Approximately Bisimilar Symbolic Models for Randomly Switched Stochastic Systems☆

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Abstract

In the past few years there has been a growing interest in the use of symbolic models for control systems. The main reason is the possibility to leverage algorithmic techniques over symbolic models to synthesize controllers that are valid for the concrete control systems. Such controllers can enforce complex logical specifications that are otherwise hard (if not impossible) to establish on the concrete models with classical control techniques. Examples of such specifications include those expressible via linear temporal logic or as automata on infinite strings. A relevant goal in this research line is in the identification of classes of systems that admit symbolic models: in particular, continuous-time systems with stochastic or hybrid dynamics have been only recently considered, due to their rather general and complex dynamics. In this work we make progress in this direction by enlarging the class of stochastic hybrid systems admitting finite, symbolic models: specifically, we show that randomly switched stochastic systems, satisfying some incremental stability assumption, admit such models.

Keywords: Stochastic hybrid systems, Randomly switched models, Symbolic models, Finite abstractions, Formal synthesis

1. Introduction

Stochastic hybrid systems represent a general class of dynamical systems that combine continuous dynamics with discrete components and that are affected by continuous probabilistic terms as well as discrete random events. Numerous real-life systems from fields such as biochemistry [1], air traffic control [2], systems biology [3], and communication networks [4], can be modeled as stochastic hybrid systems. Randomly switched stochastic systems, also known as *switching* stochastic systems [?], are a relevant sub-class of general stochastic hybrid systems. They consist of a finite family of subsystems (modes, or locations), together with a random *switching signal* that specifies the active subsystem at every time instant. Each subsystem is further endowed with continuous probabilistic dynamics, described by a control-dependent stochastic differential equation.

Quite some research has recently focused on characterizing classes of systems, involving continuous and possibly discrete components, that admit symbolic models. A symbolic model is a finite discrete approximation of a concrete model, resulting from replacing equivalent (sets of) continuous states by discrete symbols. Symbolic models are interesting because they allow the application of algorithmic machinery for controller synthesis on discrete systems [5] towards the synthesis of hybrid controllers for the corresponding concrete complex models. Such controllers are synthesized to satisfy classes of specifications that traditionally have not been considered in the context of control theory: these include specifications involving regular languages and temporal logics [6].

The search for classes of continuous-time stochastic systems admitting symbolic models include results on stochastic dynamical systems under contractivity assumptions [7], which are valid only for autonomous models (i.e. with no control input); on probabilistic rectangular automata [8] endowed with random behaviors exclusively on their discrete components and with simple continuous dynamics; on linear stochastic control systems [9], however without any quantitative relationship between abstract and concrete models; on stochastic control systems without any stability assumptions, but with no hybrid dynamics [10]; on incrementally-stable stochastic control systems without discrete components [11] and without requiring state-space discretization [12]; and finally on incrementally-stable stochastic switched systems [13] where the discrete dynamics, in the form of mode changes, are governed by a non-probabilistic control signal. The results in [10, 11, 12, 13] are based on the notion of (alternating) approximate (bi)simulation relation, introduced in [14, 15]. Notions of bisimulation for continuous-time stochastic hybrid systems have also been studied in [16], although with a different goal than that of synthesizing symbolic models: while we are interested in the construction of bisimilar models that are finite, the work in [16] uses bisimulation to relate continuous (and thus infinite) stochastic hybrid systems. Finally, there exist discretization results based on weak approximations of continuous-time stochastic control systems [17] and of continuous-time stochastic hybrid systems [18], however these do not provide any explicit approximation bound.

To the best of our knowledge there is no work on the construction of finite bisimilar abstractions for continuous-time switching stochastic systems where the discrete dynamics, in

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the form of mode changes, are governed by a random switching signal. Models for these systems have become ubiquitous in engineering applications, such as power electronics [19], manufacturing [20], economic and finance [21]: automated controller synthesis techniques for this class of models can thus lead to more reliable system development at lower costs and times.

The main contribution of this paper is to show that switching stochastic systems, under some incremental stability assumption, admit symbolic models that are alternatingly approximately bisimilar to the concrete ones, with a precision (say ε) that can be chosen a-priori, as a design parameter. More precisely, by guaranteeing the existence of an alternating ε-approximate bisimulation relation between concrete and symbolic models, one deduces that there exists a controller enforcing a desired complex specification on the symbolic model if and only if there exists a hybrid controller enforcing an ε-specification on the original switching stochastic system. We show the description of the discussed incremental stability property in terms of a so-called common Lyapunov function (with requires no probabilistic structure on the switching signal), or alternatively in terms of multiple Lyapunov functions with some fairly general probabilistic structure on the switching signal.

Building upon [11, 13], the result of this paper extends that in [11] from a single stochastic control system to a number of randomly switching stochastic systems, and the result in [13] from multiple stochastic dynamical systems with mode changes that are governed by a non-probabilistic controlled signal to multiple stochastic control systems in which mode changes are governed by a random (uncontrolled) signal. The presence of a randomly switching signal in this paper requires to provide novel symbolic models: these allow transferring the synthesized control strategies directly to the original system, regardless of the particular evolution of the switching signal.

2. Randomly Switched Stochastic Systems

2.1. Notation

The identity map on a set A is denoted by 1_A. If A is a subset of B, we denote by $\imath_A : A \hookrightarrow B$ or simply by $\imath$ the natural inclusion map taking any $a \in A$ to $\imath(a) = a \in B$. Given a set $A \subseteq \mathbb{R}^n$, the symbol $\overline{A}$ denotes the topological closure of A. The symbols $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}_0^+$ denote the set of natural, nonnegative integer, integer, real, positive, and nonnegative real numbers, respectively. The symbols 0_n and $0_{n\times m}$ denote the zero vector and matrix in $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$, respectively. Given a vector $x \in \mathbb{R}^n$, we denote by x_i the i–th element of x, and by $\|x\|$ the infinity norm of x, namely, $\|x\| = \max\{|x_1|, |x_2|, ..., |x_n|\}$, where $|x_i|$ denotes the absolute value of x_i. Given matrices $M = \{m_{ij}\} \in \mathbb{R}^{n\times m}$ and $P = \{p_{ij}\} \in \mathbb{R}^{n\times n}$, we denote by $\|M\|$ the infinity norm of M, namely, $\|M\| = \max_{1\leq i\leq n}\sum_{j=1}^m |m_{ij}|$; by $\text{Tr}(P)$ the trace of P, namely, $\text{Tr}(P) = \sum_{i=1}^n p_{ii}$; by $\|M\|_F$ the Frobenius norm of M, namely, $\|M\|_F = \sqrt{\text{Tr}\left(MM^T\right)}$; and by $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ the minimum and maximum eigenvalues of a symmetric matrix P, respectively. We denote by Δ the *diagonal set*, namely, $\Delta = \{(x, x) \mid x \in \mathbb{R}^n\}$.

The closed ball centered at $x \in \mathbb{R}^n$ with radius λ is defined by $\mathcal{B}_\lambda(x) = \{y \in \mathbb{R}^n \mid \|x - y\| \leq \lambda\}$. A set $B \subseteq \mathbb{R}^n$ is called a *box* if $B = \prod_{i=1}^n [c_i, d_i]$, where $c_i, d_i \in \mathbb{R}$ with $c_i < d_i$ for each $i \in \{1, \ldots, n\}$. The *span* of a box B is defined as $span(B) = \min\{|d_i - c_i| \mid i = 1, \ldots, n\}$. By defining $[\mathbb{R}^n]_\eta = \{a \in \mathbb{R}^n \mid a_i = k_i\eta, k_i \in \mathbb{Z}, i = 1, \ldots, n\}$, the set $\bigcup_{p\in[\mathbb{R}^n]_\eta} \mathcal{B}_\lambda(p)$ is a countable covering of $\mathbb{R}^n$ for any $\eta \in \mathbb{R}^+$ and $\lambda \geq \eta/2$. For a box B and $\eta \leq span(B)$, define the η-approximation $[B]_\eta = [\mathbb{R}^n]_\eta \cap B$. Note that $[B]_\eta \neq \varnothing$ for any $\eta \leq span(B)$ and that for any $\eta \in \mathbb{R}^+$ with $\eta \leq span(B)$ and $\lambda \geq \eta$, we have $B \subseteq \bigcup_{p\in[B]_\eta} \mathcal{B}_\lambda(p)$. We extend the notions of span and of approximation to finite unions of boxes as follows. Let $A = \bigcup_{j=1}^M A_j$, where each A_j is a box. Define $span(A) = \min\{span(A_j) \mid j = 1, \ldots, M\}$, and for any $\eta \leq span(A)$, define $[A]_\eta = \bigcup_{j=1}^M [A_j]_\eta$.

Given a set X and a metric $\mathbf{d} : X \times X \to \mathbb{R}_0^+$, we denote by $\mathbf{d}_h$ the Hausdorff pseudometric induced by $\mathbf{d}$ on 2^X; we recall that for any $X_1, X_2 \subseteq X$, $\mathbf{d}_h(X_1, X_2) := \max\left\{\vec{\mathbf{d}}_h(X_1, X_2), \vec{\mathbf{d}}_h(X_2, X_1)\right\}$, where $\vec{\mathbf{d}}_h(X_1, X_2) = \sup_{x_1\in X_1}\inf_{x_2\in X_2}\mathbf{d}(x_1, x_2)$ is the directed Hausdorff pseudometric. Given a measurable function $f : \mathbb{R}_0^+ \to \mathbb{R}^n$, the (essential) supremum (sup norm) of f is denoted by $\|f\|_\infty$; we recall that $\|f\|_\infty = (\text{ess})\sup\{\|f(t)\|, t \geq 0\}$. A continuous function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ is said to belong to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the map $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed nonzero r, the map $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$. We identify a relation $R \subseteq A \times B$ with the map $R : A \to 2^B$ defined by $b \in R(a)$ iff $(a, b) \in R$. Given a relation $R \subseteq A \times B$, R^{-1} denotes the inverse relation defined by $R^{-1} = \{(b, a) \in B \times A : (a, b) \in R\}$.

2.2. Randomly switched (a.k.a. switching) stochastic systems

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions of completeness and right-continuity [22, p. 48]. Let $\{W_t\}_{t\geq 0}$ be a $\widehat{q}$-dimensional $\mathbb{F}$-adapted Brownian motion [23].

Definition 2.1. *A switching stochastic system is a tuple* $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, \mathsf{P}, \mathcal{P}, F, G)$*, where*

- $\mathbb{R}^n$ *is the continuous state space;*
- $\mathsf{U} \subseteq \mathbb{R}^m$ *is a compact input set;*
- $\mathcal{U}$ *is a subset of the set of all measurable functions of time, from* $\mathbb{R}_0^+$ *to* U*;*
- $\mathsf{P} = \{1, \ldots, m\}$ *is a finite set of modes;*
- $\mathcal{P}$ *is a subset of the set of all piecewise constant càdlàg (i.e. right-continuous and with left limits) functions of time from* $\mathbb{R}_0^+$ *to* P*, and characterized by a finite number of discontinuities on every bounded interval in* $\mathbb{R}_0^+$ *(no Zeno behavior);*
- $F = \{f_1, \ldots, f_m\}$ *is such that, for any* $p \in \mathsf{P}$, $f_p : \mathbb{R}^n \times \mathsf{U} \to \mathbb{R}^n$ *satisfies the following Lipschitz assumption: there exist constants* $L_x^p, L_u^p \in \mathbb{R}^+$ *such that* $\|f_p(x, u) - f_p(x', u')\| \leq L_x^p\|x - x'\| + L_u^p\|u - u'\|$*, for all* $x, x' \in \mathbb{R}^n$ *and all* $u, u' \in \mathsf{U}$*;*

- $G = \{g_1, \dots, g_m\}$ *is such that, for any* $p \in \mathsf{P}$, $g_p : \mathbb{R}^n \rightarrow \mathbb{R}^{n\times\widehat{q}}$ *satisfies the following Lipschitz assumption: there exists a constant* $Z_p \in \mathbb{R}^+$ *such that, for all* $x, x' \in \mathbb{R}^n$*:* $\|g_p(x) - g_p(x')\| \leq Z_p\|x - x'\|$.

A continuous-time stochastic process $\xi : \Omega \times \mathbb{R}_0^+ \rightarrow \mathbb{R}^n$ is said to be a *solution process* of Σ if there exist $\pi \in \mathcal{P}$ and $\upsilon \in \mathcal{U}$ satisfying

$$\mathrm{d}\,\xi = f_\pi(\xi, \upsilon)\,\mathrm{d}\,t + g_\pi(\xi)\,\mathrm{d}\,W_t, \tag{2.1}$$

$\mathbb{P}$-almost surely[1] ($\mathbb{P}$-a.s.) at each time $t \in \mathbb{R}_0^+$ where π is continuous. For any given $p \in \mathsf{P}$, we denote by Σ_p the subsystem of Σ defined by the stochastic differential equation (SDE)

$$\mathrm{d}\,\xi = f_p(\xi, \upsilon)\,\mathrm{d}\,t + g_p(\xi)\,\mathrm{d}\,W_t, \tag{2.2}$$

for any $\upsilon \in \mathcal{U}$, where f_p is known as the drift and g_p as the diffusion. A solution process of Σ_p exists and is uniquely determined owing to the assumptions on f_p and on g_p [23, Theorem 5.2.1, p. 68].

In this paper, we assume that π randomly dictates in which mode the solution process ξ is found, at any time $t \in \mathbb{R}_0^+$. Notice that whenever a mode is changed (discontinuity in π), the value of the process ξ is not reset on $\mathbb{R}^n$, hence switching stochastic systems are a strict subclass of general stochastic hybrid systems, where now the solution ξ is a continuous function of time.

We further write $\xi_{a\upsilon}^{\pi}(t)$ to denote the value of the solution process at time $t \in \mathbb{R}_0^+$ under the control input $\upsilon \in \mathcal{U}$ and the switching signal π, starting from the initial condition $\xi_{a\upsilon}^{\pi}(0) = a$ $\mathbb{P}$-a.s., in which a is a random variable that is measurable in $\mathcal{F}_0$. In general the switching stochastic system Σ may start from a random initial condition. Note that a solution process of Σ_p is also a solution process of Σ corresponding to the constant switching signal $\pi(t) = p$, for all $t \in \mathbb{R}_0^+$. We also use $\xi_{a\upsilon}^{p}(t)$ to denote the value of the solution process of Σ_p at time $t \in \mathbb{R}_0^+$ under the control input $\upsilon \in \mathcal{U}$ from the initial condition $\xi_{a\upsilon}^{p}(0) = a$ $\mathbb{P}$-a.s..

3. A Notion of Incremental Stability

The main result presented in this paper requires a stability property on Σ, inspired by the one introduced in [24], as defined next.

Definition 3.1. *A switching stochastic system* Σ *is incrementally globally asymptotically stable in the qth moment (δ-GAS-*M_q*), where* $q \geq 1$*, if there exists a* $\mathcal{KL}$ *function* β *such that for any* $t \in \mathbb{R}_0^+$*, any* $\mathbb{R}^n$*-valued random variables a and a′ that are measurable in* $\mathcal{F}_0$*, any* $\upsilon \in \mathcal{U}$*, and any* $\pi \in \mathcal{P}$*, the following condition is satisfied:*

$$\mathbb{E}\left[\left\|\xi_{a\upsilon}^{\pi}(t) - \xi_{a'\upsilon}^{\pi}(t)\right\|^q\right] \leq \beta\left(\mathbb{E}\left[\left\|a - a'\right\|^q\right], t\right). \tag{3.1}$$

Note that if $f_p(0_n, 0_m) = 0_n$ and $g_p(0_n) = 0_{n\times\widehat{q}}$ for any $p \in \mathsf{P}$, then δ-GAS-M_q implies global asymptotic stability in the qth moment (GAS-M_q) [25].

One can describe δ-GAS-M_q in terms of the existence of *incremental Lyapunov functions*, as defined next.

[1] An event E in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ happens $\mathbb{P}$-almost surely if $\mathbb{P}[E] = 1$.

Definition 3.2. *Consider a stochastic subsystem* Σ_p *and a continuous function* $V_p : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_0^+$ *that is twice continuously differentiable on* $\{\mathbb{R}^n \times \mathbb{R}^n\}\backslash\Delta$*. Function* V_p *is called a δ-GAS-*M_q *Lyapunov function for* Σ_p*, where* $q \geq 1$*, if there exist* $\mathcal{K}_\infty$ *functions* $\underline{\alpha}_p$*,* $\overline{\alpha}_p$*, and a constant* $\kappa_p \in \mathbb{R}^+$*, such that*

(i) $\underline{\alpha}_p$ *(resp.* $\overline{\alpha}_p$*) is a convex (resp. concave) function;*

(ii) for any $x, x' \in \mathbb{R}^n$*,* $\underline{\alpha}_p(\|x - x'\|^q) \leq V_p(x, x') \leq \overline{\alpha}_p(\|x - x'\|^q)$*;*

(iii) for any $x, x' \in \mathbb{R}^n$*, such that* $x \neq x'$*, and any* $u \in \mathsf{U}$*,*

$$\mathcal{L}^u V_p(x, x') := \begin{bmatrix}\partial_x V_p & \partial_{x'} V_p\end{bmatrix}\begin{bmatrix} f_p(x, u) \\ f_p(x', u)\end{bmatrix} +$$

$$\frac{1}{2}Tr\left(\begin{bmatrix} g_p(x) \\ g_p(x')\end{bmatrix}\begin{bmatrix} g_p^T(x) & g_p^T(x')\end{bmatrix}\begin{bmatrix}\partial_{x,x} V_p & \partial_{x,x'} V_p \\ \partial_{x',x} V_p & \partial_{x',x'} V_p\end{bmatrix}\right) \leq -\kappa_p V_p(x, x').$$

The operator $\mathcal{L}^u$ is the infinitesimal generator associated to the SDE (2.2) [23, Section 7.3]. The symbols ∂_x and $\partial_{x,x'}$ denote first- and second-order partial derivatives with respect to x and x', respectively.

It is known that a switching system whose subsystems are all stable, may exhibit unstable behaviors under some switching signals [26]: that is, the overall system may not be stable in general. The same may happen for a switching stochastic system [25]. As a result, the δ-GAS-M_q property of switching stochastic systems can be established either by using a common δ-GAS-M_q Lyapunov function, or alternatively via multiple δ-GAS-M_q Lyapunov functions that are mode dependent and under additional conditions on the sojourn time (also known as the staying or holding time) at a given mode.

Let us introduce the $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, and the constant κ, which are used in the rest of the paper, as follows: $\underline{\alpha} = \min\left\{\underline{\alpha}_1, \dots, \underline{\alpha}_m\right\}$, $\overline{\alpha} = \max\{\overline{\alpha}_1, \dots, \overline{\alpha}_m\}$, and $\kappa = \min\{\kappa_1, \dots, \kappa_m\}$. Note that in the case of a common Lyapunov function, we have that $\underline{\alpha} = \underline{\alpha}_1 = \cdots = \underline{\alpha}_m$ and $\overline{\alpha} = \overline{\alpha}_1 = \cdots = \overline{\alpha}_m$. The following result provides a sufficient condition for a switching stochastic system Σ to be δ-GAS-M_q based on the existence of a common δ-GAS-M_q Lyapunov function.

Theorem 3.3. *Consider a switching stochastic system* Σ*. If there exists a common δ-GAS-*M_q *Lyapunov function V for all the subsystems* $\{\Sigma_1, \dots, \Sigma_m\}$*, then* Σ *is δ-GAS-*M_q*.*

Proof. The proof is similar to the proof of Theorem 3.3 in [11] and is thus omitted. □

The existence of a common Lyapunov function in Theorem 3.3 is a very conservative assumption and it may fail to hold in general. One can alternatively describe the δ-GAS-M_q property by resorting to multiple δ-GAS-M_q Lyapunov functions under a class of switching signals that is fairly general and quite natural to examine [25].

Assumption 3.4. *Consider the stochastic process* $\widehat{\pi} : \Omega \times \mathbb{R}_0^+ \rightarrow \mathsf{P}$ *on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *such that, for every fixed* $\omega \in \Omega$*,* $\pi(\cdot) = \widehat{\pi}(\omega, \cdot) : \mathbb{R}_0^+ \rightarrow \mathsf{P}$ *belongs to* $\mathcal{P}$*, and assume that* $\widehat{\pi}$ *is completely known at time* $t = 0$*. We assume that there exists some* $\lambda \in \mathbb{R}_0^+$ *such that for any* $\widehat{\pi}$*, the probability of sojourning (staying in a mode) within an infinitesimal time interval h is lower-bounded as follows, for any* $p \in \mathsf{P}$*:*

$$\mathbb{P}\left[\widehat{\pi}(t + h) = p \mid \widehat{\pi}(t) = p\right] \geq 1 - \lambda h. \tag{3.2}$$

Remark 3.5. *If the switching process $\widehat{\pi}$ is the state of a continuous-time Markov chain with a given generator matrix $Q = \{q_{ij}\} \in \mathbb{R}^{\mathsf{P}\times\mathsf{P}}$, one can obtain the lower bound on the probability in* (3.2) *with $\lambda = \max_{i\in\mathsf{P}} \sum_{i\neq j} q_{ij}$.*

For a stochastic switching process $\widehat{\pi}$, we denote the number of switches (the discontinuity points of $\widehat{\pi}$) on the interval $]0,t]$ by $N_{\widehat{\pi}}(t)$, which is measurable in $\mathcal{F}_t$. We assume $N_{\widehat{\pi}}(0) = 0$. Due to Assumption 3.4 on $\widehat{\pi}$, the probability distribution of $N_{\widehat{\pi}}(t)$ satisfies [25]:

$$\mathbb{P}\left[N_{\widehat{\pi}}(t) = k\right] \leq \frac{\mathsf{e}^{-\lambda t}(\lambda t)^k}{k!}. \tag{3.3}$$

From (3.3), one can readily verify that the probability mass function of $N_{\widehat{\pi}}(t)$ corresponds to that of a Poisson process and that $N_{\widehat{\pi}}(t)$ takes with probability one finite values for any bounded time t. We assume that $\{W_t\}_{t\geq 0}$, $\{N_{\widehat{\pi}}\}_{t\geq 0}$, and the initial condition of Σ, which is measurable in $\mathcal{F}_0$, are mutually independent. The next result provides sufficient conditions for a switching stochastic system Σ to be δ-GAS-M$_q$ based on the existence of multiple δ-GAS-M$_q$ Lyapunov functions and on Assumption 3.4.

Theorem 3.6. *Consider a switching stochastic system Σ. Suppose that Assumption 3.4 holds and that for any $p \in \mathsf{P}$, there exists a δ-GAS-M$_q$ Lyapunov function V_p for Σ_p, and in addition that there exits a constant $\mu \geq 1$ such that*

(i) for any $x, x' \in \mathbb{R}^n$, and any $p, p' \in \mathsf{P}, V_p(x,x') \leq \mu V_{p'}(x,x')$;

(ii) $(\mu - 1)\lambda - \kappa < 0$.

Then Σ is δ-GAS-M$_q$.

The proof of Theorem 3.6 is provided in the Appendix.

For stochastic subsystems Σ_p, with f_p and g_p in the form of polynomials for any $p \in \mathsf{P}$, one can resort to available software tools, such as SOSTOOLS [27, Subsection 4.2], to search for appropriate δ-GAS-M$_q$ functions V_p. Although the satisfaction of conditions (i) and (ii) of Definition 3.2 globally on $\mathbb{R}^n$ may require $\underline{\alpha}_p$ and $\overline{\alpha}_p$ to be piecewise polynomial functions, as a concave function is supposed to dominate a convex one, those conditions can be still satisfied by $\underline{\alpha}_p$ and $\overline{\alpha}_p$ of the form of polynomials as long as one is interested in dynamics of Σ_p on a compact subset of $\mathbb{R}^n$, which is always the case in practice. We refer the interested reader to the results in [11], providing special instances where these functions can be easily computed. As an example, for linear stochastic subsystems (i.e. for subsystems with linear drift and diffusion terms), one can search for appropriate δ-GAS-M$_q$ Lyapunov functions by easily solving a linear matrix inequality (LMI).

In order to show the main result of the paper, we need the following technical lemma, which provides an upper bound on the distance (in the qth moment metric) between the solution processes of subsystems Σ_p and the corresponding non-probabilistic subsystems $\overline{\Sigma}_p$ obtained by disregarding the diffusion term (g_p). From now on, we use the notation $\zeta^p_{x\upsilon}$ to denote the solution of the ordinary differential equation (ODE) $\dot{\zeta}^p_{x\upsilon} = f_p\left(\zeta^p_{x\upsilon}, \upsilon\right)$ starting from the initial condition x and under the input curve υ.

Lemma 3.7. *Consider a stochastic subsystem Σ_p such that $g_p(0_n) = 0_{n\times\widehat{q}}$. Suppose there exists a δ-GAS-M$_q$ Lyapunov function V_p for Σ_p such that its Hessian is a positive semidefinite matrix in $\mathbb{R}^{2n\times 2n}$. Considering the dynamics of Σ_p exclusively on a compact set $\mathsf{D} \subset \mathbb{R}^n$ and given any $\upsilon \in \mathcal{U}$, we have*

$$\mathbb{E}\left[\left\|\xi^p_{x\upsilon}(t) - \zeta^p_{x\upsilon}(t)\right\|^q\right] \leq h_p(g_p, t), \tag{3.4}$$

where $h_p(g_p, t) =$

$$\underline{\alpha}_p^{-1}\left(\frac{1}{2}\sup_{x,x'\in\mathsf{D}}\left\{\left\|\sqrt{\partial_{x,x}V_p(x,x')}\right\|^2\right\} n\min\{n,\widehat{q}\} Z_p^2 \mathsf{e}^{-\kappa_p t}\cdot\sup_{x\in\mathsf{D}}\|x\|^2\, t\right),$$

and Z_p is the Lipschitz constant introduced in Definition 2.1.

One can readily verify that the nonnegative function h_p tends to zero as $t \to 0$, $t \to +\infty$, or as $Z_p \to 0$.

Proof. The proof is similar to the proof of Lemma 3.7 in [11] and is thus omitted. □

The interested readers are referred to [11] providing results in line with that of Lemma 3.7 for (linear) stochastic subsystems Σ_p admitting a specific type of δ-GAS-M$_q$ Lyapunov functions.

For later use, we introduce function $h(G,t) = \max\{h_1(g_1,t),\ldots,h_m(g_m,t)\}$ for all $t \in \mathbb{R}^+_0$.

4. Systems and Approximate Equivalence Relations

We employ the notion of *systems*, introduced in [28], to provide (in Sec. 5) an alternative description of switching stochastic models that can be directly related to their corresponding symbolic models.

Definition 4.1. *A system S is a tuple $S = (X, X_0, U, \longrightarrow, Y, H)$, where*

- *X is a set of states (possibly infinite);*
- *$X_0 \subseteq X$ is a set of initial states (possibly infinite);*
- *$U = A \times B$ is a set of inputs, where*
  - *A is the set of control inputs (possibly infinite);*
  - *B is the set of adversarial inputs (possibly infinite);*
- *$\longrightarrow \subseteq X \times U \times X$ is a transition relation;*
- *Y is a set of outputs;*
- *$H : X \to Y$ is an output map.*

We write $x \xrightarrow{a,b} x'$ if $(x,(a,b),x') \in \longrightarrow$. If $x \xrightarrow{a,b} x'$, we call state x' a successor of state x. From now on, we assume that for any $x \in X$, there is some successor of x for some $(a,b) \in U$ – let us remark that this is always the case for the systems considered later in this paper. A system S is said to be

- *metric*, if the output set Y is equipped with a metric $\mathbf{d} : Y \times Y \to \mathbb{R}^+_0$;
- *countable*, if X and U are countable sets;
- *finite* (or *symbolic*), if X and U are finite sets.

For a system $S = (X, X_0, U, \longrightarrow, Y, H)$ and given any initial state $x_0 \in X_0$, a finite state run started from x_0 is a finite sequence of transitions:

$$x_0 \xrightarrow{a_0,b_0} x_1 \xrightarrow{a_1,b_1} \cdots x_{n-1} \xrightarrow{a_{n-1},b_{n-1}} x_n, \tag{4.1}$$

such that $x_i \xrightarrow{a_i,b_i} x_{i+1}$ for all $i \in \{0, \ldots, n-1\}$. A finite state run can be trivially extended to an infinite state run [28]. A finite output run is a sequence $\{y_0, y_1, \ldots, y_n\}$ such that there exists a finite state run of the form (4.1) with $y_i = H(x_i)$, for $i = 0, \ldots, n$. A finite output run can also be directly extended to an infinite output run [28].

We recall the notion of alternating approximate (bi)simulation relation, introduced in [15], which is useful to relate properties of switching stochastic systems to those of their symbolic models. Such a relation captures the different role of control and adversarial inputs in the system, by treating the former as cooperative and the latter as non-cooperative. We refer the interested reader to [15, Example 3.4], discussing the usefulness of the notion of alternating approximate (bi)simulation relation over that of approximate (bi)simulation relation [14], which instead treats adversarial inputs as cooperative (rather than non-cooperative).

Definition 4.2. *Let $S_1 = (X_1, X_{10}, A_1 \times B_1, \underset{1}{\longrightarrow}, Y_1, H_1)$ and $S_2 = (X_2, X_{20}, A_2 \times B_2, \underset{2}{\longrightarrow}, Y_2, H_2)$ be metric systems with the same output sets $Y_1 = Y_2$ and metric* $\mathbf{d}$. *For $\varepsilon \in \mathbb{R}_0^+$, a relation $R \subseteq X_1 \times X_2$ is said to be an alternating ε-approximate simulation relation from S_1 to S_2 if the following three conditions are satisfied:*

(i) for every $x_{10} \in X_{10}$, there exists $x_{20} \in X_{20}$: $(x_{10}, x_{20}) \in R$;

(ii) for every $(x_1, x_2) \in R$, $\mathbf{d}(H_1(x_1), H_2(x_2)) \leq \varepsilon$;

(iii) for every $(x_1, x_2) \in R$, $\forall a_1 \in A_1$ $\exists a_2 \in A_2$ $\forall b_2 \in B_2$ $\exists b_1 \in B_1$ such that $x_1 \xrightarrow[1]{a_1,b_1} x_1'$ and $x_2 \xrightarrow[2]{a_2,b_2} x_2'$ with $(x_1', x_2') \in R$.

A relation $R \subseteq X_1 \times X_2$ is said to be an alternating ε-approximate bisimulation relation between S_1 and S_2 if R is an alternating ε-approximate simulation relation from S_1 to S_2 and R^{-1} is an alternating ε-approximate simulation relation from S_2 to S_1.

System S_1 is alternatingly ε-approximately simulated by S_2, or S_2 alternatingly ε-approximately simulates S_1, denoted by $S_1 \preceq^{\varepsilon}_{\mathcal{AS}} S_2$, if there exists an alternating ε-approximate simulation relation from S_1 to S_2. System S_1 is alternatingly ε-approximatly bisimilar to S_2, denoted by $S_1 \cong^{\varepsilon}_{\mathcal{AS}} S_2$, if there exists an alternating ε-approximate bisimulation relation between S_1 and S_2.

5. Symbolic Models for Switching Stochastic Systems

This section contains the main contribution of the article. We show that for any δ-GAS-M$_q$ switching stochastic system Σ and for any precision level $\varepsilon \in \mathbb{R}^+$, there exists a finite abstraction that is alternatingly ε-approximately bisimilar to Σ as long as we are interested in its dynamics within a bounded set. In order to do so, we use systems as abstract representations of switching stochastic systems. More precisely, given a switching stochastic system Σ, we define an associated metric system $S(\Sigma) = (X, X_0, U, \longrightarrow, Y, H)$, where:

- X is the set of all $\mathbb{R}^n$-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
- X_0 is the set of all $\mathbb{R}^n$-valued random variables that are measurable over the trivial sigma-algebra $\mathcal{F}_0$, i.e. the system starts from a non-probabilistic initial condition;
- $U = A \times B$, where $A = \mathcal{U}$ and $B = \mathcal{P}$;
- $x \xrightarrow{\upsilon,\pi} x'$ if x and x' are measurable in $\mathcal{F}_t$ and $\mathcal{F}_{t+\tau}$, respectively, for some $t \in \mathbb{R}_0^+$ and $\tau \in \mathbb{R}^+$, and there exists a solution process $\xi : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}^n$ of Σ satisfying $\xi(t) = x$ and $\xi^{\pi}_{x\upsilon}(\tau) = x'$ $\mathbb{P}$-a.s.;
- Y is the set of all $\mathbb{R}^n$-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
- $H = 1_X$.

We assume that the output set Y is equipped with the metric $\mathbf{d}(y, y') = (\mathbb{E}[\|y - y'\|^q])^{\frac{1}{q}}$, for any $y, y' \in Y$ and some $q \geq 1$. Let us remark that the set of states and inputs of $S(\Sigma)$ are uncountable, hence $S(\Sigma)$ is an uncountable system. Note that $S(\Sigma)$ captures all the information contained in Σ. Notice that A and B are the sets of cooperative and non-cooperative input signals, respectively.

In subsequent developments, we will work with a sub-system of $S(\Sigma)$ obtained by selecting those transitions of $S(\Sigma)$ describing trajectories of duration τ, where τ is a given fixed sampling time. This can be seen as a time discretization or a sampled-data version of $S(\Sigma)$. This restriction is practically motivated by the fact that the original model Σ has to be controlled by a digital platform with a given clock period τ. More precisely, given a switching stochastic system Σ and a sampling time $\tau \in \mathbb{R}^+$, we define the associated system $S_\tau(\Sigma) = (X_\tau, X_{\tau 0}, U_\tau, \underset{\tau}{\longrightarrow}, Y_\tau, H_\tau)$, where $X_\tau = X$, $X_{\tau 0} = X_0$, $Y_\tau = Y$, $H_\tau = H$, and

- $U_\tau = A_\tau \times B_\tau$, where
  - $A_\tau = \{\upsilon \in \mathcal{U} \mid$ the domain of υ is $[0, \tau[\}$;
  - $B_\tau = \{\pi \in \mathcal{P} \mid$ the domain of π is $[0, \tau[\}$;
- $x_\tau \xrightarrow[\tau]{\upsilon_\tau,\pi_\tau} x_\tau'$ if x_τ and x_τ' are measurable, respectively, in $\mathcal{F}_{k\tau}$ and $\mathcal{F}_{(k+1)\tau}$ for some $k \in \mathbb{N}_0$, and there exists a solution process $\xi : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}^n$ of Σ satisfying $\xi(k\tau) = x_\tau$ and $\xi^{\pi_\tau}_{x_\tau \upsilon_\tau}(\tau) = x_\tau'$ $\mathbb{P}$-a.s..

Note that a finite state run $x_0 \xrightarrow[\tau]{\upsilon_0,\pi_0} x_1 \xrightarrow[\tau]{\upsilon_1,\pi_1} \cdots \xrightarrow[\tau]{\upsilon_{N-1},\pi_{N-1}} x_N$ of $S_\tau(\Sigma)$, where $\upsilon_{i-1} \in A_\tau$, $\pi_{i-1} \in B_\tau$, and $x_i = \xi^{\pi_{i-1}}_{x_{i-1}\upsilon_{i-1}}(\tau)$ $\mathbb{P}$-a.s. for $i = 1, \ldots, N$, captures the trajectory of the switching stochastic system Σ at times $t = 0, \tau, \ldots, N\tau$. This trajectory starts from the non-probabilistic initial condition x_0 and results from the control input υ and the adversarial input (or switching signal) π, obtained by the concatenation of the control and adversarial inputs υ_{i-1} and π_{i-1}, respectively, (that is,

$\upsilon((i-1)\tau+s) = \upsilon_{i-1}(s)$ and $\pi((i-1)\tau+s) = \pi_{i-1}(s)$ for any $s \in [0,\tau[$, for $i = 1,\ldots,N$.

Given a switching stochastic system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, \mathsf{P}, \mathcal{P}, F, G)$, we define for subsequent analysis the corresponding switching system $\overline{\Sigma} = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, \mathsf{P}, \mathcal{P}, F)$ obtained by discarding the diffusion terms G, defined by the ODE: $\dot{\zeta} = f_\pi(\zeta,\upsilon)$, for any $\upsilon \in \mathcal{U}$ and any $\pi \in \mathcal{P}$. Note that due to the assumptions on f_p, for any $p \in \mathsf{P}$, each subsystem $\overline{\Sigma}_p$ of $\overline{\Sigma}$ is forward complete [29], i.e. every trajectory is defined on the interval $[0,\infty[$. Moreover, due to the assumptions on the switching signals $\pi \in \mathcal{P}$, one can conclude that the overall non-probabilistic switching system $\overline{\Sigma}$ is forward complete[2] [30]. For $\overline{\Sigma}$, we write $\zeta^\pi_{x\upsilon}(t)$ to denote the point reached at time $t \in \mathbb{R}^+_0$ under the control input $\upsilon \in \mathcal{U}$ and the switching signal π from the initial condition $\zeta^\pi_{x\upsilon}(0) = x$.

In order to construct a symbolic model for any δ-GAS-M$_q$ switching stochastic system Σ, we will extract a finite set of states X_{q} and inputs U_{q} from X_τ and U_τ, respectively, in such a way that the resulting symbolic model is finite if we are interested in the dynamics of Σ in a bounded set. Note that the approximation of the set of inputs U_τ of $S_\tau(\Sigma)$ requires the notion of reachable set, as defined next. Given a switching non-probabilistic system $\overline{\Sigma}$, any $\tau \in \mathbb{R}^+$, and $x \in \mathbb{R}^n$, the reachable set of $\overline{\Sigma}$ with initial condition $x \in \mathbb{R}^n$ after τ seconds is the set $\mathcal{R}(\tau, x)$ of endpoints $\zeta^\pi_{x\upsilon}(\tau)$ for any $\upsilon \in A_\tau$ and $\pi \in B_\tau$ or, equivalently,

$$\mathcal{R}(\tau, x) := \{y \in \mathbb{R}^n \mid y = \zeta^\pi_{x\upsilon}(\tau), \upsilon \in A_\tau, \pi \in B_\tau\}. \tag{5.1}$$

Moreover, the reachable set of $\overline{\Sigma}$ with initial condition $x \in \mathbb{R}^n$ and control input $\upsilon \in A_\tau$ after τ seconds is the set $\mathcal{R}(\tau, x, \upsilon)$ of endpoints $\zeta^\pi_{x\upsilon}(\tau)$ for any $\pi \in B_\tau$, i.e.,

$$\mathcal{R}(\tau, x, \upsilon) := \{y \in \mathbb{R}^n \mid y = \zeta^\pi_{x\upsilon}(\tau), \pi \in B_\tau\}. \tag{5.2}$$

The reachable sets in (5.1) and (5.2) are well defined because $\overline{\Sigma}$ is forward complete. Given any desired precision $\mu \in \mathbb{R}^+$ and any $\tau \in \mathbb{R}^+$, define the following sets:

$$\mathsf{A}_\mu(\tau, x_{\mathsf{q}}) := \left\{P \in 2^{[\mathbb{R}^n]_\mu} \mid \exists \upsilon \in A_\tau \text{ s.t. } \mathbf{d}_h\left(P, \mathcal{R}(\tau, x_{\mathsf{q}}, \upsilon)\right) \leq \mu\right\}, \tag{5.3}$$

$$\mathsf{B}_\mu(\tau, x_{\mathsf{q}}, \upsilon) := \left\{x'_{\mathsf{q}} \in [\mathbb{R}^n]_\mu \mid \exists \pi \in B_\tau \text{ s.t. } \left\|x'_{\mathsf{q}} - \zeta^\pi_{x_{\mathsf{q}}\upsilon}(\tau)\right\| \leq \mu\right\}, \tag{5.4}$$

where $\mathbf{d}_h$ is the Hausdorff pseudometric induced by the infinity norm on $\mathbb{R}^n$. Note that for any $P \in \mathsf{A}_\mu(\tau, x_{\mathsf{q}})$ and any $x'_{\mathsf{q}} \in \mathsf{B}_\mu(\tau, x_{\mathsf{q}}, \upsilon)$, there may exist a (possibly uncountable) set of control inputs $\upsilon \in A_\tau$ and a (possibly uncountable) set of switching signals $\pi \in B_\tau$ such that $\mathbf{d}_h\left(P, \mathcal{R}(\tau, x_{\mathsf{q}}, \upsilon)\right) \leq \mu$ and $\left\|x'_{\mathsf{q}} - \zeta^\pi_{x_{\mathsf{q}}\upsilon}(\tau)\right\| \leq \mu$, respectively. One can construct countable (possibly finite) sets of control inputs and switching signals by collecting representative signals, as explained in the following. Let us define the functions

$$\psi^{\tau,x_{\mathsf{q}}}_\mu : \mathsf{A}_\mu(\tau, x_{\mathsf{q}}) \to A_\tau, \quad \varphi^{\tau,x_{\mathsf{q}},\upsilon}_\mu : \mathsf{B}_\mu(\tau, x_{\mathsf{q}}, \upsilon) \to B_\tau, \tag{5.5}$$

where

[2] Note that if one allows for Zeno behavior in $\overline{\Sigma}$, it may cause a finite escape time even if all the subsystems are forward complete.

- $\psi^{\tau,x_{\mathsf{q}}}_\mu$ associates to any $P \in \mathsf{A}_\mu(\tau, x_{\mathsf{q}})$ one control input $\upsilon = \psi^{\tau,x_{\mathsf{q}}}_\mu(P) \in A_\tau$ so that $\mathbf{d}_h\left(P, \mathcal{R}(\tau, x_{\mathsf{q}}, \upsilon)\right) \leq \mu$;
- $\varphi^{\tau,x_{\mathsf{q}},\upsilon}_\mu$ associates to any $x'_{\mathsf{q}} \in \mathsf{B}_\mu(\tau, x_{\mathsf{q}}, \upsilon)$ one switching signal $\pi = \varphi^{\tau,x_{\mathsf{q}},\upsilon}_\mu(x'_{\mathsf{q}}) \in B_\tau$ so that $\left\|x'_{\mathsf{q}} - \zeta^\pi_{x_{\mathsf{q}}\upsilon}(\tau)\right\| \leq \mu$.

Note that functions $\psi^{\tau,x_{\mathsf{q}}}_\mu$ and $\varphi^{\tau,x_{\mathsf{q}},\upsilon}_\mu$ are not uniquely defined. Let us now introduce sets $A^\mu(x_{\mathsf{q}})$ and $B^\mu(x_{\mathsf{q}}, \upsilon)$ as follows:

$$A^\mu(x_{\mathsf{q}}) := \psi^{\tau,x_{\mathsf{q}}}_\mu\left(\mathsf{A}_\mu(\tau, x_{\mathsf{q}})\right), \tag{5.6}$$

$$B^\mu(x_{\mathsf{q}}, \upsilon) := \varphi^{\tau,x_{\mathsf{q}},\upsilon}_\mu\left(\mathsf{B}_\mu(\tau, x_{\mathsf{q}}, \upsilon)\right). \tag{5.7}$$

We remark again that, since $\overline{\Sigma}$ is forward complete, the sets $\mathsf{A}_\mu(\tau, x_{\mathsf{q}})$ and $\mathsf{B}_\mu(\tau, x_{\mathsf{q}}, \upsilon)$ in (5.3) and (5.4) are not empty, hence $A^\mu(x_{\mathsf{q}})$ and $B^\mu(x_{\mathsf{q}}, \upsilon)$ in (5.6) and (5.7) are not empty.

We now have all the ingredients to introduce a symbolic model for $S_\tau(\Sigma)$. Consider a switching stochastic system Σ, and a triple $\mathsf{q} = (\tau, \eta, \mu)$ of quantization parameters, where τ is the sampling time, η is the state space quantization, and μ is an additional design parameter. Given Σ and q, consider the following system: $S_{\mathsf{q}}(\Sigma) = (X_{\mathsf{q}}, X_{\mathsf{q}0}, U_{\mathsf{q}}, \underset{\mathsf{q}}{\longrightarrow}, Y_{\mathsf{q}}, H_{\mathsf{q}})$, where $X_{\mathsf{q}} = [\mathbb{R}^n]_\eta$, $X_{\mathsf{q}0} = [\mathbb{R}^n]_\eta$, and

- $U_{\mathsf{q}} = A_{\mathsf{q}} \times B_{\mathsf{q}}$, where
$$A_{\mathsf{q}} = \cup_{x_{\mathsf{q}} \in X_{\mathsf{q}}} A^\mu(x_{\mathsf{q}}),\ B_{\mathsf{q}} = \cup_{x_{\mathsf{q}} \in X_{\mathsf{q}}} \cup_{\upsilon \in A^\mu(x_{\mathsf{q}})} B^\mu(x_{\mathsf{q}}, \upsilon),$$
and the sets $A^\mu(x_{\mathsf{q}})$ and $B^\mu(x_{\mathsf{q}}, \upsilon)$ are defined in (5.6) and (5.7), respectively;
- $x_{\mathsf{q}} \xrightarrow[\mathsf{q}]{\upsilon_{\mathsf{q}},\pi_{\mathsf{q}}} x'_{\mathsf{q}}$ if $\upsilon_{\mathsf{q}} \in A^\mu(x_{\mathsf{q}})$, $\pi_{\mathsf{q}} \in B^\mu(x_{\mathsf{q}}, \upsilon_{\mathsf{q}})$, and there exists $x'_{\mathsf{q}} \in X_{\mathsf{q}}$ such that $\left\|\zeta^{\pi_{\mathsf{q}}}_{x_{\mathsf{q}}\upsilon_{\mathsf{q}}}(\tau) - x'_{\mathsf{q}}\right\| \leq \eta$;
- $Y_{\mathsf{q}} = Y_\tau$ (i.e. the set of all $\mathbb{R}^n$-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$);
- $H_{\mathsf{q}} = \imath : X_{\mathsf{q}} \hookrightarrow Y_{\mathsf{q}}$.

Note that in the definition of H_{q}, the inclusion map $\imath$ is meant, with slight abuse of notation, as a mapping from a grid point to a random variable with a Dirac probability distribution centered at that grid point.

The transition relation of $S_{\mathsf{q}}(\Sigma)$ is well defined in the sense that for every $x_{\mathsf{q}} \in [\mathbb{R}^n]_\eta$, every $\upsilon_{\mathsf{q}} \in A^\mu(x_{\mathsf{q}})$, and every $\pi_{\mathsf{q}} \in B^\mu(x_{\mathsf{q}}, \upsilon_{\mathsf{q}})$, there always exists $x'_{\mathsf{q}} \in [\mathbb{R}^n]_\eta$ such that $x_{\mathsf{q}} \xrightarrow[\mathsf{q}]{\upsilon_{\mathsf{q}},\pi_{\mathsf{q}}} x'_{\mathsf{q}}$. This can be seen since by definition of $[\mathbb{R}^n]_\eta$, for any $\widehat{x} \in \mathbb{R}^n$ there always exists a state $\widehat{x}' \in [\mathbb{R}^n]_\eta$ such that $\|\widehat{x} - \widehat{x}'\| \leq \eta$. Hence, for $\zeta^{\pi_{\mathsf{q}}}_{x_{\mathsf{q}}\upsilon_{\mathsf{q}}}(\tau)$ there always exists a state $x'_{\mathsf{q}} \in [\mathbb{R}^n]_\eta$ satisfying $\left\|\zeta^{\pi_{\mathsf{q}}}_{x_{\mathsf{q}}\upsilon_{\mathsf{q}}}(\tau) - x'_{\mathsf{q}}\right\| \leq \eta$.

Before showing the main result of the paper, we need the following technical result.

Proposition 5.1. *Consider a switching non-probabilistic system $\overline{\Sigma} = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, \mathsf{P}, \mathcal{P}, F)$. For any $x \in \mathbb{R}^n$, the reachable set $\mathcal{R}(\tau, x)$, defined in* (5.1)*, is bounded.*

Proof. One can characterize a switching non-probabilistic system $\overline{\Sigma} = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, \mathsf{P}, \mathcal{P}, F)$ with a non-probabilistic control system $\widetilde{\Sigma} = (\mathbb{R}^n, \mathsf{U} \times \mathsf{P}, \mathcal{U} \times \mathcal{P}, f)$, where

- $\mathbb{R}^n$ is the state space;
- $\mathsf{U} \times \mathsf{P}$ is the input set;
- $\mathcal{U} \times \mathcal{P}$ is the set of input curves;
- $f : \mathbb{R}^n \times \mathsf{U} \times \mathsf{P} \to \mathbb{R}^n$ is a continuous map, defined as $f(x, u, p) := \sum_{i=1}^m f_i(x, u)\delta_{ip}$, where $\delta_{ip} := \begin{cases} 1 & i = p; \\ 0 & i \neq p. \end{cases}$

Note that the function f satisfies the Lipschitz assumption: $\|f(x, u, p) - f(x', u, p)\| \leq L_x\|x - x'\|$, for all $x, x' \in \mathbb{R}^n$, $u \in \mathsf{U}$, and $p \in \mathsf{P}$, where $L_x = \max\left\{L_x^1, \ldots, L_x^m\right\}$. It can be readily verified that any trajectory ζ_{xv}^{π} of $\overline{\Sigma}$ is also a trajectory of $\widetilde{\Sigma}$, satisfying $\dot{\zeta}_{xv}^{\pi} = f(\zeta_{xv}^{\pi}, \upsilon, \pi)$, and vice versa. Since $\overline{\Sigma}$ is forward complete, $\widetilde{\Sigma}$ is a forward complete control system. The rest of the proof follows from the proof of Proposition 5.1 in [31]. □

Note that X_{q} is a countable set. Since $\mathcal{R}(\tau, x)$, defined in (5.1), is bounded (cf. Proposition 5.1) and using Proposition 4.4 in [15], one can readily verify that U_{q} is also a countable set. Therefore, $S_{\mathsf{q}}(\Sigma)$ is countable. Moreover, if we are interested in the dynamics of Σ in a bounded set, which is often the case in many practical situations, $S_{\mathsf{q}}(\Sigma)$ is finite.

We can now present the main result of the paper, which shows that any δ-GAS-M$_q$ switching stochastic system Σ admits an alternatingly approximately bisimilar symbolic model.

Theorem 5.2. *Consider a δ-GAS-M$_q$ switching stochastic system Σ, satisfying the result of Lemma 3.7. For any $\varepsilon \in \mathbb{R}^+$, and any triple $\mathsf{q} = (\tau, \eta, \mu)$ of quantization parameters satisfying*

$$(\beta(\varepsilon^q, \tau))^{\frac{1}{q}} + (h(G, \tau))^{\frac{1}{q}} + 2\mu + \eta < \varepsilon, \tag{5.8}$$

we have $S_{\mathsf{q}}(\Sigma) \cong_{\mathcal{AS}}^{\varepsilon} S_{\tau}(\Sigma)$.

It can be readily seen that when we are interested in the dynamics of Σ in a compact $\mathsf{D} \subset \mathbb{R}^n$ of the form of finite union of boxes and for a given precision ε, there always exists a sufficiently large value of τ and small values of η and μ such that $\eta \leq span(\mathsf{D})$ and the condition in (5.8) are satisfied.

Proof. The proof is inspired by that in [15, Theorem 4.6]. We start by proving $S_{\tau}(\Sigma) \preceq_{\mathcal{AS}}^{\varepsilon} S_{\mathsf{q}}(\Sigma)$. Consider the relation $R \subseteq X_{\tau} \times X_{\mathsf{q}}$ defined by $\left(x_{\tau}, x_{\mathsf{q}}\right) \in R$ if and only if

$$\left(\mathbb{E}\left[\left\|H_{\tau}(x_{\tau}) - H_{\mathsf{q}}(x_{\mathsf{q}})\right\|^q\right]\right)^{\frac{1}{q}} = \left(\mathbb{E}\left[\left\|x_{\tau} - x_{\mathsf{q}}\right\|^q\right]\right)^{\frac{1}{q}} \leq \varepsilon.$$

Since $X_{\tau 0} \subseteq \bigcup_{p \in [\mathbb{R}^n]_{\eta}} \mathcal{B}_{\eta}(p)$, for every $x_{\tau 0} \in X_{\tau 0}$ there always exists $x_{\mathsf{q}0} \in X_{\mathsf{q}0}$ such that $\left\|x_{\tau 0} - x_{\mathsf{q}0}\right\| \leq \eta$. Then,

$$\left(\mathbb{E}\left[\left\|x_{\tau 0} - x_{\mathsf{q}0}\right\|^q\right]\right)^{\frac{1}{q}} = \left(\left\|x_{\tau 0} - x_{\mathsf{q}0}\right\|^q\right)^{\frac{1}{q}} \leq \eta \leq \varepsilon,$$

because of (5.8). Hence, $\left(x_{\tau 0}, x_{\mathsf{q}0}\right) \in R$ and condition (i) in Definition 4.2 is satisfied. Now consider any $\left(x_{\tau}, x_{\mathsf{q}}\right) \in R$. Condition (ii) in Definition 4.2 is satisfied by the definition of R. Let us now show that condition (iii) in Definition 4.2 holds. Since $\overline{\Sigma}$ is forward complete, the reachable sets defined in (5.1) and (5.2) are well defined, for any $\tau \in \mathbb{R}^+$, $x \in \mathbb{R}^n$, and $\upsilon \in A_{\tau}$. Consider any $\upsilon_{\tau} \in A_{\tau}$. Given $\mu \in \mathbb{R}^+$, by Lemma 4.2 in [15], there exists $P \subseteq [\mathbb{R}^n]_{\mu}$ such that

$$\mathbf{d}_h\left(P, \mathcal{R}(\tau, x_{\mathsf{q}}, \upsilon_{\tau})\right) \leq \mu. \tag{5.9}$$

By inequality (5.9), one concludes $P \in \mathsf{A}_{\mu}\left(\tau, x_{\mathsf{q}}\right)$ and then let υ_{q} be given by $\upsilon_{\mathsf{q}} = \psi_{\mu}^{\tau, x_{\mathsf{q}}}(P) \in A^{\mu}(x_{\mathsf{q}})$. By (5.9), the definition of $\psi_{\mu}^{\tau, x_{\mathsf{q}}}$, and the triangle inequality property of $\mathbf{d}_h$, we have:

$$\begin{aligned} &\mathbf{d}_h\left(\mathcal{R}(\tau, x_{\mathsf{q}}, \upsilon_{\tau}), \mathcal{R}(\tau, x_{\mathsf{q}}, \upsilon_{\mathsf{q}})\right) \\ &\quad \leq \mathbf{d}_h\left(P, \mathcal{R}(\tau, x_{\mathsf{q}}, \upsilon_{\tau})\right) + \mathbf{d}_h\left(P, \mathcal{R}(\tau, x_{\mathsf{q}}, \upsilon_{\mathsf{q}})\right) \leq 2\mu. \end{aligned} \tag{5.10}$$

Consider now any switching signal $\pi_{\mathsf{q}} \in B^{\mu}(x_{\mathsf{q}}, \upsilon_{\mathsf{q}}) \subset B_{\tau}$ and set $z = \zeta_{x_{\mathsf{q}}\upsilon_{\mathsf{q}}}^{\pi_{\mathsf{q}}}(\tau) \in \mathcal{R}(\tau, x_{\mathsf{q}}, \upsilon_{\mathsf{q}})$. By inequality (5.10) and the definition of $\mathbf{d}_h$, there exists $z_1 \in \overline{\mathcal{R}(\tau, x_{\mathsf{q}}, \upsilon_{\tau})}$ such that

$$\|z_1 - z\| \leq 2\mu. \tag{5.11}$$

The vector[3] z_1 can be either in $\mathcal{R}(\tau, x_{\mathsf{q}}, \upsilon_{\tau})$ or in $\overline{\mathcal{R}(\tau, x_{\mathsf{q}}, \upsilon_{\tau})} \backslash \mathcal{R}(\tau, x_{\mathsf{q}}, \upsilon_{\tau})$; in both cases, for any $\sigma \in \mathbb{R}^+$, there exists $z_2 \in \mathcal{R}(\tau, x_{\mathsf{q}}, \upsilon_{\tau})$ such that

$$\|z_1 - z_2\| \leq \sigma. \tag{5.12}$$

Particularly, if $z_1 \in \mathcal{R}(\tau, x_{\mathsf{q}}, \upsilon_{\tau})$, one can choose $z_1 = z_2$. Choose $\pi_{\tau} \in B_{\tau}$ such that $z_2 = \zeta_{x_{\mathsf{q}}\upsilon_{\tau}}^{\pi_{\tau}}(\tau)$. Notice that since $z_2 \in \mathcal{R}(\tau, x_{\mathsf{q}}, \upsilon_{\tau})$, such $\pi_{\tau} \in B_{\tau}$ does exist.

Consider the transition $x_{\tau} \xrightarrow[\tau]{\upsilon_{\tau}, \pi_{\tau}} x_{\tau}' = \xi_{x_{\tau}\upsilon_{\tau}}^{\pi_{\tau}}(\tau)$ $\mathbb{P}$-a.s. in $S_{\tau}(\Sigma)$. It follows from the δ-GAS-M$_q$ assumption on Σ that:

$$\mathbb{E}\left[\left\|x_{\tau}' - \xi_{x_{\mathsf{q}}\upsilon_{\tau}}^{\pi_{\tau}}(\tau)\right\|^q\right] \leq \beta\left(\mathbb{E}\left[\left\|x_{\tau} - x_{\mathsf{q}}\right\|^q\right], \tau\right) \leq \beta(\varepsilon^q, \tau). \tag{5.13}$$

Since $\mathbb{R}^n \subseteq \bigcup_{p \in [\mathbb{R}^n]_{\eta}} \mathcal{B}_{\eta}(p)$, there exists $x_{\mathsf{q}}' \in X_{\mathsf{q}}$ such that

$$\left\|z - x_{\mathsf{q}}'\right\| \leq \eta, \tag{5.14}$$

which, by the definition of $S_{\mathsf{q}}(\Sigma)$, implies the existence of $x_{\mathsf{q}} \xrightarrow[\mathsf{q}]{\upsilon_{\mathsf{q}}, \pi_{\mathsf{q}}} x_{\mathsf{q}}'$ in $S_{\mathsf{q}}(\Sigma)$. Using Lemma 3.7, (5.11), (5.12), (5.13), (5.14), and triangle inequality, we obtain

$$\begin{aligned} &\left(\mathbb{E}\left[\left\|x_{\tau}' - x_{\mathsf{q}}'\right\|^q\right]\right)^{\frac{1}{q}} \\ &= \left(\mathbb{E}\left[\left\|x_{\tau}' - \xi_{x_{\mathsf{q}}\upsilon_{\tau}}^{\pi_{\tau}}(\tau) + \xi_{x_{\mathsf{q}}\upsilon_{\tau}}^{\pi_{\tau}}(\tau) - z_2 + z_2 - z_1 + z_1 - z + z - x_{\mathsf{q}}'\right\|^q\right]\right)^{\frac{1}{q}} \\ &\leq \left(\mathbb{E}\left[\left\|x_{\tau}' - \xi_{x_{\mathsf{q}}\upsilon_{\tau}}^{\pi_{\tau}}(\tau)\right\|^q\right]\right)^{\frac{1}{q}} + \left(\mathbb{E}\left[\left\|\xi_{x_{\mathsf{q}}\upsilon_{\tau}}^{\pi_{\tau}}(\tau) - z_2\right\|^q\right]\right)^{\frac{1}{q}} \\ &\quad + \|z_2 - z_1\| + \|z_1 - z\| + \left\|z - x_{\mathsf{q}}'\right\| \\ &\leq (\beta(\varepsilon^q, \tau))^{\frac{1}{q}} + (h(G, \tau))^{\frac{1}{q}} + \sigma + 2\mu + \eta. \end{aligned}$$

By inequality (5.8), there exists a sufficiently small value of $\sigma \in \mathbb{R}^+$ such that $(\beta(\varepsilon^q, \tau))^{\frac{1}{q}} + (h(G, \tau))^{\frac{1}{q}} + \sigma + 2\mu + \eta \leq \varepsilon$. Therefore, we conclude that $\left(x_{\tau}', x_{\mathsf{q}}'\right) \in R$ and that condition (iii) in Definition 4.2 holds.

In a similar way, we can prove that $S_{\mathsf{q}}(\Sigma) \preceq_{\mathcal{AS}}^{\varepsilon} S_{\tau}(\Sigma)$ by showing that R^{-1} is an ε-approximate simulation relation from $S_{\mathsf{q}}(\Sigma)$ to $S_{\tau}(\Sigma)$ which completes the proof. □

[3] Notice that the reachable set $\mathcal{R}(\tau, x_{\mathsf{q}}, \upsilon_{\tau})$ is not closed, in general, and hence inequality (5.10) does not guarantee the existence of $z_1 \in \mathcal{R}(\tau, x_{\mathsf{q}}, \upsilon_{\tau})$, satisfying inequality (5.11). However, by the definition of $\mathbf{d}_h$, the vector z_1 is guaranteed to exist in the topological closure of $\mathcal{R}(\tau, x_{\mathsf{q}}, \upsilon_{\tau})$.

Remark 5.3. *Let us remark that in order to show the result in Theorem 5.2, one does not require any probabilistic structure on the switching signals $\pi \in \mathcal{P}$, as long as the switching stochastic system Σ admits a common δ-GAS-M$_q$ Lyapunov function, or as it satisfies property* (3.1) *with some $\mathcal{KL}$ function β. Alternatively, Assumption 3.4 allows us to compute the $\mathcal{KL}$ function β satisfying* (3.1)*, by resorting to multiple δ-GAS-M$_q$ Lyapunov functions.*

Remark 5.4. *Further notice that, in order to construct the proposed finite abstraction, one requires to compute the reachable sets in* (5.1) *and* (5.2)*, leveraging a well developed theory for this goal. For instance, one may leverage flow-based techniques [32] or alternatively Monte-Carlo simulations [33].*

Let us finally remark that the proposed finite abstraction is computed by discretizing the state-space, which suffers severely from the curse of dimensionality related to the discretization of the continuous space. One can leverage the results in [12] to provide finite abstractions for switching stochastic systems without state-space discretization.

6. Conclusions

In this paper we have shown the existence of symbolic models that are alternatingly approximately bisimilar to δ-GAS-M$_q$ switching stochastic systems, for any $q \geq 1$, when their dynamics lie in a bounded set (this is always the case in practice). Moreover, we have provided a description of the δ-GAS-M$_q$ property using a common δ-GAS-M$_q$ Lyapunov function or, alternatively, using multiple δ-GAS-M$_q$ Lyapunov functions under some fairly general assumption on the switching signals.

In future work we plan to focus on constructive approaches to obtain the symbolic models of which we have discussed the existence in this work. Note that the construction of the symbolic models in this paper relies on the computation of sets of reachable states in (5.1) and (5.2), which is a tolling task in general. The authors are currently investigating several different techniques to mitigate this limitation, allowing for the use of the proposed technique on practical models for cyber-physical systems operating in uncertain or noisy environments.

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7. Appendix

Proof (of Theorem 3.6). The proof is inspired by that of Theorem 3.1 in [34] and is a consequence of the application of Gronwall's inequality and of Ito's lemma [23, p. 80 and 123]. Let a and a' be any $\mathbb{R}^n$-valued random variables that are measurable in $\mathcal{F}_0$, $t_0 = 0$, and $\upsilon \in \mathcal{U}$ be any control input. Let the sequence $\{t_1, t_2, \ldots, t_s\}$ denote the time instances when a switching between modes occurs before arbitrary time $t \in \mathbb{R}_0^+$. We denote by $t_s \leq t$ the time instance of the last mode switching and by p_s the active mode index for the time interval $t_s \leq t < t_{s+1}$. Let $\mathcal{H}_t \subset \mathcal{F}_t$ be the sigma-subalgebra generated by $N_{\widehat{\pi}}(s)$, $s \leq t$. We introduce the conditional expectation of $V_{p_s}\left(\xi^{\widehat{\pi}}_{a\upsilon}(t), \xi^{\widehat{\pi}}_{a'\upsilon}(t)\right)$, given $\mathcal{H}_t$ at time t, as the following:

$$\begin{aligned}
&\mathbb{E}\left[V_{p_s}\left(\xi^{\widehat{\pi}}_{a\upsilon}(t), \xi^{\widehat{\pi}}_{a'\upsilon}(t)\right) | \mathcal{H}_t\right] \leq \mathbb{E}\Bigg[\int_{t_s}^{t}\left(\mathcal{L}^{\upsilon(\tau)} V_{p_s}\left(\xi^{\widehat{\pi}}_{a\upsilon}(\tau), \xi^{\widehat{\pi}}_{a'\upsilon}(\tau)\right)\right) d\tau \\
&\quad + \int_{t_s}^{t}\left[\partial_x V_{p_s}\ \partial_{x'} V_{p_s}\right]\begin{bmatrix} g_{p_s}\left(\xi^{\widehat{\pi}}_{a\upsilon}(\tau)\right) \\ g_{p_s}\left(\xi^{\widehat{\pi}}_{a'\upsilon}(\tau)\right)\end{bmatrix} \mathrm{d} W_\tau + V_{p_s}\left(\xi^{\widehat{\pi}}_{a\upsilon}(t_s), \xi^{\widehat{\pi}}_{a'\upsilon}(t_s)\right) | \mathcal{H}_t\Bigg] \\
&\leq \mathbb{E}\left[\int_{t_s}^{t}\left(-\kappa_{p_s} V_{p_s}\left(\xi^{\widehat{\pi}}_{a\upsilon}(\tau), \xi^{\widehat{\pi}}_{a'\upsilon}(\tau)\right)\right) d\tau | \mathcal{H}_t\right] + \mathbb{E}\left[V_{p_s}\left(\xi^{\widehat{\pi}}_{a\upsilon}(t_s), \xi^{\widehat{\pi}}_{a'\upsilon}(t_s)\right) | \mathcal{H}_t\right] \\
&\leq -\kappa \int_{t_s}^{t} \mathbb{E}\left[V_{p_s}\left(\xi^{\widehat{\pi}}_{a\upsilon}(\tau), \xi^{\widehat{\pi}}_{a'\upsilon}(\tau)\right) | \mathcal{H}_t\right] d\tau + \mathbb{E}\left[V_{p_s}\left(\xi^{\widehat{\pi}}_{a\upsilon}(t_s), \xi^{\widehat{\pi}}_{a'\upsilon}(t_s)\right) | \mathcal{H}_t\right].
\end{aligned} \tag{7.1}$$

Note that

$$\mathbb{E}\left[\int_{t_s}^{t}\left[\partial_x V_{p_s}\ \partial_{x'} V_{p_s}\right]\begin{bmatrix} g_{p_s}\left(\xi^{\widehat{\pi}}_{a\upsilon}(\tau)\right) \\ g_{p_s}\left(\xi^{\widehat{\pi}}_{a'\upsilon}(\tau)\right)\end{bmatrix} \mathrm{d} W_\tau | \mathcal{H}_t\right] = 0,$$

because W_τ, $\tau \leq t$, is independent of $\mathcal{H}_t$. Using (7.1) and by virtue of Gronwall's inequality, we obtain

$$\mathbb{E}\left[V_{p_s}\left(\xi^{\widehat{\pi}}_{a\upsilon}(t), \xi^{\widehat{\pi}}_{a'\upsilon}(t)\right) | \mathcal{H}_t\right] \leq \mathbb{E}\left[V_{p_s}\left(\xi^{\widehat{\pi}}_{a\upsilon}(t_s), \xi^{\widehat{\pi}}_{a'\upsilon}(t_s)\right) | \mathcal{H}_t\right] \mathrm{e}^{-\kappa(t-t_s)}. \tag{7.2}$$

In inequality (7.2), t_s is an instance of a mode switching. Hence, from assumption (i) of the theorem, one obtains:

$$\begin{aligned}
\mathbb{E}\left[V_{p_s}\left(\xi^{\widehat{\pi}}_{a\upsilon}(t), \xi^{\widehat{\pi}}_{a'\upsilon}(t)\right) | \mathcal{H}_t\right] &\leq \mathbb{E}\left[\mu V_{p_{s-1}}\left(\xi^{\widehat{\pi}}_{a\upsilon}(t_s), \xi^{\widehat{\pi}}_{a'\upsilon}(t_s)\right) | \mathcal{H}_t\right] \mathrm{e}^{-\kappa(t-t_s)} \\
&\leq \mu \mathbb{E}\left[V_{p_{s-1}}\left(\xi^{\widehat{\pi}}_{a\upsilon}(t_s), \xi^{\widehat{\pi}}_{a'\upsilon}(t_s)\right) | \mathcal{H}_t\right] \mathrm{e}^{-\kappa(t-t_s)}.
\end{aligned} \tag{7.3}$$

Similar to inequality (7.2) and using the continuity of $V_{p_{s-1}}$ and of the solution process ξ, one can derive the inequality

$$\begin{aligned}
&\mathbb{E}\left[V_{p_{s-1}}\left(\xi^{\widehat{\pi}}_{a\upsilon}(t_s), \xi^{\widehat{\pi}}_{a'\upsilon}(t_s)\right) | \mathcal{H}_t\right] \\
&\quad \leq \mathbb{E}\left[V_{p_{s-1}}\left(\xi^{\widehat{\pi}}_{a\upsilon}(t_{s-1}), \xi^{\widehat{\pi}}_{a'\upsilon}(t_{s-1})\right) | \mathcal{H}_t\right] \mathrm{e}^{-\kappa(t_s-t_{s-1})}.
\end{aligned} \tag{7.4}$$

Substituting (7.4) into (7.3), one gets

$$\begin{aligned}
&\mathbb{E}\left[V_{p_s}\left(\xi^{\widehat{\pi}}_{a\upsilon}(t), \xi^{\widehat{\pi}}_{a'\upsilon}(t)\right) | \mathcal{H}_t\right] \\
&\quad \leq \mu \mathbb{E}\left[V_{p_{s-1}}\left(\xi^{\widehat{\pi}}_{a\upsilon}(t_{s-1}), \xi^{\widehat{\pi}}_{a'\upsilon}(t_{s-1})\right) | \mathcal{H}_t\right] \mathrm{e}^{-\kappa(t-t_{s-1})}.
\end{aligned} \tag{7.5}$$

Repeating the calculation and the substitution, similar to (7.5), we arrive at

$$\mathbb{E}\left[V_{p_s}\left(\xi^{\widehat{\pi}}_{a\upsilon}(t), \xi^{\widehat{\pi}}_{a'\upsilon}(t)\right) | \mathcal{H}_t\right] \leq \mu^s \mathbb{E}\left[V_{p_0}(a, a') | \mathcal{H}_t\right] \mathrm{e}^{-\kappa t}.$$

Because $s = N_{\widehat{\pi}}(t)$, one gets

$$\mathbb{E}\left[V_{p_s}\left(\xi^{\widehat{\pi}}_{a\upsilon}(t), \xi^{\widehat{\pi}}_{a'\upsilon}(t)\right) | \mathcal{H}_t\right] \leq \mu^{N_{\widehat{\pi}}(t)} \mathbb{E}\left[V_{p_0}(a, a') | \mathcal{H}_t\right] \mathrm{e}^{-\kappa t}.$$

Since initial conditions a, a' are independent of $N_{\widehat{\pi}}(t)$, we have

$$\begin{aligned}
\mathbb{E}\left[V_{p_s}\left(\xi^{\widehat{\pi}}_{a\upsilon}(t), \xi^{\widehat{\pi}}_{a'\upsilon}(t)\right)\right] &\leq \mathbb{E}\left[\mu^{N_{\widehat{\pi}}(t)}\right] \mathbb{E}\left[V_{p_0}(a, a')\right] \mathrm{e}^{-\kappa t} \\
&\leq \left(\sum_{k=0}^{\infty} \mu^k \mathbb{P}\left[N_{\widehat{\pi}}(t) = k\right]\right) \mathbb{E}\left[V_{p_0}(a, a')\right] \mathrm{e}^{-\kappa t} \\
&\leq \left(\mathrm{e}^{-\lambda t} \sum_{k=0}^{\infty} \mu^k \frac{(\lambda t)^k}{k!}\right) \mathbb{E}\left[V_{p_0}(a, a')\right] \mathrm{e}^{-\kappa t} \\
&= \mathbb{E}\left[V_{p_0}(a, a')\right] \mathrm{e}^{((\mu-1)\lambda-\kappa)t}.
\end{aligned} \tag{7.6}$$

Note that we used the Taylor series of the exponential function to obtain the inequality (7.6), i.e. $\mathrm{e}^{\mu\lambda t} = \sum_{k=0}^{\infty} \mu^k \frac{(\lambda t)^k}{k!}$. Using assumptions (i) and (ii) in Definition (3.2), functions $\underline{\alpha}$, $\overline{\alpha}$, and Jensen's inequality, we obtain

$$\begin{aligned}
\underline{\alpha}\left(\mathbb{E}\left[\left\|\xi^{\widehat{\pi}}_{a\upsilon}(t) - \xi^{\widehat{\pi}}_{a'\upsilon}(t)\right\|^q\right]\right) &\leq \underline{\alpha}_{p_s}\left(\mathbb{E}\left[\left\|\xi^{\widehat{\pi}}_{a\upsilon}(t) - \xi^{\widehat{\pi}}_{a'\upsilon}(t)\right\|^q\right]\right) \\
&\leq \mathbb{E}\left[\underline{\alpha}_{p_s}\left(\left\|\xi^{\widehat{\pi}}_{a\upsilon}(t) - \xi^{\widehat{\pi}}_{a'\upsilon}(t)\right\|^q\right)\right] \leq \mathbb{E}\left[V_{p_s}\left(\xi^{\widehat{\pi}}_{a\upsilon}(t), \xi^{\widehat{\pi}}_{a'\upsilon}(t)\right)\right] \\
&\leq \mathbb{E}\left[V_{p_0}(a, a')\right] \mathrm{e}^{((\mu-1)\lambda-\kappa)t} \leq \mathbb{E}\left[\overline{\alpha}_{p_0}\left(\|a - a'\|^q\right)\right] \mathrm{e}^{((\mu-1)\lambda-\kappa)t} \\
&\leq \overline{\alpha}_{p_0}\left(\mathbb{E}\left[\|a - a'\|^q\right]\right) \mathrm{e}^{((\mu-1)\lambda-\kappa)t} \leq \overline{\alpha}\left(\mathbb{E}\left[\|a - a'\|^q\right]\right) \mathrm{e}^{((\mu-1)\lambda-\kappa)t}.
\end{aligned}$$

Since $\underline{\alpha}$ is a $\mathcal{K}_\infty$ function, we have

$$\mathbb{E}\left[\left\|\xi^{\widehat{\pi}}_{a\upsilon}(t) - \xi^{\widehat{\pi}}_{a'\upsilon}(t)\right\|^q\right] \leq \underline{\alpha}^{-1}\left(\overline{\alpha}\left(\mathbb{E}\left[\|a - a'\|^q\right]\right) \mathrm{e}^{((\mu-1)\lambda-\kappa)t}\right).$$

Therefore, condition (3.1) holds with the function

$$\beta(r, s) := \underline{\alpha}^{-1}\left(\overline{\alpha}(r) \mathrm{e}^{((\mu-1)\lambda-\kappa)s}\right),$$

which is a $\mathcal{KL}$ function because by assumption (ii) of the theorem $(\mu - 1)\lambda - \kappa < 0$. Therefore, the switching stochastic system Σ is δ-GAS-M$_q$. □