \text{cl}(\mathbf{K}_2'')$. Therefore $\mathbf{K}_1' \cap \mathbf{K}_1''$ and $\mathbf{K}_1' \cap \mathbf{K}_2''$ both contain points arbitrarily close to x, whence $\mathbf{K}_1' \cap \mathbf{K}_1'' \setminus \cup \mathbb{P}\langle p \rangle$ and $\mathbf{K}_1' \cap \mathbf{K}_2'' \setminus \cup \mathbb{P}\langle p \rangle$ are both nonempty. By the same type of argument $\mathbf{K}_2' \setminus \cup \mathbb{P}\langle p \rangle$ is also nonempty. But if we pick one point from each of these sets then these three points must lie in distinct polygonally connected components of $\mathbf{N}(p) \setminus \cup \mathbb{P}\langle p \rangle$. So we are home by Proposition 1 (Corollary). □

PROPOSITION 3. *Suppose p is an $(\alpha, \bar{\beta})$ surface point of the set S, and suppose further that $\bar{S}\langle p \rangle$ has exactly two β-components. Let $\mathbb{P}$ be a set of plates which is $\bar{\beta}$-natural wrt $(S, \mathbf{N}(p))$ and which satisfies the $\bar{\beta}$-form of Condition 1 and the $(\alpha, \bar{\beta})$-form of Condition 2 wrt (S, p). Then $\mathbb{P}\langle p \rangle$ is the plate set of a single plate cycle at p, and every α-neighbor of p that is contained in S is a vertex of some plate in $\mathbb{P}\langle p \rangle$.*

Remark. The proof makes use of the fact that if ab is an edge of a plate π then if ϵ is positive and sufficiently small $(B(a, \epsilon) \cap \mathbf{N}(b)) \setminus \pi$ is polygonally connected. The reader can **either** prove this **or** include an extra hypothesis in the statement of the proposition to the effect that all plates in $\mathbb{P}$ have this property. (It is a trivial matter to check that every plate introduced in the next chapter satisfies this condition.)

Proof of Proposition 3. Suppose the hypotheses are satisfied. Then $\mathbb{P}\langle p \rangle$ is $\bar{\beta}$-natural wrt $(S, \mathbf{N}(p))$. Let v be p or an α-neighbor of p that is contained in S: then by the definition of an $(\alpha, \bar{\beta})$ surface point there are β-neighbors y_1, y_2 of v which are in different β-components of $\bar{S}\langle p \rangle$. So, by Proposition 1, y_1 and y_2 are in different polygonally connected components of $\mathbf{N}(p) \setminus \cup \mathbb{P}\langle p \rangle$. By Condition 2b the straight line segments $y_1 v$ and $y_2 v$ can only meet $\cup \mathbb{P}\langle p \rangle$ at v. Hence $v \in \cup \mathbb{P}\langle p \rangle$, so v is a vertex of a plate π in $\mathbb{P}\langle p \rangle$. Further, if vp is an edge of a plate then v is a vertex of a second plate in $\mathbb{P}\langle p \rangle$. For suppose otherwise: then choose $r > 0$ such that $(B(v, \epsilon) \cap \mathbf{N}(p)) \setminus \pi$ is polygonally connected whenever $0 < \epsilon < r$ (see the remark above). Now choose $n(> 1)$ so large that $B(v, r/n)$ meets no plate in $\mathbb{P}\langle p \rangle$ except π; then $B(v, r/n) \cap \mathbf{N}(p) \setminus \cup \mathbb{P}\langle p \rangle$ is polygonally connected, so there is a path in $\mathbf{N}(p) \setminus \cup \mathbb{P}\langle p \rangle$ from y_1 to y_2, contrary to the definition of these two points.

Next suppose $e = xp$ is an edge of a plate π in $\mathbb{P}\langle p \rangle$. Then by Condition 2c x is α-adjacent to p, so by the previous paragraph x is a vertex of a plate π' in $\mathbb{P}\langle p \rangle$, where $\pi' \neq \pi$. Condition 2a implies that xp is also an edge of π'. So we have shown that any edge of a plate in $\mathbb{P}\langle p \rangle$ that has p as an endpoint is an edge of at least two plates in $\mathbb{P}\langle p \rangle$. Now subconditions a and d of $\bar{\beta}$-naturalness, Condition 2a and the previous sentence together imply that $\mathbb{P}\langle p \rangle$ satisfies the preconditions i, ii, and iii of the cycle-finding algorithm. On applying this algorithm we get a plate cycle at p. Suppose for the purpose of getting a contradiction that there is a plate in $\mathbb{P}\langle p \rangle$ which is not in this plate cycle.

There *may* be two plates π_0, π_1 in $\mathbb{P}\langle p \rangle$ such that π_0 is in the cycle, π_1 is *not* in the cycle, and π_0 and π_1 have an edge in common: if so then apply the cycle-finding algorithm to $\mathbb{P}\langle p \rangle$ starting from (π_0, π_1). If two such plates do not exist then apply the cycle-finding algorithm to $\mathbb{P}\langle p \rangle$ starting from any pair of plates which share an edge containing p but which are not in the cycle already found. In either case we obtain a plate cycle that is different from the first one. Let $\mathbb{P}'$ and $\mathbb{P}''$ be the sets of plates in the two cycles: then by condition 1b $\bar{S}\langle p \rangle$ has at least three β-components, contrary to hypothesis. This contradiction proves Proposition 3. □

Remark. Proposition 3 is a very powerful tool, as will be seen in Sections 4 and 5.

PROPOSITION 4. *Suppose $p \in S \subseteq \mathbb{Z}^3$ and $\mathbb{P}$ is the plate set of a plate cycle at p satisfying the hypotheses* i, ii, iii *of Proposition* 2. *Suppose further that* $\mathbf{D}$ *is a subset of* $\mathrm{N}(p)$ *such that* $\mathrm{int}(\mathbf{D})$ *meets* $\cup\mathbb{P}$ *and* $\mathbf{D} \setminus \cup \mathbb{P}$ *has exactly two polygonally connected components* $\mathbf{C}_1$, $\mathbf{C}_2$ *both of which meet* $\bar{S}$. *Then* $\mathbf{C}_1 \cap \bar{S}\langle p \rangle$ *and* $\mathbf{C}_2 \cap \bar{S}\langle p \rangle$ *are contained in different β-components of* $\bar{S}\langle p \rangle$.

Proof. Suppose the conditions hold. Pick x in $\mathrm{int}(\mathbf{D}) \cap (\cup \mathbb{P})$: then by Proposition 2 $x \in \mathrm{cl}(\mathbf{K}_1)$ and $x \in \mathrm{cl}(\mathbf{K}_2)$, where $\mathbf{K}_1$ and $\mathbf{K}_2$ are the two polygonally connected components of $\mathrm{N}(p) \setminus \cup \mathbb{P}$. So $\mathrm{int}(\mathbf{D}) \cap \mathbf{K}_1$ and $\mathrm{int}(\mathbf{D}) \cap \mathbf{K}_2$ are nonempty. Since $\mathbf{D}$ meets both $\mathbf{K}_1$ and $\mathbf{K}_2$ it is impossible for either one of $\mathbf{K}_1$ and $\mathbf{K}_2$ to contain both $\mathbf{C}_1$ and $\mathbf{C}_2$. So WLOG $\mathbf{C}_1 \subseteq \mathbf{K}_1$ and $\mathbf{C}_2 \subseteq \mathbf{K}_2$. The result now follows from Proposition 1. □

4. WHAT DO $(\alpha, \bar{\beta})$ SURFACE POINTS LOOK LIKE?

DEFINITION. If p is an $(\alpha, \bar{\beta})$ surface point of W then $\bar{A}(p)$ and $\bar{B}(p)$ denote the two β-components of $\bar{S}\langle p \rangle$ that are β-adjacent to p. (We do not care *which* of the two β-components is $\bar{A}(p)$ and which is $\bar{B}(p)$: the purpose of this notation is merely to allow us to refer to each β-component separately.)

PROPOSITION 5. *Let p be an $(\alpha, \bar{\beta})$ surface point of S and let K be a unit cell in* $\mathbb{Z}^3\langle p \rangle$. *In the cases where $\beta = 6$ suppose further that each 6-neighbor of p that is contained in $S \cap K$ is also an $(\alpha, \bar{6})$ surface point of S. Then $\bar{S} \cap K$ is β-adjacent to p.*

Proof. Case I. $\beta = 18$ or 26

WLOG $p = 111$ and K is the cell $\{xyz \mid x, y, z \in \{1, 2\}\}$. Suppose the result fails. Then $221, 121, 122, 212, 211, 111, 112$ are in S. As $\{102, 202, 201, 101\}$ cannot meet both $\bar{A}(111)$ and $\bar{B}(111)$ we may assume WLOG that it *does not* meet $\bar{A}(111)$. Then by $(SP(S), \mathbb{Z}^3\langle 111 \rangle, 112)$ one of $002, 012, 022, 001, 011, 021$ is in $\bar{A}(111)$, and by $(SP(S), \mathbb{Z}^3\langle 111 \rangle, 211)$ one of $200, 210, 220, 100, 110, 120$ is in $\bar{A}(111)$. Therefore, since $\bar{A}(111)$ is β-connected one of $001, 011, 021$ is in $\bar{A}(111)$, and one of $100, 110, 120$ is in $\bar{A}(111)$. Hence none of the nine points in $\mathbb{Z}^3\langle 111 \rangle$ with $y = 1$ can be in $\bar{B}(111)$, so **either** $y = 0$ for all points in $\bar{B}(111)$, **or** $y = 2$ for all points in $\bar{B}(111)$. The former is impossible because $121 \in S$ implies 121 is β-adjacent to $\bar{B}(111)$, but the latter implies that 101 is neither in $\bar{B}(111)$ nor β-adjacent to $\bar{B}(111)$, whence $101 \in \bar{A}(111)$ contrary to our earlier assumption.

Case II. $\beta = 6$

WLOG $p = 111$ and K is the same cell as before. Suppose the result fails. Then $121, 112, 211$ are $(\alpha, \bar{6})$ surface points of S. WLOG $011 \in \bar{A}(111)$ and $101 \in \bar{B}(111)$. Then by $(SP(S), \mathbb{Z}^3\langle 211 \rangle, 111)$ $110 \in \bar{S}$, so WLOG $110 \in \bar{B}(111)$. Since $011 \in \bar{A}(111)$ none of $010, 021, 012, 001$ can be in $\bar{B}(111)$. Hence no 6-path in $\bar{B}(111)$ from 101 to 110 can go through $\mathbb{Z}^3\langle 111 \rangle \setminus \mathbb{Z}^3 \langle 211 \rangle$. So there is a 6-path in $\bar{B}(111)$ from 101 to 110 which lies entirely within $\mathbb{Z}\langle 111 \rangle \cap \mathbb{Z}^3\langle 211 \rangle$. This implies that 101 and 110 are in the same 6-component of $\bar{S}\langle 211 \rangle$, so $(SP(S), \mathbb{Z}^3\langle 211 \rangle, 111)$ is violated. □

COROLLARY. *If p is an $(\alpha, \bar{\beta})$ surface point of a set S, where $\beta = 18$ or 26 then $\bar{S}\langle p \rangle$ has exactly two β-components, and both are β-adjacent to p.*

Proof. The result is trivial if $\beta = 26$. If $\beta = 18$ and K is any unit cell in $\mathbb{Z}^3\langle p \rangle$ then there is some point q in $K \cap \bar{S}\langle p \rangle$ which is 18-adjacent to p. Since every element of $K \cap \bar{S}\langle p \rangle$ is 18-adjacent to p or q it follows that every 18-component of $\bar{S}\langle p \rangle$ is 18-adjacent to p, so the corollary follows from the definition of an $(\alpha, \overline{18})$ surface point. □

$\overline{26}$ Surface Points

DEFINITION. Let S be any set of points in $\mathbb{Z}^3$ such that no unit cell contains eight points in S. Then we define $\mathbb{F}_{26}(S)$ to be the set of 1×1 squares whose corners all lie in S.

We claim that if $p \in S$ then $\mathbb{F}_{26}(S)$ satisfies the $\overline{26}$-form of Condition 1 and the $(\alpha, \overline{26})$-form of Condition 2 wrt (S, p) with $\alpha = 6$, 18, or 26. We further claim that $\mathbb{F}_{26}(S)$ is $\overline{26}$-natural wrt S. It is easy to confirm the naturalness of $\mathbb{F}_{26}(S)$ and the validity of Condition 2. Furthermore, $\mathbb{F}_{26}(S)$ satisfies the hypotheses i, ii, iii of Proposition 2 with $\beta = 26$, so $\mathbb{F}_{26}(S)$ satisfies the $\overline{26}$-form of Condition 1 wrt (S, p).

PROPOSITION 6. *If $p \in S \subseteq \mathbb{Z}^3$ then p is an $(\alpha, \overline{26})$ surface point of S* **iff** *the following all hold*:

(i) *No unit cell in* $\mathbf{N}(p)$ *contains eight points in S.*

(ii) $\mathbb{F}_{26}(S\langle p\rangle)\langle p\rangle$ *is the plate set of a single plate cycle at p.*

(iii) *If q is an α-neighbor of p that is contained in S then q is a vertex of some plate in* $\mathbb{F}_{26}(S\langle p\rangle)\langle p\rangle$.

Proof. "*only if*": i follows from Proposition 5. If p is an $(\alpha, \overline{26})$ surface point of S then $\bar{S}\langle p\rangle$ has exactly two β-components, so ii and iii follow from Proposition 3.

"*if*": Suppose i, ii, and iii all hold. Then by ii and Condition 1a $\bar{S}\langle p\rangle$ has exactly two 26-components. Let v be p or any α-neighbor of p that is contained in S; by iii v is a vertex of some plate π in $\mathbb{F}_{26}(S\langle p\rangle)\langle p\rangle$. WLOG $p = 111$, and the vertices of π are 111, 121, 122, 112. Let $\mathbf{D}_1$ and $\mathbf{D}_2$ denote the closed unit cells in $\mathbf{N}(111)$ containing 022 and 222, respectively. Each of $\mathbf{D}_1$ and $\mathbf{D}_2$ meets $\bar{S}\langle 111\rangle$ by i. Further, $\mathbf{D}_1 \setminus \cup\, \mathbb{F}_{26}(S\langle 111\rangle)\langle 111\rangle$ and $\mathbf{D}_2 \setminus \cup\, \mathbb{F}_{26}(S\langle 111\rangle)\langle 111\rangle$ are both polygonally connected sets, so on applying Proposition 4 to $\mathbf{D}_1 \cup \mathbf{D}_2$ we deduce that $\mathbf{D}_1 \cap \bar{S}\langle 111\rangle$ and $\mathbf{D}_2 \cap \bar{S}\langle 111\rangle$ are contained in different 26-components of $\bar{S}\langle 111\rangle$. But every point in these two sets is 26-adjacent to each vertex of π. So $p = 111$ is an $(\alpha, \overline{26})$ surface point of S, as required. □

COROLLARY. *If $S \subseteq \mathbb{Z}^3$ then S is an $(\alpha, \overline{26})$ digital surface* **iff** i, ii, *and* iii *hold for all p in S.* □

$\overline{18}$ Surface Points

DEFINITION. If X is a unit cell in $\mathbb{Z}^3$, g is the centroid of X, and x, y are any two diametrically opposite points in X, then the set $\cup\{\Delta uvg \mid u, v$ are 6-adjacent points in $X \setminus \{x, y\}\}$ will be called a **compound plate**.

A compound plate is shown in Fig. 2.

DEFINITION. Let S be a subset of $\mathbb{Z}^3$ such that no unit cell contains eight points in S. Then we define $\mathbb{F}_{18}(S)$ to be the set of plates such that $\pi \in \mathbb{F}_{18}(S)$ **iff** one of the following is true:

either π is a 1×1 square whose corners are all in S,

or π is a compound plate whose vertices are all in S **and** the unit cell containing π contains no point in S that is not a vertex of π.

We claim that if $p \in S$ then $\mathbb{F}_{18}(S)$ satisfies the $\overline{18}$-form of Condition 1 and the $(\alpha, 18)$-form of Condition 2 wrt (S, p) with $\alpha = 6$, 18, or 26. We further claim that $\mathbb{F}_{18}(S)$ is $\overline{18}$-natural wrt S. It is easy to confirm the naturalness of $\mathbb{F}_{18}(S)$ and the

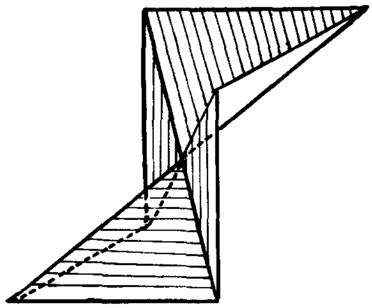

FIG. 2. A compound plate.

validity of Condition 2. To see that $\mathbb{F}_{18}(S)$ satisfies Condition 1 wrt (S, p) define a function f on $\mathbb{F}_{18}(S)\langle p\rangle$ as follows:

(i) If π is *not* a compound plate then $i(\pi) = \pi$.

(ii) If π is a compound plate with vertices $p, v_1, v_2, v_3, v_4, v_5$, where v_i is 6-adjacent to v_{i+1} and p is 6-adjacent to v_1 and v_5, then $f(\pi) = (\Delta p v_1 v_3) \cup (\Delta p v_5 v_3)$.

Now the set $f(\mathbb{F}_{18}(S)\langle p\rangle)$ satisfies the hypotheses i, ii, iii of Proposition 2 with $\beta = 18$. So by Proposition 2 $f(\mathbb{F}_{18}(S)\langle p\rangle)$ satisfies the $\overline{18}$-form of Condition 1 wrt (S, p). But if $\mathbb{P}' \subseteq \mathbb{F}_{18}(S)\langle p\rangle$ and $\mathbb{P}'$ is the plate set of a plate cycle at p then $f(\mathbb{P}')$ is also the plate set of a plate cycle at p. Hence $\mathbb{F}_{18}(S)$ also satisfies the $\overline{18}$-form of Condition 1 wrt (S, p), as asserted.

PROPOSITION 7. *Suppose p is an $(\alpha, \overline{18})$ surface point of a set S, and all α-neighbors of p that lie in S are also $(\alpha, \overline{18})$ surface points of S. Then the following all hold*:

(i) *Each unit cell in* $\mathbb{N}(p)$ *contains at most six points in S.*

(ii) $\mathbb{F}_{18}(S\langle p\rangle)\langle p\rangle$ *is the plate set of a single plate cycle at p.*

(iii) *If q is an α-neighbor of p that is contained in S then q is a vertex of some plate in* $\mathbb{F}_{18}(S\langle p\rangle)\langle p\rangle$.

Conversely, if $p \in S \subseteq \mathbb{Z}^3$ and i, ii, iii *all hold then p is an $(\alpha, \overline{18})$ surface point of S.*

Proof. If $p \in S \subseteq \mathbb{Z}^3$ and p and all its α-neighbors in S are $(\alpha, \overline{18})$ surface points of S then i follows from Proposition 5 while ii and iii follow from Proposition 5 (Corollary) and Proposition 3.

Conversely, suppose S satisfies i, ii, and iii. Then WLOG $p = 111$. Then by ii and Condition 1a $\bar{S}\langle 111\rangle$ has precisely two 18-components. Let v be 111 or an α-neighbor of 111 that lies in S: then by iii there is a plate π in $\mathbb{F}_{18}(S\langle 111\rangle)\langle 111\rangle$ that contains v. If π is a compound plate then WLOG the vertices of π are 111, 211, 212, 222, 122, and 121. Let $\mathbf{D}$ be the closed unit cell of $\mathbb{N}(111)$ that contains 222. Then $\mathbf{D} \setminus \cup f(\mathbb{F}_{18}(\langle 111\rangle)\langle 111\rangle)$ has precisely two polygonally connected components, $\mathbf{C}_1, \mathbf{C}_2$ say, and each of these contains one point in $\bar{S}$. By Proposition 4 (applied to $\mathbf{D}$ and $f(\mathbb{F}_{18}(S\langle 111\rangle)\langle 111\rangle)$ the points in $\mathbf{C}_1 \cap \bar{S}$ and $\mathbf{C}_2 \cap \bar{S}$ are in different 18-components of $\bar{S}\langle 111\rangle$. If, on the other hand, π is a square plate then WLOG the vertices of π are 111, 121, 122, and 112. Let $\mathbf{D}_1$ and $\mathbf{D}_2$ be the closed unit cells in $\mathbf{N}(111)$ that contain 222 and 022. Then by Proposition 4 $\mathbf{D}_1 \cap \bar{S}$ and $\mathbf{D}_2 \cap \bar{S}$ are contained in different 18-components of $\bar{S}\langle 111\rangle$. By i each of these two sets contains a point 18-adjacent to v. So in each case V is 18-adjacent to both 18-components of $\bar{S}\langle p\rangle$. Hence $p = 111$ is an $(\alpha, \overline{18})$ surface point of S. □

COROLLARY. *If $S \subseteq \mathbf{Z}^3$ then S is an $(\alpha, \overline{18})$ digital surface* **iff** i, ii, *and* iii *hold for every p in S.* □

$\bar{6}$ Surface Points

PROPOSITION 8. *Suppose p is an $(18, \bar{6})$ surface point of a set S and K is a unit cell in $\mathbf{Z}^3\langle p\rangle$ such that every 18-neighbor of p that lies in $S \cap K$ is also an $(18, \bar{6})$ surface point of S. Then K is identical (after a suitable rotation or reflection) to one of the nine cells in Fig.* 3.

Proof. In this proof "surface point" means "$(18, \bar{6})$ surface point of S."

Suppose the hypotheses are satisfied. WLOG K is the unit cell in $\mathbf{Z}^3\langle 111\rangle$ that contains 222. We shall prove the result by showing that the following four situations **cannot** arise.

(1) $S \cap K$ contains four corners of a regular tetrahedron.

(2) $S \cap K$ contains more than four points.

(3) $S \cap K$ consists of three of the four corners of a $1 \times \sqrt{2}$ rectangle.

(4) $S \cap K$ contains exactly two points, and these are diagonally opposite corners of a face of K.

(Proposition 5 implies that if $S \cap K$ contains exactly four points and (1) does not occur then K is identical to (a), (b), (d) or (e) in Fig. 3.)

Suppose (1) occurs. WLOG 111, 212, 122, 221 are in S. Then by Proposition 5 p must be one of these four points and so WLOG $p = 111$. Proposition 5 now implies that none of 112, 121, and 211 is in S. Thus WLOG 112 and 211 are in the same

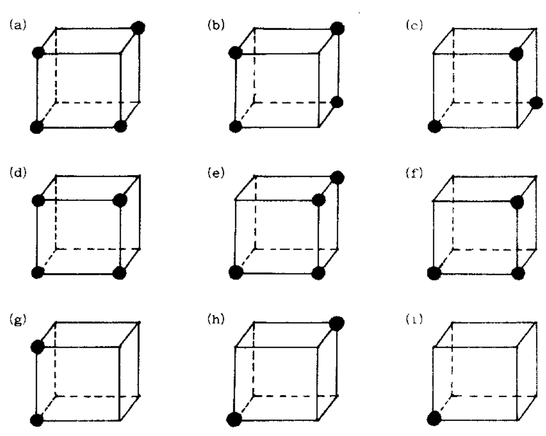

FIG. 3. Unit cells of types (a) to (i). (The points marked ● are in S.)

6-component of $\bar{S}\langle 111\rangle$. But now $\{102, 201\}$ either meets $\bar{S}$ or does not meet $\bar{S}$, and in both cases 212 is 6-adjacent to just one 6-component of $\bar{S}\langle 111\rangle$ that is 6-adjacent to 111. #

Next, suppose that (2) occurs. Let us first eliminate the possibility that K contains four points in S which are the corners of a face of K: if so then by symmetry we may assume that 111, 211, 221, 121, and 112 are in S. By hypothesis **either** 221 is a surface point, in which case $(SP(S), \mathbb{Z}^3\langle 221\rangle, 111)$ is violated, **or** $p = 112$, in which case 111, 121, and 211 are also surface points, so that Proposition 5 is contradicted at 111. Now consider the case in which $S \cap K$ does **not** contain four corners of a unit square. Since (1) does not occur WLOG 111, 112, 212, 221, and 121 are all in S. By symmetry we may assume that p is one of these five points. Then 111 is a surface point, and one of 112 and 121 is also a surface point. Assume WLOG that 121 is a surface point. By $(SP(S), \mathbb{Z}^3\langle 121\rangle, 112)$ $022 \in \bar{S}$. Hence 201 and 210 are in S and $211 \in \bar{S}$ by $(SP(S), \mathbb{Z}^3\langle 111\rangle, 212)$ and $(SP(S), \mathbb{Z}^3\langle 111\rangle, 221)$. So $\{211\}$ is a 6-component of $\bar{S}\langle 111\rangle$ that is not 6-adjacent to 112, whence 111 is not a surface point. #

Suppose finally that (3) or (4) occurs. Then WLOG 111 and 212 are in S and 112, 122, 222, 221, 211 are all in $\bar{S}$. By hypothesis 111 must be a surface point. But now $\{102, 201\}$ either meets $\bar{S}$ or does not meet $\bar{S}$, and in both cases 212 is 6-adjacent to just one 6-component of $\bar{S}\langle 111\rangle$ that is 6-adjacent to 111. # □

If a unit cell is identical to cell (a) in Fig. 3 then we shall say that it is a cell of *type* (a); similarly cells will be said to be of types (b), (c),... (h) or (i).

PROPOSITION 9. *If p is an $(18, \bar{6})$ surface point of a set S and every 18-neighbor of p that lies in S is also an $(18, \bar{6})$ surface point of S then each 6-component of $\bar{S}\langle p\rangle$ is 6-adjacent to p.*

Proof. Suppose the hypotheses are satisfied, but there is a 6-component of $\bar{S}\langle p\rangle$ which is not 6-adjacent to p. By inspection of Fig. 3 we see that this 6-component cannot contain any of the eight corners of $\mathbb{Z}^3\langle p\rangle$, so it must consist of a single point which is 18-adjacent but not 6-adjacent to p. Then WLOG $p = 011$ and $\{112\}$ is a 6-component of $\bar{S}\langle 011\rangle$. This means that 012, 102, 122, and 111 are in S, so (by Proposition 8) 101, 001, 002, 121, 021, and 022 are in $\bar{S}$.

Suppose WLOG that $112 \in \bar{A}(111)$: then $212 \in \bar{A}(111)$ (else 011 is not 6-adjacent to $\bar{A}(111)$, a contradiction), and therefore 022, 021, and 121 are in $\bar{B}(111)$ (for if 022, 021, and 121 are in $\bar{A}(111)$ then 122 is not 6-adjacent to $\bar{B}(111)$, a contradiction). Similarly 002, 001, and 101 are in $\bar{B}(111)$. So by $(SP(S), \mathbb{Z}^3\langle 111\rangle, 011)$ $010 \in \bar{A}(111)$, whence there is a 6-path in $\bar{A}(111)$ from 010 to 212. Consequently $110 \in \bar{A}(111)$, which implies that $\bar{B}(111)$ is not 6-connected. This contradiction proves the proposition. □

COROLLARY 1. *If p is an $(18, \bar{6})$ surface point of a set S and every 18-neighbor of p that lies in S is also an $(18, \bar{6})$ surface point of S then $\bar{S}\langle p\rangle$ has exactly two 6-components, and both are 6-adjacent to p.* □

COROLLARY 2. (This is not a genuine corollary.) *If p is an $(18, \bar{6})$ surface point of a set S and all 18-neighbors of p that lie in S are also $(18, \bar{6})$ surface points of S then no $2 \times 1 \times 1$ cell in $\mathbf{N}(p)$ is identical to the cell in Figure 4.*

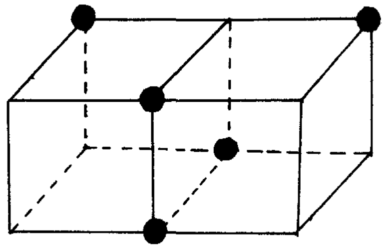

FIG. 4. A "forbidden" configuration. (The points marked ● are in S.)

Proof. Suppose the hypotheses are satisfied and such a $2 \times 1 \times 1$ cell exists. WLOG the cell contains 002, 022, and 111, and WLOG 011, 012, 102, 122, 111 are in S. Then $p = 011$, 012, or 111 and so 111 must be an $(18, \bar{6})$ surface point of S. Hence the argument given in the second paragraph of the proof of Proposition 9 produces the required contradiction. □

PROPOSITION 10. *If p is a $(26, \bar{6})$ surface point of a set S and every 26-neighbor of p that lies in S is also a $(26, \bar{6})$ surface point of S then every unit cell in $\mathbb{Z}^3\langle p \rangle$ is identical to* (b), (c), (d), (g), *or* (i) *in Fig.* 3.

Proof. Suppose the hypotheses are satisfied. Then WLOG $p = 111$. By virtue of Proposition 8 it suffices to establish the following two assertions:

(i) If two diametrically opposite corners of a unit cell in $\mathbb{Z}^3\langle p \rangle$ are both in S then the cell is identical to cell (b).

(ii) No cell in $\mathbb{Z}^3\langle p \rangle$ can be identical to cell (f).

To prove i, suppose 222 is in S. We claim this implies that the unit cell containing 111 and 222 is identical to cell (b). For 222 must be 6-adjacent to $\bar{A}(111)$ and to $\bar{B}(111)$, so WLOG $122 \in \bar{A}(111)$, $221 \in \bar{B}(111)$, and $121 \in S$. Then, since 111 is 6-adjacent to two 6-components of $\bar{S}\langle 222 \rangle$, 112 and 211 are in $\bar{S}$ and 212 is in S. So our claim is justified.

To prove ii, suppose on the contrary that 111, 211, 212 are in S and 112, 122, 222, 221, and 121 are in $\bar{S}$. Then by i, 101, 102, and 202 are in $\bar{S}$, so 212 is 6-adjacent to only one 6-component of $\bar{S}\langle 111 \rangle$, a contradiction. □

DEFINITION. Let S be a subset of $\mathbb{Z}^3$ such that no unit cell contains more than four points in S and every cell containing four points in S is identical (after a suitable rotation) to cell (a), (b), (d), or (e) in Fig. 3. Suppose further that no $2 \times 1 \times 1$ cell is identical to the $2 \times 1 \times 1$ cell in Fig. 4. Then we define $\mathbb{F}_6(S)$ to be the set of plates such that $\pi \in \mathbb{F}_6(S)$ iff one of the following [(a)–(d)] applies:

(a) π is the union of two triangles $\triangle ABC$ and $\triangle BCD$, where A, B, C, and D are the four points in a cell of type a which are in S, and $BC = \sqrt{3}$.

(b) π is a $1 \times \sqrt{2}$ rectangle whose corners are the four points in a cell of type b which are in S.

(c) π is a $(\sqrt{2}, \sqrt{2}, \sqrt{2})$ triangle whose corners are the three points in a cell of type c which are in S.

(d) π is a 1×1 square whose corners are the four points in a cell of type d which are in S.

π will be called a plate of type a, b, c, or d depending on which of the above applies.

We claim that if $p \in S$ then $\mathbb{F}_6(S)$ satisfies the $\bar{6}$-form of Condition 1 and the $(\alpha, \bar{6})$-form of Condition 2 wrt (S, p), with $\alpha = 18$ or 26. We further claim that $\mathbb{F}_6(S)$ is $\bar{6}$-natural wrt S. It is easy to confirm the naturalness of $\mathbb{F}_6(S)$ and the validity of Condition 2 by inspection of Fig. 3. But $\mathbb{F}_6(S)$ satisfies hypotheses i, ii, and iii of Proposition 2 with $\beta = 6$. Hence by Proposition 2 $\mathbb{F}_6(S)$ satisfies the $\bar{6}$-form of Condition 1 wrt (S, p).

PROPOSITION 11. *Suppose p is an $(18, \bar{6})$ surface point of a set S and all 18-neighbors of p that lie in S are also $(18, \bar{6})$ surface points of S. Then the following all hold*:

(i) *No unit cell in $\mathbf{N}(p)$ contains more than four points in S and every cell containing four points in S is identical to cell* (a), (b), (d), *or* (e) *in Fig.* 3.

(ii) *No $2 \times 1 \times 1$ cell in $\mathbf{N}(p)$ is as in Fig.* 4.

(iii) *$\mathbb{F}_6(S\langle p\rangle)\langle p\rangle$ is the plate set of a single plate cycle at p.*

(iv) *If q is an 18-neighbor of p that lies in S then q is a vertex of some plate in $\mathbb{F}_6(S\langle p\rangle)\langle p\rangle$.*

Conversely if $p \in S \subseteq \mathbb{Z}^3$ and i, ii, iii, and iv all hold then p is an $(18, \bar{6})$ surface point of S.

Proof. If p is an $(18, \bar{6})$ surface point of a set S and all 18-neighbors of p that lie in S are also $(18, \bar{6})$ surface points of S then i and ii follow from Propositions 8 and 9 (Corollary 2), while iii and iv follow from Proposition 9 (Corollary 1) and Proposition 3.

Conversely, suppose $p \in S \subseteq \mathbb{Z}^3$ and i, ii, iii, and iv all hold. Then, since $\mathbb{F}_6(S\langle p\rangle)$ satisfies the $\bar{6}$-form of Condition 1 wrt (S, p), iii implies that $\bar{S}\langle p\rangle$ has exactly two 6-components. Let v be p or any 18-neighbor of p that lies in S. Then by iv v is a vertex of a plate π in $\mathbb{F}_6(S\langle p\rangle)\langle p\rangle$. We assert that v is 6-adjacent to both 6-components of $\bar{S}\langle p\rangle$.

If π is a plate of type b or c then this result can be obtained by applying Proposition 4 to $\mathbb{F}_6(S\langle p\rangle)$ (taking $\mathbf{D}$ to be the closed unit cell in $\mathbf{N}(p)$ containing the plate). If π is of type a or d, then WLOG $p = 111$, and by symmetry it is enough to establish the result in the following four cases:

I. The vertices of π are 111, 212, 222, and 122.

II. The vertices of π are 111, 112, 222, and 121.

III. The vertices of π are 111, 112, 122, and 221.

IV. The vertices of π are 111, 121, 122, and 112.

In each case the desired result can be deduced from Proposition 4 (applied to $\mathbb{F}_6(S\langle p\rangle)\langle p\rangle$): in case I take $\mathbf{D}$ to be the closed unit cell in $\mathbf{N}(p)$ that contains 222; in the other cases take $\mathbf{D}$ to be the union of the closed unit cells in $\mathbf{N}(p)$ which contain 022 and 222 (note that by i and ii the interior of the cell containing 022 meets no plate in $\mathbb{F}_6(S\langle p\rangle)$ and that this cell always contains a point in $\bar{S}$ that is 6-adjacent to v). This argument justifies our assertion, which implies that p is an $(18, \bar{6})$ surface point of S. $\square$

COROLLARY. *If $S \subseteq \mathbb{Z}^3$ then S is an $(18, \bar{6})$ digital surface* **iff** i, ii, iii, *and* iv *hold for all p in S.* □

PROPOSITION 12. *Suppose p is a $(26, \bar{6})$ surface point of a set S and every 26-neighbor of p that lies in S is also a $(26, \bar{6})$ surface point of S. Then the following all hold*:

(i) *No unit cell in* $\mathbf{N}(p)$ *contains more than four points in S, and every cell containing four points in S is identical to* (b) *or* (d) *in Fig.* 3.

(ii) $\mathbb{F}_6(S\langle p\rangle)\langle p\rangle$ *is the plate set of a single plate cycle at p.*

(iii) *If q is a 26-neighbor of p that lies in S then q is a vertex of some plate in* $\mathbb{F}_6(S\langle p\rangle)\langle p\rangle$.

Conversely, if $p \in S \subseteq \mathbb{Z}^3$ and (*i*), (*ii*), *and* (*iii*) *all hold then p is a* $(26, \bar{6})$ *surface point of S.*

Proof. If p is a $(26, \bar{6})$ surface point of a set S and every 26-neighbor of p that lies in S is also a $(26, \bar{6})$ surface point of S then i is Proposition 10 while ii and iii follow from Proposition 9 (Corollary 1) and Proposition 3.

Conversely, if $p \in S \subseteq \mathbb{Z}^3$ and i, ii, and iii all hold then by Proposition 1 p is an $(18, \bar{6})$ surface point of S. Suppose v is a 26-neighbor of p that lies in S and v is not 18-adjacent to p. Then by i and iii v and p are diametrically opposite corners of a cell of type b, so on applying Proposition 4 to this cell we deduce that v is 6-adjacent to two different 6-components of $\bar{S}\langle p\rangle$, both of which are 6-adjacent to p. Therefore p is a $(26, \bar{6})$ surface point of S. □

COROLLARY. *If $S \subseteq \mathbb{Z}^3$ then S is a $(26, \bar{6})$ digital surface* **iff** i, ii, *and* iii *all hold for every p in S.* □

5. FUNDAMENTAL PROPERTIES OF SURFACE POINTS AND DIGITAL SURFACES

In the previous section we obtained simple "visual interpretations" of eight of the nine different kinds of surface point (the exception being the $(6, \bar{6})$ surface points). We will now use these ideas to establish a number of basic results about surface points and digital surfaces. Special cases of Propositions 13 and 18 were proved by Morgenthaler, Reed, and Rosenfeld in [5], [7], and [6]. The complexity and subtlety of their arguments (which did not make use of "filling-in" algorithms) would seem to highlight the benefits of our approach.

PROPOSITION 13. *Let S be an α-connected $(\alpha, \bar{\beta})$ digital surface, where α and β are not both equal to* 6. *Suppose $S \subseteq$* int($\mathbf{Q}$) *where $\mathbf{Q}$ is a cuboid whose corners are all in $\mathbb{Z}^3$ and whose edges are all parallel to the coordinate axes. Then $\bar{S} \cap \mathbf{Q}$ has exactly two β-components, and each β-component is β-adjacent to every point in S.*

Proof. Suppose the hypotheses are satisfied. Then, by Propositions 6ii, 7ii, 11iii, and 12ii, $\cup\mathbb{F}_\beta(S)$ is a polyhedral surface without boundary contained in int($\mathbf{Q}$). We claim that $\cup\mathbb{F}_\beta(S)$ is strongly connected. To see this, let $\mathbf{P}$ be a **maximal** subset of $\mathbb{F}_\beta(S)$ such that $\cup\mathbf{P}$ is strongly connected. Let p and q be any pair of α-adjacent points in S. Then $\mathbb{F}_\beta(S)\langle p\rangle \cap \mathbb{F}_\beta(S)\langle q\rangle$ is nonempty, by Propositions 6iii, 7iii, 11iv, and 12iii. Furthermore, $q \in S$ implies $\mathbb{F}_\beta(S)\langle q\rangle$ is strongly connected (this follows from Propositions 6ii, 7ii, 11iii, 12ii, and the fact that every plate cycle is strongly connected). Therefore if $\mathbb{F}_\beta(S)\langle p\rangle \subseteq \mathbb{P}$ then $\mathbb{F}_\beta(S)\langle q\rangle \subseteq \mathbb{P}$. Since (p, q) is an arbitrary pair of α-neighbors in S, and S is α-connected, it follows that $\mathbb{F}_\beta(S)\langle r\rangle \subseteq \mathbb{P}$ for all r in S. Hence $\mathbb{P} = \mathbb{F}_\beta(S)$, and so $\cup\mathbb{F}_\beta(S)$ is strongly connected.

Hence by Proposition 0 $\mathbf{Q} \setminus \cup \mathbb{F}_\beta(S)$ has precisely two polygonally connected components, $\mathbf{C}_1$ and $\mathbf{C}_2$ say, and every point in S is in the closure of both components. But $\mathbb{F}_\beta(S)$ is $\bar{\beta}$-natural wrt S, so we deduce from subcondition c of naturalness that each of $\mathbf{C}_1$ and $\mathbf{C}_2$ contains a point in $\bar{S}\langle p\rangle$ for all p in S. Furthermore, Proposition 1 implies that $\mathbf{C}_1 \cap \bar{S}$ and $\mathbf{C}_2 \cap \bar{S}$ are distinct β-components of $\bar{S} \cap \mathbf{Q}$. Combining these two observations with Proposition 5 (Corollary) and Proposition 9 (Corollary 1) we get the required result. □

Remark. The basic goal of [7] and [6] was to prove this proposition for $(6, \overline{26})$ and $(26, \bar{6})$ digital surfaces.

PROPOSITION 14.

(i) *Let S be an $(\alpha, \bar{\beta})$ digital surface, where $\beta = 18$ or 26 and let p be any point in S. Let $\mathbf{W}$ be the polygonally connected component of $\cup\mathbb{F}_\beta(S)$ that contains p, and let $T = \mathbf{W} \cap \mathbb{Z}^3$. Then T is 6-connected and T is an α-component of S. Furthermore, if $q \in T$ then $\mathbb{F}_\beta(T)\langle q\rangle = \mathbb{F}_\beta(S)\langle q\rangle$.*

(ii) *Let S be an $(\alpha, \bar{6})$ digital surface, where $\alpha = 18$ or 26, and let p be any point in S. Let $\mathbf{W}$ be the polygonally connected component of $\cup\mathbb{F}_6(S)$ that contains p, and let $T = \mathbf{W} \cap \mathbb{Z}^3$. Then T is 18-connected and T is an α-component of S. Furthermore, if $q \in T$ then $\mathbb{F}_6(T)\langle q\rangle = \mathbb{F}_6(S)\langle q\rangle$.*

Proof. Let $\mathbb{P}$ be the subset of $\mathbb{F}_\beta$(S) such that $\cup\mathbb{P} = \mathbf{W}$.

(i) Suppose the hypotheses are satisfied. If p and q are two points in $\mathbf{W} \cap \mathbb{Z}^3$ then (since $\mathbf{W}$ is polygonally connected) there is a sequence $(\pi_i | 0 \le i \le n)$ of plates in $\mathbb{P}$ such that p is a vertex of π_0, q is a vertex of π_n and π_i has a vertex in common with π_{i+1} $(0 \le i < n)$. But the set of vertices of each π_i is 6-connected, so here is a 6-path in $\mathbf{W} \cap \mathbb{Z}^3$ from p to q. Hence $\mathbf{W} \cap \mathbb{Z}^3$ is 6-connected. T is an α-component of S, for if $u \in T$ and v is a point in S that is α-adjacent to u then there is a plate π in $\mathbb{F}_\beta(S)$ that contains both u and v (by Propositions 6iii and 7iii): since $u \in \mathbf{W}$ it follows that $\pi \subseteq \mathbf{W}$, which implies $v \in T$.

Now let q be an arbitrary point in T and let $\mathbf{K}$ be any closed unit cell in $\mathbf{N}(q)$. If $\mathbf{K}$ contains a plate in $\mathbb{F}_\beta(S)$ then $\mathbf{K} \cap S$ must be α-connected, so since T is an α-component of S it follows that $\mathbf{K} \cap S = \mathbf{K} \cap T$, whence the sets $\{\pi \in \mathbb{F}_\beta(T) | \pi \subseteq \mathbf{K}\}$ and $\{\pi \in \mathbb{F}_\beta(S) | \pi \subseteq \mathbf{K}\}$ are the same. If $\mathbf{K}$ does not contain a plate in $\mathbb{F}_\beta(S)$ then since $T \subseteq S$ it is easily seen that $\mathbf{K}$ contains no plate in $\mathbb{F}_\beta(T)$. So, since $\mathbf{K}$ is an arbitrary cell in $\mathbf{N}(q)$, $\mathbb{F}_\beta(S)\langle q\rangle = \mathbb{F}_\beta(T)\langle q\rangle$, as asserted.

(ii) The proof is obtained from the proof of i by substituting "18-connected" and "18-path" for "6-connected" and "6-path," and invoking Propositions 11iv and 12iii instead of 6iii and 7iii. □

COROLLARY. *Suppose α and β are not both equal to 6. Then S is an $(\alpha, \bar{\beta})$ digital surface* **iff** *every α-component of S is itself an $(\alpha, \bar{\beta})$ digital surface.*

Proof. Let T be an α-component of an $(\alpha, \bar{\beta})$ digital surface S, and let q be any point in T. Then by the proposition above $\mathbb{F}_\beta(T)\langle q\rangle = \mathbb{F}_\beta(S)\langle q\rangle$, so since S is an

$(\alpha, \bar{\beta})$ digital surface the "only if" part is an immediate consequence of Propositions 6, 7, 11, or 12 (depending on the values of α and β). To prove the "if" part, suppose every α-component of a set S is an $(\alpha, \bar{\beta})$ digital surface. Then $\mathbb{F}_\beta(S)$ exists: for suppose there is a unit cell of a "forbidden" type (wrt S) (such as a cell with eight points in S); then this cell meets S in an α-connected set and so it is also a "forbidden" cell wrt some α-component of S #. Let q be any point in S, and let T be the α-component of S that contains q. Then $\mathbb{F}_\beta(S)\langle q\rangle = \mathbb{F}_\beta(T)\langle q\rangle$, by the argument given in the second paragraph of the proof of part i of the proposition. The corollary now follows from Propositions 6, 7, 11, or 12 (depending on the values of α and β). □

Remark. This corollary is the main reason why we did *not* require that an $(\alpha, \bar{\beta})$ digital surface should be α-connected.

Our next proposition shows that the corresponding result for $(6, \bar{6})$ digital surfaces is false. (Our proof is essentially the same as the proof of Proposition 16 in [5].)

PROPOSITION 15. *A finite 6-component of a $(6, \bar{6})$ digital surface is* **never** *a $(6, \bar{6})$ digital surface.*

Proof. Let T be a finite 6-component of a $(6, \bar{6})$ digital surface. Then since T is bounded we may assume WLOG that the z coordinate of each point in T is *positive*, and WLOG $111 \in T$. Suppose T contains a point p whose z coordinate is strictly greater than 1. Then since there is a 6-path in T from 111 to p there must be x, y, such that $(x, y, 1)$ and $(x, y, 2)$ are both in T. Then $(x, y, 1)$ is 6-adjacent to exactly one 6-component of $\bar{S}\langle(x, y, 1)\rangle$, so T is *not* a $(6, \bar{6})$ digital surface. This argument shows that if a 6-component of a $(6, \bar{6})$ digital surface is itself a $(6, \bar{6})$ digital surface then every point in that 6-component has the same z coordinate: similarly every point in the 6-component has the same x coordinate, and the same y coordinate, whence the 6-component consists of just one point, and so it is not a $(6, \bar{6})$ digital surface. □

Remark. If we omit the word "finite" in the statement of Proposition 15 then the result is obviously false, since if P is any coordinate plane then $\mathbb{Z}^3 \cap P$ is a 6-connected $(6, \bar{6})$ digital surface. Nontrivial examples of 6-connected $(6, \bar{6})$ digital surfaces seem to be quite "rare." But they do exist, as Fig. 5 shows.

PROPOSITION 16.

(i) *Suppose p is an $(\alpha, \bar{\beta})$ surface point of S, where $\beta = 18$ or 26. Then p is α-adjacent to exactly one 6-component of $S\langle p\rangle \setminus \{p\}$.*

(ii) *Suppose p is an $(\alpha, \bar{6})$ surface point of S, and every α-neighbor of p that lies in S is also an $(\alpha, \bar{6})$ surface point of S where $\alpha = 18$ or 26. Then p is α-adjacent to exactly one 18-component of $S\langle p\rangle \setminus \{p\}$.*

Proof.

(i) Suppose the hypotheses are satisfied. Then $\mathbb{F}_\beta(S\langle p\rangle)\langle p\rangle$ exists and is the plate set of a single plate cycle at p by Propositions 3 and 5 (Corollary). But if $\pi \in \mathbb{F}_\beta(S\langle p\rangle)\langle p\rangle$ then π is either a square plate or a compound plate, and in both cases $\pi \cap S \setminus \{p\}$ is 6-connected and 6-adjacent to p. So, since the intersection of the vertex sets of two consecutive plates in a plate cycle must contain a point in $S \setminus \{p\}$, $(\cup\mathbb{F}_\beta(S\langle p\rangle)\langle p\rangle) \cap S \setminus \{p\}$ is 6-connected and 6-adjacent to p. By

If N is any positive integer then the Nth layer looks like this:
(Extend this picture to infinity in the obvious way.)

$$\begin{array}{ccccccccccc}
0&0&0&0&0&0&\mathbf{1}&0&0&0&0\\
0&0&0&0&0&0&\mathbf{1}&0&0&0&0\\
0&0&0&0&0&0&\mathbf{1}&0&0&0&0\\
0&0&0&0&0&0&\mathbf{1}&0&0&0&0\\
\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}&0&0&\mathbf{1}&0&0&0&0\\
0&0&0&0&\mathbf{1}&0&\mathbf{1}&0&0&0&0\\
0&0&0&0&\mathbf{1}&0&0&\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}\\
0&0&0&0&\mathbf{1}&0&0&0&0&0&0\\
0&0&0&0&\mathbf{1}&0&0&0&0&0&0\\
0&0&0&0&\mathbf{1}&0&0&0&0&0&0\\
0&0&0&0&\mathbf{1}&0&0&0&0&0&0
\end{array}$$

The 0th layer looks like this:
(Extend the picture to infinity.)

$$\begin{array}{ccccccccccc}
0&0&0&0&0&0&\mathbf{1}&0&0&0&0\\
0&0&0&0&0&0&\mathbf{1}&0&0&0&0\\
0&0&0&0&0&0&\mathbf{1}&0&0&0&0\\
0&0&0&0&0&0&\mathbf{1}&0&0&0&0\\
\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}&0&\mathbf{1}&0&0&0&0&0\\
0&0&0&0&\mathbf{1}&\mathbf{1}&\mathbf{1}&0&0&0&0\\
0&0&0&0&0&\mathbf{1}&0&\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}\\
0&0&0&0&\mathbf{1}&0&0&0&0&0&0\\
0&0&0&0&\mathbf{1}&0&0&0&0&0&0\\
0&0&0&0&\mathbf{1}&0&0&0&0&0&0\\
0&0&0&0&\mathbf{1}&0&0&0&0&0&0
\end{array}$$

If N is a negative integer then the Nth layer looks like this:
(Extend the picture to infinity.)

$$\begin{array}{ccccccccccc}
0&0&0&0&0&0&\mathbf{1}&0&0&0&0\\
0&0&0&0&0&0&\mathbf{1}&0&0&0&0\\
0&0&0&0&0&0&\mathbf{1}&0&0&0&0\\
0&0&0&0&0&0&\mathbf{1}&0&0&0&0\\
\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}\\
0&0&0&0&\mathbf{1}&0&0&0&0&0&0\\
0&0&0&0&\mathbf{1}&0&0&0&0&0&0\\
0&0&0&0&\mathbf{1}&0&0&0&0&0&0\\
0&0&0&0&\mathbf{1}&0&0&0&0&0&0
\end{array}$$

FIG. 5. A 6-connected $(6, \bar{6})$ digital surface. (The points on the digital surface are marked **1**.)

Propositions 3 and 5 (Corollary) every α-neighbor of p that lies in S is contained in $(\cup \mathbb{F}_\beta(S\langle p \rangle)\langle p \rangle) \cap S \setminus \{p\}$, so p is α-adjacent to just one 6-component of $S\langle p \rangle \setminus \{p\}$, as asserted.

(ii) This is analogous to the proof of i: the result follows from Proposition 3, Proposition 9 (Corollary 1), and the fact that if $p \in S$ and $\pi \in \mathbb{F}_6(S\langle p \rangle)\langle p \rangle$ then $\pi \cap S \setminus \{p\}$ is 18-connected and 18-adjacent to p. □

COROLLARY. *If the hypotheses of* i **or** ii *hold then p is α-adjacent to exactly one α-component of $S\langle p \rangle \setminus \{p\}$.* □

Remark. All the earlier authors defined a "simple surface point" of a set S to be a point p such that p is an $(\alpha, \bar{\beta})$ surface point of S **and** p is α-adjacent to precisely one α-component of $S\langle p \rangle \setminus \{p\}$. Proposition 16 (Corollary) shows that if one of α and β is not equal to 6 then the extra axiom is unnecessary in the sense that it is automatically satisfied by every point of an $(\alpha, \bar{\beta})$ digital surface. (If $\alpha = \beta = 6$ then the extra axiom is **not** redundant, as can be seen from any of the five central points in Fig. 5.)

PROPOSITION 17.

(i) *The only* $(18, \overline{26})$ *digital surfaces are the sets* $\mathbb{Z}^3 \cap P$ *where* P *is a coordinate plane.*

(ii) *If* S *is a* $(26, \overline{18})$ *digital surface which is* not *of the form* $\mathbb{Z}^3 \cap P$ *for some coordinate plane* P *then* $\mathbb{F}_{18}(S)$ *contains only compound plates.*

Proof. If an $(\alpha, \overline{\beta})$ digital surface S is a subset of a coordinate plane P then it is plain that $S = \mathbb{Z}^3 \cap P$. So in proving i and ii we need only consider digital surfaces that are not subsets of any coordinate plane.

(i) Suppose S is an $(18, \overline{26})$ digital surface. Then $\mathbb{F}_{26}(S)$ is a polyhedral surface without boundary. So if $\mathbb{F}_{26}(S)$ is not a subset of a coordinate plane there must be two plates π_1 and π_2 in $\mathbb{F}_{26}(S)$ such that π_1 and π_2 have an edge in common **and** π_1 and π_2 are perpendicular. WLOG the vertices of π_1 and π_2 are $\{111, 121, 122, 112\}$ and $\{111, 121, 221, 211\}$. Then by Proposition 6iii 112 and 211 are contained in a single plate in $\mathbb{F}_{26}(S)$ and so $212 \in S$. Similarly $222 \in S$. This contradicts Proposition 6i and so the result is proved.

(ii) Suppose S is a $(26, \overline{18})$ digital surface. Then $\mathbb{F}_{18}(S)$ is a polyhedral surface without boundary. Suppose $\mathbb{F}_{18}(S)$ contains a square plate: then if $\mathbb{F}_{18}(S)$ is not a subset of a coordinate plane there are two plates π_1 and π_2 in $\mathbb{F}_{18}(S)$ such that π_1 and π_2 have an edge in common, π_1 is a square plate and **either** π_2 is a square plate perpendicular to π_1 **or** π_2 is a compound plate. In the former case there are vertices x, y of π_1 and π_2, respectively, such that x and y are diametrically opposite corners of a unit cell. This contradicts Proposition 7iii. In the latter case WLOG the vertices of π_1 are $\{111, 121, 122, 112\}$ and WLOG 111 and 112 are also vertices of π_2. Now one of 012, 011, 212, and 211 is a vertex of π_2, so WLOG $012 \in S$. Then there is no plate in $\mathbb{F}_{18}(S)$ which contains both 012 and 121, and this contradiction to Proposition 7iii proves the result. □

COROLLARY. *The only* $(26, \overline{26})$ *digital surfaces are the sets* $\mathbb{Z}^3 \cap P$ *where* P *is a coordinate plane.* □

PROPOSITION 18. *Suppose* $\mathbf{Q}$ *is a closed cuboid whose corners are in* $\mathbb{Z}^3$ *and whose edges are parallel to the coordinate axes. Let* S *be a subset of* $\mathbb{Z}^3$ *and let* T *be an* α*-component of* $S \cap \operatorname{int}(\mathbf{Q})$ *such that each point in* T *is an* $(\alpha, \overline{\beta})$ *surface point of* S, *where* $\beta = 18$ *or* 26 (*the same value of* β *is used for each point in* T). *Then there are two distinct* β*-components of* $\overline{S} \cap \mathbf{Q}$ *each of which is* β*-adjacent to every point in* T.

Proof. Suppose the hypotheses are satisfied. Then T is α-connected. So by the definition of an $(\alpha, \overline{\beta})$ surface point any β-component of $\overline{S} \cap \mathbf{Q}$ that is β-adjacent to one point in T is β-adjacent to all points in T. Thus it suffices to prove that each point in T is β-adjacent to two different β-components of $\overline{S} \cap \mathbf{Q}$.

Define $\mathbb{P} = \cup\{\mathbb{F}_\beta(S\langle t\rangle) \mid t \in T\}$ and define $\Sigma = \cup \mathbb{P}$. ($\mathbb{P}$ exists by Proposition 5.) Then Σ is a polyhedral surface. By Propositions 3 and 5 (Corollary) no point in T can be on $\partial\Sigma$.

Now let π be an arbitrary plate in $\mathbb{P}$. It is readily confirmed that $\pi \cap \mathbb{Z}^3 \cap \operatorname{int}(\mathbf{Q})$ is 6-connected regardless of which type of plate π is and irrespective of the position of π in $\mathbf{Q}$, so every vertex of π is **either** in T **or** is on $\partial\mathbf{Q}$. Hence $\partial\Sigma \setminus \partial\mathbf{Q}$ contains

Layer 1	Layer 2	Layer 3	Layer 4	Layer 5
0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	1 1 1 1 0	0 0 0 0 0
0 0 0 0 0	0 0 0 0 0	1 1 1 1 0	0 0 0 0 1	0 0 0 0 0
0 0 0 0 0	0 0 0 0 0	1 1 1 1 0	0 0 0 0 1	0 0 0 0 0
0 0 0 0 0	0 0 0 0 0	1 1 1 1 0	0 0 0 0 1	0 0 0 0 0
0 0 0 0 0	0 0 0 0 0	1 1 1 1 0	0 0 0 0 1	0 0 0 0 0

0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 1 1 0	0 0 0 0 0
0 0 0 0 0	0 0 0 0 0	0 0 1 1 0	0 1 0 0 1	0 0 1 0 0
0 0 0 0 0	0 0 0 0 0	0 0 1 1 0	0 1 0 0 1	0 0 1 1 0
0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 1 1 0	0 0 0 0 0
0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0

FIG. 6. Two cross caps.

no vertices of π. But by definition of $\mathbb{F}_\beta$ $\partial\Sigma$ is a union of straight line segments each of which joins two 6-adjacent vertices of a plate in $\mathbb{P}$. Hence $\partial\Sigma \subseteq \partial\mathbf{Q}$.

Let p be any point in T. Then $p \in \Sigma \cap \text{int}(\mathbf{Q})$, so by Proposition 0 p is in the closure of two different polygonally connected components of $\mathbf{Q} \setminus \Sigma$. Call these polygonally connected components $\mathbf{C}_1$ and $\mathbf{C}_2$. We know $\mathbb{P}$ is $\bar{\beta}$-natural wrt $(S, \mathbf{N}(p))$ so, by subcondition c of naturalness, $\mathbf{C}_1 \cap \bar{S}\langle p \rangle$ and $\mathbf{C}_2 \cap \bar{S}\langle p \rangle$ are both nonempty. By Propositions 1 and 5 (Corollary) these two sets are distinct β-components of $\bar{S}\langle p \rangle$ and they are both β-adjacent to p. But each of $\mathbf{C}_1 \cap \bar{S}$ and $\mathbf{C}_2 \cap \bar{S}$ is a union of β-components of $\bar{S} \cap \mathbf{Q}$, because it is easy to see that the straight line segment joining two β-adjacent points in $\bar{S}$ cannot meet Σ. So the result is proved. □

Proposition 18 is a natural generalization of the main theorem of [7]. The proposition may fail if we allow β to be equal to 6, because there can be a "gap" between $\partial \cup (\cup\{\mathbb{F}_6(S\langle t\rangle) | t \in T\})$ and the surface of $\mathbf{Q}$. A counterexample to Proposition 18 (with $\beta = 6$) in which $\mathbf{Q}$ is a cube with sides of length four and T contains the point at the center of $\mathbf{Q}$ is called a **cross cap** (following Morgenthaler, Reed, and Rosenfeld). [7] contains one example, and Fig. 6 shows two others. (In Fig. 6 α can be 6, 18, or 26. The points marked **1** are in S.) Readers should have no difficulty at all in constructing their own cross caps using the concepts discussed in this paper.

In this paper we have said very little about the structure of $(6, \bar{6})$ digital surfaces. In fact $(6, \bar{6})$ digital surfaces are really quite unlike the other kinds of digital surface, as Proposition 15 has already shown. A further illustration of the strangeness of $(6, \bar{6})$ surface points is provided by the following example of a $(6, \bar{6})$ digital surface whose complement in $\mathbf{Z}^3$ is 6-connected—we shall call it a **global cross cap**. (Recall that by Propositions 13 and 14 (Corollary) the complement of an $(\alpha, \bar{\beta})$ digital surface cannot be β-connected if α and β are not both equal to 6.) The global cross cap referred to is a subset of $\{(x, y, z) \in \mathbf{Z}^3 || \le 6, |y| \le 6, |z| \le 3\}$. In Fig. 7 "Level n" is the set $\{(x, y, n) | x$ and y are in $\mathbf{Z}^3$, $|x| \le 6, |y| \le 6\}$. The points on the global cross cap are labeled "**1**".

6. SUMMARY AND CONCLUSIONS

If W is an arbitrary subset of $\mathbf{Z}^3$ then define the continuous analog of W to be the union of the unit cubes whose corners are all in W, the unit squares whose corners are all in W, and the line segments of unit length whose endpoints are both in W.

```
0 1 1 1 0 0 0 1 1 1 1 1 0
1 0 0 0 1 0 1 0 0 0 0 0 1
1 0 0 0 1 0 1 0 1 0 0 0 1
1 0 0 1 0 1 0 1 0 1 0 0 1
1 0 1 0 1 0 0 0 1 0 1 1 0
1 0 0 1 0 0 0 0 0 1 0 0 0
0 1 1 0 0 0 0 0 0 0 1 1 0
0 0 0 1 0 0 0 0 0 1 0 0 1
0 1 1 0 1 0 0 0 1 0 1 0 1
1 0 0 1 0 1 0 1 0 1 0 0 1
1 0 0 0 1 0 1 0 1 0 0 0 1
1 0 0 0 0 0 1 0 1 0 0 0 1
0 1 1 1 1 1 0 0 0 1 1 1 0
```

Level 0

```
0 1 1 1 1 0 1 1 1 1 1 1 0
1 0 0 0 0 1 0 0 0 0 0 0 1
1 0 0 0 0 1 0 0 1 0 0 0 1
1 0 0 1 0 0 0 1 0 1 0 0 1
1 0 1 0 1 0 0 0 1 0 0 0 1
1 0 0 1 0 0 0 0 0 0 1 1 0
1 0 0 0 0 0 0 0 0 0 0 0 1
0 1 1 0 0 0 0 0 0 1 0 0 1
1 0 0 0 1 0 0 0 1 0 1 0 1
1 0 0 1 0 1 0 0 0 1 0 0 1
1 0 0 0 1 0 0 1 0 0 0 0 1
1 0 0 0 0 0 0 1 0 0 0 0 1
0 1 1 1 1 1 1 0 1 1 1 1 0
```

Level 1

```
0 1 1 1 1 1 1 1 1 1 1 1 0
1 0 0 0 0 0 0 0 0 0 0 0 1
1 0 0 0 0 0 0 0 1 0 0 0 1
1 0 0 1 0 0 0 1 0 1 0 0 1
1 0 1 0 1 0 0 0 1 0 0 0 1
1 0 0 1 0 0 0 0 0 0 0 0 1
1 0 0 0 0 0 0 0 0 0 0 0 1
1 0 0 0 0 0 0 0 0 1 0 0 1
1 0 0 0 1 0 0 0 1 0 1 0 1
1 0 0 1 0 1 0 0 0 1 0 0 1
1 0 0 0 1 0 0 0 0 0 0 0 1
1 0 0 0 0 0 0 0 0 0 0 0 1
0 1 1 1 1 1 1 1 1 1 1 1 0
```

Level 2

```
0 0 0 0 0 0 0 0 0 0 0 0 0
0 1 1 1 1 1 1 1 1 1 1 1 0
0 1 1 1 1 1 1 1 0 1 1 1 0
0 1 1 0 1 1 1 0 0 0 1 1 0
0 1 0 0 0 1 1 1 0 1 1 1 0
0 1 1 0 1 1 1 1 1 1 1 1 0
0 1 1 1 1 1 1 1 1 1 1 1 0
0 1 1 1 1 1 1 1 1 0 1 1 0
0 1 1 1 0 1 1 1 0 0 0 1 0
0 1 1 0 0 0 1 1 1 0 1 1 0
0 1 1 1 0 1 1 1 1 1 1 1 0
0 1 1 1 1 1 1 1 1 1 1 1 0
0 0 0 0 0 0 0 0 0 0 0 0 0
```

Level 3

FIG. 7i. A $(6, \bar{6})$ global cross-cap. (Levels 0, 1, 2, 3.)

Suppose $(a, b, c) \in W \subseteq \mathbf{Z}^3$ and $\mathbf{K} = \{(x, y, z) | \max(|x - a|, |y - b|, |z - c|) < 1\}$. (So $\mathbf{K}$ is the interior of a cube with edges of length two.) In this paper we have shown that if we use 6- and 26-connectivity for W and $\overline{W}$, respectively, then (a, b, c) is a "simple surface point" of W in the sense of Morgenthaler, Reed, and Rosenfeld [5, 7, and 6] **if and only if** the part of the continuous analog of W that lies in $\mathbf{K}$ is a surface (*in the naive sense*) which separates $\mathbf{K}$ into two regions.

This result was precisely stated in Section 4 (as Proposition 6), and we have proved similar but slightly more complicated results (in Propositions 7, 11, and 12) for seven of the eight other varieties of "simple surface point" that were (in some cases implicitly) defined by the earlier authors. These results led us to a new and more "intuitive" proof of the main theorems of [5], [7], and [6]. (In fact Propositions 13 and 18 are considerably stronger than the results proved in [7] and [6].) We have also discovered (see Proposition 16) that one of the three axioms used by the earlier authors to define these surface points is redundant in that it can be derived from the other two axioms—unless we use 6-connectivity both for S and for $\bar{S}$.

Unfortunately it is not immediately clear from our "visual interpretation" of surface points how one might use them in the design of surface detection algorithms. Nevertheless, we hope useful applications of the "surface-point" concept will be discovered, and we hope our results will play a part in the discovery of such

0	1	1	1	1	0	1	1	1	1	1	1	0
1	0	0	0	0	1	0	0	0	0	0	0	1
1	0	0	0	0	1	0	0	0	0	0	0	1
1	0	0	0	0	0	1	0	1	0	0	0	1
1	0	0	1	0	1	0	1	0	0	0	0	1
1	0	0	0	1	0	0	0	1	0	1	1	0
1	0	0	1	0	0	0	0	0	1	0	0	1
0	1	1	0	1	0	0	0	1	0	0	0	1
1	0	0	0	0	1	0	1	0	1	0	0	1
1	0	0	0	1	0	1	0	0	0	0	0	1
1	0	0	0	0	0	0	1	0	0	0	0	1
1	0	0	0	0	0	0	1	0	0	0	0	1
0	1	1	1	1	1	1	0	1	1	1	1	0

Level -1

0	1	1	1	1	1	1	1	1	1	1	1	0
1	0	0	0	0	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	0	0	0	0	0	1
1	0	0	0	0	0	1	0	0	0	0	0	1
1	0	0	0	0	1	0	1	0	0	0	0	1
1	0	0	0	1	0	0	0	1	0	0	0	1
1	0	0	0	0	1	0	1	0	0	0	0	1
1	0	0	0	0	0	1	0	0	0	0	0	1
1	0	0	0	0	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	0	0	0	0	0	1
0	1	1	1	1	1	1	1	1	1	1	1	0

Level -2

0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	1	1	1	1	1	1	1	1	1	1	0
0	1	1	1	1	1	1	1	1	1	1	1	0
0	1	1	1	1	1	1	1	1	1	1	1	0
0	1	1	1	1	1	0	1	1	1	1	1	0
0	1	1	1	1	0	0	0	1	1	1	1	0
0	1	1	1	0	0	0	0	0	1	1	1	0
0	1	1	1	1	0	0	0	1	1	1	1	0
0	1	1	1	1	1	0	1	1	1	1	1	0
0	1	1	1	1	1	1	1	1	1	1	1	0
0	1	1	1	1	1	1	1	1	1	1	1	0
0	1	1	1	1	1	1	1	1	1	1	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0

Level -3

FIG. 7ii. A $(6, \bar{6})$ global cross cap. (Levels -1, -2, -3.)

applications. (The earlier authors suggested that surface points might be relevant to the theory of "thinning"—[9] is a recent paper on thinning.)

But this paper is intended to be more than just a thorough investigation of surface points; we believe that it also illustrates an interesting way of proving theorems in digital topology, namely the transformation of the problems we wish to solve into problems of continuous topology. Propositions 13 and 18 are two results of digital topology that are readily established using this approach, but which are surprisingly difficult to prove directly. In a future paper we shall use a similar method to give a proof of Theorem 5 in [8]—in fact we shall prove the more general result that if $S \subseteq \mathbf{Z}^3$, C is an α-component of S and D is a β-component of $\bar{S}$, where α and β are not both equal to 6, then the set of all points in C that are β-adjacent to D is α-connected.

APPENDIX: A PROOF OF PROPOSITION 0

We shall need the following three-dimensional analog of the Jordan Curve Theorem:

THE JORDAN–BROUWER SEPARATION THEOREM FOR A STRONGLY CONNECTED POLYHEDRAL SURFACE WITHOUT BOUNDARY. *If Σ is a strongly connected polyhedral* *surface without boundary then $\mathbb{R}^3 \setminus \Sigma$ has precisely two components, and one of the components is bounded. Σ is the boundary of each component.*

Proof. See [1]. □

PROPOSITION 0. *Let $\mathbf{X}$ be a closed cuboid in $\mathbb{R}^3$, and let Σ be a polyhedral surface contained in $\mathbf{X}$ such that $\Sigma \cap \operatorname{int}(\mathbf{X}) \neq \varnothing$ and $\partial\Sigma \subseteq \partial\mathbf{X}$. Let p be any point in $\Sigma \cap \operatorname{int}(\mathbf{X})$. Then p is in the closure of two different polygonally connected components of $\mathbf{X} \setminus \Sigma$. Furthermore, if Σ is strongly connected and $\partial\Sigma = \Sigma \cap \partial\mathbf{X}$ then $\mathbf{X} \setminus \Sigma$ has exactly two polygonally connected components, and Σ is a subset of the closure of each of these two sets.*

Proof. We shall assume throughout this proof that $\mathbf{X}$ and Σ are as defined in the first sentence of the proposition. *We shall also assume that Σ is strongly connected, for the result is certainly true if it holds for each "strongly connected component" of Σ.* All the sets we consider in this proof are "locally polygonally connected," so the terms "connected," "path connected," and "polygonally connected" are interchangeable.

Suppose $\partial\Sigma = \Sigma \cap \partial\mathbf{X}$. Then **either** $\partial\Sigma = \varnothing$ **or** $\partial\Sigma = \gamma_1 \cup \gamma_2 \cdots \cup \gamma_n$ where $n \geq 1$ and the γ_i are simple closed (polygonal) curves on $\partial\mathbf{X}$ such that $\gamma_i \cap \gamma_j$ contains only finitely many points whenever $i \neq j$. Pick an arbitrary point q in $\partial\mathbf{X} \setminus \partial\Sigma$. By the Jordan Curve Theorem $\partial\mathbf{X} \setminus \gamma_i$ has exactly two components: say that a point x is outside γ_i **iff** x is in the same component of $\partial\mathbf{X} \setminus \gamma_i$ as q. Let $\mathbf{C}_1$ be the set of points which are *outside* an odd number of the γ_i, and let $\mathbf{C}_2$ be the set of points which are *outside* an even number of the γ_i (if $\partial\Sigma = \varnothing$ then $\mathbf{C}_1 = \varnothing$ and $\mathbf{C}_2 = \partial\mathbf{X}$). The Jordan Curve Theorem states that γ_i is contained in the closure of both components of $\partial\mathbf{X} \setminus \gamma_i$. It follows that $\gamma_i \subseteq \operatorname{cl}(\mathbf{C}_1) \cap \operatorname{cl}(\mathbf{C}_2)$, and it is easy to see from this that $\mathbf{C}_1 \cup \Sigma$ is a strongly connected polyhedral surface without boundary. So by the Jordan–Brouwer Separation Theorem $\mathbb{R}^3 \setminus (\mathbf{C}_1 \cup \Sigma)$ has exactly two components, one bounded and one unbounded. Let $\mathbf{B}$ be the bounded component and let $\mathbf{U}$ be the unbounded component. It is plain that $\mathbf{B}$ is a subset of $\mathbf{X}$, and it is also clear that $\mathbf{C}_2$ is a subset of $\mathbf{U}$.

We claim that neither one of $\mathbf{B} \cup \mathbf{C}_1$ and $\mathbf{U} \cap \mathbf{X}$ can meet the closure of the other. This is plain if $\mathbf{C}_1 = \varnothing$, so suppose $\mathbf{C}_1 \neq \varnothing$ and pick x in $\mathbf{C}_1$. By the Jordan–Brouwer Separation Theorem $x \in \operatorname{cl}(\mathbf{B})$. Now pick ϵ so small that $B(x, \epsilon) \cap \Sigma = \varnothing$. Then $(B(x,\epsilon) \cap \mathbf{X}) \setminus (\mathbf{C}_1 \cup \Sigma)$ is connected and so must be entirely contained in $\mathbf{B}$. Hence $x \notin \operatorname{cl}(\mathbf{U} \cap \mathbf{X})$. Therefore $\mathbf{C}_1 \cap \operatorname{cl}(\mathbf{U} \cap \mathbf{X}) = \varnothing$. As $\mathbf{C}_1 \cup \Sigma$ is a polyhedral surface it is closed so $\operatorname{cl}(\mathbf{C}_1) \cap (\mathbf{U} \cap \mathbf{X}) = \varnothing$.

We assert that $\mathbf{C}_2$ is contained in a single component of $\mathbf{X} \setminus \Sigma$. This is certainly the case if $\Sigma \cap \partial\mathbf{X} = \varnothing$, for then $\mathbf{C}_2 = \partial\mathbf{X}$. If $\Sigma \cap \partial\mathbf{X} \neq \varnothing$ then $\mathbf{C}_2 \cup \Sigma$ is a strongly connected polyhedral surface without boundary, so by the Jordan–Brouwer Separation Theorem $\mathbb{R}^3 \setminus (\mathbf{C}_2 \cup \Sigma)$ has a bounded component $\mathbf{B}'$ and $\mathbf{C}_2$ is contained in $\operatorname{cl}(\mathbf{B}')$: since $\mathbf{B}'$ must plainly be contained in $\mathbf{X}$ our assertion is proved.

Let x and y be any two points in $\mathbf{U} \cap \mathbf{X}$, and let w be any point in $\mathbf{U} \setminus \mathbf{X}$. Then the paths in $\mathbf{U}$ from x and y to w must meet $\mathbf{C}_2$. Hence by the previous paragraph there is a path in $\mathbf{U} \cap \mathbf{X}$ from x to y. This argument shows that $\mathbf{U} \cap \mathbf{X}$ is connected, and so $\mathbf{X} \setminus \Sigma$ has precisely two components, which are $\mathbf{B} \cup \mathbf{C}_1$ and $\mathbf{U} \cap \mathbf{X}$. By the

Jordan–Brouwer Separation Theorem $\Sigma \subseteq \mathrm{cl}(\mathbf{U})$. So $\Sigma \cap \mathrm{int}(\mathbf{X}) \subseteq \mathrm{cl}(\mathbf{U} \cap \mathbf{X})$. By the Jordan Curve Theorem $\partial\Sigma \subseteq \mathrm{cl}(\mathbf{C}_2) \subseteq \mathrm{cl}(\mathbf{U} \cap \mathbf{X})$. Hence $\Sigma \subseteq \mathrm{cl}(\mathbf{U} \cap \mathbf{X})$. By the Jordan–Brouwer Separation Theorem $\Sigma \subseteq \mathrm{cl}(\mathbf{B}) \subseteq \mathrm{cl}(\mathbf{B} \cup \mathbf{C}_1)$. So we have established the proposition in the case when $\partial\Sigma = \Sigma \cap \partial\mathbf{X}$.

It remains to consider the case in which $\partial\Sigma$ is a proper subset of $\Sigma \cap \partial\mathbf{X}$. Suppose WLOG that the centroid of **X** is the origin. Define a map f on $\mathbb{R}^3$ such that $f(x, y, z) \equiv (2x, 2y, 2z)$. Let $\mathbf{Y} = f(\mathbf{X})$, and let L be the set whose members are the straight line segments joining each point on $\partial\Sigma$ to the image of that point under f. Let $\Sigma' = \Sigma \cup \bigcup L$. Then it is easy to see that Σ' is a strongly connected polyhedral surface and $\partial\Sigma' = \Sigma' \cap \partial\mathbf{Y}$. So by what we proved in the previous paragraph $\mathbf{Y} \setminus \partial\Sigma'$ has exactly two components, and if p is a point in $\Sigma \cap \mathrm{int}(\mathbf{X})$ then p is in the closure of both components, whence p is in the closure of two different components of $\mathbf{X} \setminus \Sigma$. □

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REFERENCES

1. P. S. Aleksandrov, *Combinatorial Topology*, Vol. 3, p. 94. Graylock Press, Albany, N. Y., 1960.
2. E. Artzy, G. Frieder, and G. T. Herman, The theory, design, implementation, and evaluation of a three-dimensional surface-detection algorithm, *Computer Graphics and Image Processing* **15**, 1981, 1–24.
3. R. Gordan and G. T. Herman, Three-dimensional reconstruction from projections: A review of algorithms, in *International Review of Cytology* (G. H. Bourne and J. F. Danielli, Eds.), Vol. 38, pp. 111–115. Academic Press, New York, 1974.
4. G. T. Herman and D. Webster, Surfaces of organs in discrete three-dimensional space, in *Mathematical Aspects of Computerized Tomography* (G. T. Herman and F. Natterer, Eds.). Springer-Verlag, Berlin, 1980.
5. D. G. Morgenthaler and A. Rosenfeld, Surfaces in three dimensional digital images, *Information and Control* **51**, 1981, 227–247.
6. G. M. Reed, On the characterization of simple closed surfaces in three-dimensional digital images, *Computer Graphics and Image Processing* **25**, 1984, 226–235.
7. G. M. Reed and A. Rosenfeld, Recognition of Surfaces in Three-Dimensional Digital Images, TR-1210, Computer Vision Laboratory, University of Maryland, College Park, MD 20742, August 1982, *Information and Control* **53**, 1982, 108–120.
8. A. Rosenfeld, Three-dimensional digital topology, *Information and Control* **50**, 1981, 119–127.
9. Y. F. Tsao and K. S. Fu, A parallel thinning algorithm for 3-D pictures, *Computer Graphics and Image Processing* **17**, 1981, 315–331.
10. J. K. Udupa, S. N. Srihari, and G. T. Herman, *Boundary Detection in Multidimensions*, Technical report No. MIPG-31, Department of Computer Science, State University of New York at Buffalo, 1979.