Approximation of General Stochastic Hybrid Systems by Switching Diffusions with Random Hybrid Jumps

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Abstract. In this work we propose an approximation scheme to transform a general stochastic hybrid system (SHS) into a SHS without forced transitions due to spatial guards. Such switching mechanisms are replaced by spontaneous transitions with state-dependent transition intensities (jump rates). The resulting switching diffusion process with random hybrid jumps is shown to converge in distribution to the original stochastic hybrid system execution. The obtained approximation can be useful for various purposes such as, on the computational side, simulation and reachability analysis, as well as for the theoretical investigation of the model. More generally, it is suggested that SHS which are endowed exclusively with random jumping events are *simpler* than those that present spatial forcing transitions.
In the opening of this work, the general SHS model is presented, a few of its basic properties are discussed, and the concept of generator is introduced. The second part of the paper describes the approximation procedure, introduces the new SHS model, and proves, under some assumptions, its weak convergence to the original system.

We describe the general stochastic hybrid system model introduced in [1].

Definition 1 (General Stochastic Hybrid System). *A* General Stochastic Hybrid System *(GSHS) is a collection $\mathcal{S}_g = (\mathcal{Q}, n, A, B, \Gamma, R^{\Gamma}, \Lambda, R^{\Lambda}, \pi)$, where*

- $\mathcal{Q} = \{q_1, q_2, \ldots, q_m\}, m \in \mathbb{N}$, *is a countable set of discrete modes;*
- $n : \mathcal{Q} \to \mathbb{N}$ *is a map such that, for $q \in \mathcal{Q}$, the continuous state space is the Euclidean space $\mathbb{R}^{n(q)}$. The hybrid state space is then $\mathcal{S} = \cup_{q \in \mathcal{Q}} \{q\} \times \mathbb{R}^{n(q)}$;*
- $A = \{a(q, \cdot) : \mathbb{R}^{n(q)} \to \mathbb{R}^{n(q)}, q \in \mathcal{Q}\}$ *is a collection of drift terms;*
- $B = \{b(q, \cdot) : \mathbb{R}^{n(q)} \to \mathbb{R}^{n(q) \times n(q)}, q \in \mathcal{Q}\}$ *is a collection of diffusion terms;*
- $\Gamma = \cup_{q \in \mathcal{Q}} \{q\} \times \Gamma_q \subset \mathcal{S}$*, where $\Gamma_q = \cup_{q' \neq q \in \mathcal{Q}} \gamma_{qq'}$ is a closed set composed of $m-1$ disjoint guard sets $\gamma_{qq'}$ causing forced transitions from q to $q' \neq q$;*
- $R^{\Gamma} : \mathcal{B}(\mathbb{R}^{n(\cdot)}) \times \mathcal{Q} \times \Gamma \to [0, 1]$ *is the reset stochastic kernel associated with Γ. Specifically, $R^{\Gamma}(\cdot | q', (q, x))$ is a probability measure concentrated on $\mathbb{R}^{n(q')} \setminus \Gamma_{q'}$, which describes the probabilistic reset of the continuous state when a jump from mode q to q' occurs from $x \in \gamma_{qq'}$;*

– $\Lambda : \mathcal{S}\setminus\Gamma\times\mathcal{Q} \to \mathbb{R}^+$ *is the transition intensity function governing spontaneous transitions. Specifically, for any* $q \neq q' \in \mathcal{Q}$, $\lambda_{qq'}(x) := \Lambda((q,x),q')$ *is the jump rate from mode* q *to mode* q' *when* $x \in \mathbb{R}^{n(q)} \setminus \Gamma_q$*;*
– $R^\Lambda : \mathcal{B}(\mathbb{R}^{n(\cdot)}) \times \mathcal{Q} \times \mathcal{S} \setminus \Gamma \to [0,1]$ *is the reset stochastic kernel associated with* Λ*. In particular,* $R^\Lambda(\cdot|q',(q,x))$ *is a probability measure concentrated on* $\mathbb{R}^{n(q')} \setminus \Gamma_{q'}$ *that describes the probabilistic reset of the continuous state when a jump from mode* q *to* q' *occurs from* $x \in \mathbb{R}^{n(q)} \setminus \Gamma_q$*;*
– $\pi : \mathcal{B}(\mathcal{S}) \to [0,1]$ *is a measure on* $\mathcal{S}\setminus\Gamma$ *describing the initial distribution.* $\square$

In the definition above, $\mathcal{B}(\mathcal{S})$ denotes the σ-field on $\mathcal{S}$.

Assumption 1 (on the system dynamics) *1. The drift and diffusion terms* $a(q,\cdot)$ *and* $b(q,\cdot), q \in \mathcal{Q}$*, are bounded and uniformly Lipschitz continuous.*
2. The jump rate function $\Lambda : \mathcal{S}\setminus\Gamma\times\mathcal{Q} \to \mathbb{R}^+$ *satisfies the following conditions:*
  – *it is measurable and bounded;*
  – *for any* $q, q' \in \mathcal{Q}, q \neq q'$*, and any sample path* $\omega^{(q,x)}(t)$*,* $t \geq 0$*, of the process solving the SDE in* q*, initialized at* $x \in \mathbb{R}^{n(q)} \setminus \Gamma_q$*, there exists* $\epsilon_{qq'}(x) > 0$ *such that* $\lambda_{qq'}(\omega^{(q,x)}(t))$ *is integrable over* $[0, \epsilon_{qq'}(x))$*.*
3. For all $C \in \mathcal{B}(\mathbb{R}^{n(\cdot)})$*,* $R^\Gamma(C|\cdot)$ *and* $R^\Lambda(C'|\cdot)$ *are measurable.*
4. For any execution associated with $\pi = \delta_s$*,* $s \in \mathcal{S}\setminus\Gamma$*, the expected value of the number of jumps within the time interval* $[0,t]$ *is bounded for all* $t \geq 0$*.* $\square$

Intuitively, Assumption 1.1 guarantees the existence and uniqueness of the $n(q)$-dimensional solution to the SDE associated with $q \in \mathcal{Q}$, $d\mathbf{v}(t) = a(q,\mathbf{v}(t))dt + b(q,\mathbf{v}(t))d\mathbf{w}_q(t)$, where $\mathbf{w}_q$ is a $n(q)$-dimensional standard Wiener process.

The semantic definition of the GSHS $\mathcal{S}_g$, given via the notion of *execution* (a stochastic process $\{\mathbf{s}(t) = (\mathbf{q}(t), \mathbf{x}(t)),\ t \geq 0\}$, with values in $\mathcal{S}$, solution of $\mathcal{S}_g$), can be done as in [1]. Note that a sample-path of a GSHS execution is a right-continuous, $\mathcal{S}$-valued function on $[0,\infty)$, with left-limits on $(0,\infty)$ (*càdlàg*). Furthermore, the following property holds.

Proposition 1. *Consider a GSHS* $\mathcal{S}_g$*. Under assumptions 1.1-1.2-1.3-1.4, the execution* $\mathbf{s}(t), t \geq 0$*, of* $\mathcal{S}_g$ *is a càdlàg strong Markov process.* $\square$

It is interesting to associate to the set of real-valued functions f, acting on Markov processes defined on a Borel space, a *strong generator* $\mathcal{L}$, and a weaker, yet more general, *extended generator* [2]. Denote with $C_b^2(\mathcal{S})$ the class of real-valued, twice continuously differentiable and bounded functions on $\mathcal{S}$. Let $\frac{\partial f(q,x)}{\partial x} a(q,x) = \sum_{i=1}^{n(q)} \frac{\partial f(q,x)}{\partial x_i} a_i(q,x)$ be the Lie derivative of $f(q,\cdot)$ along $a(q,\cdot)$, and $H_f(q,x) = \left[\frac{\partial^2 f(q,x)}{\partial x_i \partial x_j}\right]_{i,j=1,2,\ldots,n(q)}$ be the Hessian of $f(q,\cdot)$.

Proposition 2 (Extended Generator of $\mathcal{S}_g$). *The extended generator* $\mathcal{L}_g : \mathcal{D}(\mathcal{L}_g) \to \mathcal{B}_b(\mathcal{S})$ *associated with the executions of* $\mathcal{S}_g$ *is, for* $s = (q,x) \in \mathcal{S}\setminus\Gamma$*:*

$$\mathcal{L}_g f(s) = \mathcal{L}_g^d f(s) + I_{\mathcal{S}\setminus\Gamma}(s) \sum_{q' \in \mathcal{Q}, q' \neq q} \lambda_{qq'}(x) \int_{\mathbb{R}^{n(q')}} \big(f((q',z)) - f(s)\big) R^\Lambda(dz|q',s),$$

where $\mathcal{L}_g^d f(s) = \sum_{q\in\mathcal{Q}} \frac{\partial f(q,x)}{\partial x} a(q,x) + \frac{1}{2} Tr\big(b(q,x)b(q,x)^T H_f(q,x)\big)$.

The domain $\mathcal{D}(\mathcal{L}_g)$ *of* $\mathcal{L}_g$ *is the set of functions* $f \in C_b^2(\mathcal{S})$ *satisfying the condition:* $f(s) = \sum_{q'\in\mathcal{Q}, q'\neq q} \int_{\mathbb{R}^{n(q')}} f((q',z)) R^{\Gamma}(dz|q',s), \quad s \in \Gamma.$ □

Consider the GSHS system $\mathcal{S}_g$ in Definition 1. The guard set of $\mathcal{S}_g$ within mode $q \in \mathcal{Q}$ is made up of $\gamma_{qq'} \subset \mathbb{R}^{n(q)}$, $q' \in \mathcal{Q}$, $q' \neq q$. Assume that each set $\gamma_{qq'}$ can be expressed as a zero sub-level set of a continuous function $h_{qq'} : \mathbb{R}^{n(q)} \to \mathbb{R}$:

$$\gamma_{qq'} = \{x \in \mathbb{R}^{n(q)} : h_{qq'}(x) \leq 0\}.$$

Pick a small enough $\delta > 0$, and by the continuity of $h_{qq'}$, introduce the sets

$$\gamma_{qq'}^{-\delta} = \{x \in \mathbb{R}^{n(q)} : h_{qq'}(x) \leq -\delta\} \subseteq \gamma_{qq'} \subseteq \gamma_{qq'}^{\delta} = \{x \in \mathbb{R}^{n(q)} : h_{qq'}(x) \leq \delta\}.$$

For any $q \in \mathcal{Q}$, define the set of functions $\lambda_{qq'}^{\delta} : \mathbb{R}^{n(q)} \to \mathbb{R}^+$, $q' \in \mathcal{Q}$, $q' \neq q$,

$$\lambda_{qq'}^{\delta}(x) = \begin{cases} \left(\dfrac{1}{d(x,\gamma_{qq'}^{-\delta})} - \dfrac{1}{\sup\limits_{y:h_{qq'}(y)=\delta} d(y,\gamma_{qq'}^{-\delta})} \right) \wedge \left(\dfrac{1}{\sup\limits_{y:h_{qq'}(y)=0} d(y,\gamma_{qq'}^{-\delta})} \right), & x \in \gamma_{qq'}^{\delta} \\ 0, & x \in \mathbb{R}^{n(q)} \setminus \gamma_{qq'}^{\delta} \end{cases}$$

where $a \wedge b = \min\{a,b\}$, whereas $d(z,A) = \inf_{y\in A} ||z-y||$, $z \in \mathbb{R}^{n(q)}$, $A \subset \mathbb{R}^{n(q)}$.

We associate to $\mathcal{S}_g$ a new stochastic hybrid system $\mathcal{S}_\delta$, which is made up of the elements of $\mathcal{S}_g$, except for the following:

- The spatial guards set is empty, $\Gamma = \emptyset$;
- The transition intensity function Λ, whose domain of definition is $\mathcal{S} \setminus \Gamma \times \mathcal{Q}$, is replaced by $\Lambda^{\delta} : \mathcal{S} \times \mathcal{Q} \to \mathbb{R}^+$ given by $\Lambda^{\delta}((q,x),q') := \lambda_{qq'}^{\delta}(x) + \lambda_{qq'}(x)$ where for any $q' \neq q \in \mathcal{Q}$, the original jump rate $\lambda_{qq'}(\cdot)$ is extended to $\mathbb{R}^{n(q)}$ by setting it to zero over Γ_q;
- The stochastic reset kernel $R^{\Lambda^{\delta}} : \mathcal{B}(\mathbb{R}^{n(\cdot)}) \times \mathcal{Q} \times \mathcal{S} \to [0,1]$ associated with Λ^{δ} is given by $R^{\Lambda^{\delta}}(C_{q'}|q',(q,x)) = R^{\Lambda}(C_{q'}|q',(q,x)) + R^{\Gamma}(C_{q'}|q',(q,x))$, for any Borel set $C_{q'}$ of $\mathbb{R}^{n(q')}$, where the original stochastic reset kernels $R^{\Lambda}(C_{q'}|q',\cdot)$ and $R^{\Lambda}(C_{q'}|q',\cdot)$ are extended to $\mathcal{S}$ by setting them to zero outside their original domain of definition.

Notice that the conclusions in Proposition 1 hold true also for the SHS $\mathcal{S}_\delta$. The extended generator $\mathcal{L}_\delta$ of $\mathcal{S}_\delta$ can be derived in a similar way as for that for $\mathcal{S}_g$, but has no condition on the points of the guard set. Its domain $\mathcal{D}(\mathcal{L}_\delta)$ is the set of functions $f \in C_b^2(\mathcal{S})$. This implies that $\mathcal{D}(\mathcal{L}_g) \subseteq \mathcal{D}(\mathcal{L}_\delta)$, for $\delta > 0$.

Let us formally show that, as $\delta \to 0$, the sequence of stochastic processes $\{\mathbf{s}_\delta(t)\}_{\delta>0}$ converges, in some sense, to $\mathbf{s}(t)$, for any $t \geq 0$. The forthcoming notions are found in [3].The concept of extended generator can be useful in showing that a sequence of Markov processes converges to a given Markov process. Qualitatively, given a sequence of $\mathcal{S}$-valued processes $\{\mathbf{X}_n\}_{n\geq 1}$ and a process $\mathbf{X}$, with extended generators $(A_n, \mathcal{D}(A_n))$ and $(A, \mathcal{D}(A))$ respectively, to prove that

$\mathbf{X}_n \Rightarrow \mathbf{X}$ (convergence in the weak sense), it is sufficient to show that for all functions $f \in \mathcal{D}(\mathcal{A})$, there exist $f_n \in \mathcal{D}(\mathcal{A}_n)$, such that $f_n \to f$ and $A_n f_n \to Af$. The following fact, needed in Theorem 2, is verified:

Theorem 1 (Compact Containment Condition). *Consider the GSHS $\mathcal{S}_g$, the SHS $\mathcal{S}_\delta$, and their corresponding unique global solutions $\mathbf{s}(t)$ and $\mathbf{s}_\delta(t), t \geq 0$. The stochastic processes $\mathbf{s}_\delta(t)$ are such that, for any $\epsilon > 0, N > 0$, there exists a compact set $K_{\epsilon,N} \subset \mathcal{S}$ such that*

$$\liminf_{\delta \downarrow 0} \mathcal{P}\left[\mathbf{s}_\delta(t) \in K_{\epsilon,N}, \forall\, 0 \leq t \leq N\right] \geq 1 - \epsilon.$$

Similarly for the stochastic process $\mathbf{s}(t)$. □

Given a sequence of entities $\{c_n\}_{n \geq 1}$ and a scalar c, let us denote as $\lim_n^\star c_n = c$ the conditions $\lim_{n\to\infty} c_n = c$ and $(\vee_n \|c_n\|) \vee \|c\| < \infty$, where $\|\cdot\|$ is the sup norm. Similarly if the indexing parameter tends to zero ($\delta = 1/n$). A process $\mathbf{X}$ is said to be a solution of the local martingale problem for a linear operator (A, π) if $\mathcal{P} \circ \mathbf{X}(0)^{-1} = \pi$, and for each $f \in \mathcal{D}(A)$, $f(\mathbf{X}(t)) - f(\mathbf{X}(0)) - \int_0^t Af(\mathbf{X}(s))ds$ is a local martingale, $\forall t \geq 0$. In order to complete the proof of the following Theorem 2, it is necessary to raise the following

Assumption 2 *1. Given a GSHS, as in Definition 1, assume that the probabilistic reset kernels $R^\Gamma(\cdot|q', (q,x))$ are continuous in x, for any $q' \neq q \in \mathcal{Q}$.*
2. The local martingale problem for $(\mathcal{L}_g, \mathcal{D}(\mathcal{L}_g))$ is well posed, that is, it admits a unique *solution.* □

The following theorem is based on results from [4, Theorem 4.4].

Theorem 2 (Weak Convergence of $\mathcal{S}_\delta$ to $\mathcal{S}_g$). *Consider the SHS model $\mathcal{S}_\delta$, the GSHS model $\mathcal{S}_g$ under Assumption 2.1, and their associated $\mathcal{S}$-valued unique solution processes $\mathbf{s}_\delta(t)$ and $\mathbf{s}(t), t \geq 0$, where $\mathbf{s}_\delta(0) = \mathbf{s}(0) = (q_0, x_0) \in \mathcal{S}$. Consider further their extended generators $(\mathcal{L}_\delta, \mathcal{D}(\mathcal{L}_\delta))$ and $(\mathcal{L}_g, \mathcal{D}(\mathcal{L}_g))$, and conjecture that Assumption 2.2 is valid. It holds that*

- $\mathcal{L}_g \subset C_b^0(\mathcal{S}) \times C^0(\mathcal{S})$;
- *For all* $f \in \mathcal{D}(\mathcal{L}_g), \exists f_\delta \in \mathcal{D}(\mathcal{L}_\delta)$, *such that* $\lim_\delta^\star f_\delta = f, \lim_\delta \mathcal{L}_\delta f_\delta = \mathcal{L} f$;
- $\mathcal{D}(\mathcal{L})$ *is dense in* $C_b^0(\mathcal{S})$ *with respect to* $\lim^\star$.

By Theorem 1, as the approximation step $\delta \downarrow 0$, the solution of the SHS $\mathcal{S}_\delta$ weakly converges to that of the GSHS $\mathcal{S}_g$: $\mathbf{s}_\delta(t) \Rightarrow \mathbf{s}(t), \forall t \geq 0$. □

References

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