A Mean Field Equilibrium for a Model of Interbank Lending

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Abstract— **We study the formation of short-term interest rates in the interbank lending market where banks are modeled as agents with bounded rationality. We propose a novel model which is based on bilateral contracts between risk-neutral profit-maximizing agents. To render the model tractable for large financial networks, we assume that banks' beliefs about the borrowing alternatives in the market are governed by a common reference interest rate, which is a function of the rates offered in the bilateral contracts. We show how this reference rate can be determined endogenously from a suitable mean field equilibrium and provide sufficient conditions for the existence of such an equilibrium together with an algorithm to compute it. Using simulation, we study the dependence of the equilibrium on the model parameters.**

I. INTRODUCTION

Models of financial networks are receiving significant attention from the research community [12]. The need for a better understanding of such networks and their potential vulnerabilities has been illustrated in the Financial Crisis of 2007-08 [6] and in the ongoing Eurozone Crisis [10].

Financial institutions can be seen as nodes in a network, connected with each other via mutual financial obligations, e.g. interbank loans. While such connectivity can help in distributing risks over the network to better absorb small external shocks, it may also have a contagious effect, spreading large shocks caused by a default or bankruptcy of a node through the network and causing defaults of other nodes.

For financial regulators it is thus essential to study financial stability of the system by identifying which network structures are more successful in absorbing risks than others. From a policy perspective, it is also important to understand how monetary policies (reserve requirements, capital requirements, etc.) affect the properties of such networks. Furthermore, for financial institutions themselves it is crucial to make prudent decisions when facing counterparty risk. For instance, in order to determine the conditions of loans they offer, banks need to be able to accurately assess the credit risk of their potential borrower, which depends on the topology and parameters of the network.

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Among the aforementioned issues, the question of resilience of financial network structures has received the most attention the literature, with a focus on interbank lending markets, see e.g. [12] and references therein. Typically, work on this topic assumes a given network structure, and connectivity parameters that are relevant for stability are identified by applying exogenous shocks to the nodes of interest and studying the resulting propagation of defaults through the network. Much less work has been devoted to the problem of the network formation [5], i.e. studying how the characteristics of a network develop without assuming a particular network structure a priori. A model commonly adopted for either of the above problems is to assume that agents (i.e. banks) act according to some (not necessarily rational) pre-specified behavior ([1] being a notable exception). Furthermore, most existing models take important variables as exogenous. As a result, while they provide valuable insights into the role of financial networks, these models lack predictive power about how structures emerge as the result of the interaction of individual rational agents.

In this paper we propose a novel model for interbank lending, which is based on bilateral contracts and considers banks as agents with a particular kind of bounded rationality as defined below. While this framework admits a wide range of applications, here we use it to study the endogenous formation of short-term interbank interest rates. This serves as a first step towards a more elaborate analysis of financial network formation in the future. In contrast to much of the previous work, our approach allows explaining how important model characteristics, such as the banks' investment and lending decisions, emerge endogenously as a result of competitive behavior of individual banks.

We assume that each lending bank makes an offer based on its beliefs regarding the borrowing alternatives of a representative bank in the market. These beliefs depend on the reference rate $\mathsf{r}^{\mathsf{ref}}$ obtained as an aggregate of bilateral rates offered by individual lending banks: an example for $\mathsf{r}^{\mathsf{ref}}$ is the London Interbank Offered Rate (LIBOR) [2]. In our model, the equilibrium value of $\mathsf{r}^{\mathsf{ref}}$ is determined endogenously via the concept of mean field equilibrium [9], [7], [15]. That is, the conditions offered in the individual contracts are optimal with respect to $\mathsf{r}^{\mathsf{ref}}$, and, in turn, $\mathsf{r}^{\mathsf{ref}}$ is consistent with the individually offered rates. We establish sufficient conditions for the existence of an equilibrium, provide a numerical procedure to compute it and investigate the dependence of the equilibrium rate on parameters of the model in a numerical case study.

Outline: Section II discusses the contract design problem between two banks, which is extended to the case with borrowing alternatives in Section III-A. Section III-B gives the definition of and main results on the proposed equilibrium interest rate, which is illustrated by a numerical example in Section IV. Section V concludes the paper. Proofs are omitted for brevity.

II. Optimal Loans in a Two-Bank Model

The primary reason for the existence of the interbank lending market is the liquidity imbalance of banks. Consider bank 1 that currently has a surplus of liquid assets (e.g. cash), but does not have access to promising investment opportunities. In contrast, bank 2 has just invested a significant share of its cash assets in a venture which will only start paying back after a year. To ensure its liquidity, i.e. to be able to pay back maturing liabilities (such as deposits that are withdrawn), bank 2 may be willing to borrow cash from bank 1, which in turn is willing to lend as it has liquidity surplus with no ventures to invest in. In this section we focus on the contract design problem bank 1 faces when lending money to bank 2. For simplicity, we consider a single one-stage decision making problem in which the banks have no existing obligations w.r.t. to each other.

A. Investment Opportunities

The state of each bank is given by its balance sheet consisting of the assets and liabilities, c.f. Figure 1. The asset side of bank i consists of cash $C_i \geq 0$, risk-free assets (e.g. government bonds) $B_i \geq 0$ and risky assets (e.g. stocks or loans). Both banks can invest in risk-free bonds with rate of return $\mathsf{r}^{\mathsf{b}} > 0$, so that the return of bank i on bonds is deterministic and given by $B_i \cdot \mathsf{r}^{\mathsf{b}}$. We suppose that only bank 2 has access to risky investments, whose random rate of return we denote by ξ. By limited liability, the bank cannot lose more than what it invested, so $\xi \geq -1$. With S_2 the invested amount, the return on the risky investment is given by $S_2 \cdot \xi$. We denote the distribution of ξ by μ_ξ and its CDF by F_ξ. For bank 2 to have an incentive to make non-trivial risky investments we assume that $\mathbb{E}[\xi] > \mathsf{r}^{\mathsf{b}}$.

Initially, bank i holds only cash in the amount of $Q_i > 0$, and the only initial liabilities of bank i are deposits $D_i \geq 0$. Thus the initial equity of bank i (the difference between assets and liabilities) is given by $E_i^0 = Q_i - D_i$, which we assume to be positive for both banks.

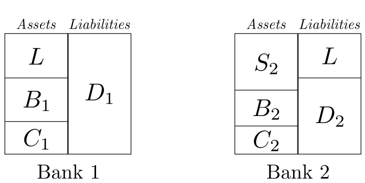

Fig. 1. Balance sheets of the two banks

B. Loans, Default and Clearing Mechanism

Since bank 1 cannot invest in stocks, it has an incentive to lend some cash $L \geq 0$ to bank 2 at an interest rate $r > \mathsf{r}^{\mathsf{b}}$. In doing so, it must take the credit risk into account, since there is a possibility bank 2 may default on such loan. Figure 1 shows the resulting balance sheets of the two banks.

In our model bank i defaults when its terminal equity E_i after realization of the uncertain return ξ is negative, i.e. when its liabilities are larger than its assets[1]. In case of a default of a borrowing bank, it is often possible for the lending bank to recover at least part of a loan. However, this may take a considerably long time [4], so for the purpose of this paper we assume that if the borrowing bank defaults, the lending bank loses the loan to the full amount L.

We regard a loan as a contract between a lending and a borrowing bank, which is fully specified by the amount L to be lent and the associated interest rate r. We assume that both banks are risk-neutral rational agents that maximize expected terminal equity, and that the lending bank has complete bargaining power. That is, bank 1 specifies the contract (L, r) and bank 2 can only accept it as it is or reject it; in the latter case bank 2 is said to exercise its outside option. We make the standard assumption that contracts are enforceable, i.e. bank 2 always pays $L(1+r)$ to bank 1 unless bank 2 defaults after realization of the uncertain return ξ.

The set of decision variables for bank 1 is $\{C_1, B_1, L, r\}$. Given that it accepts the offered loan, the set of decision variables for bank 2 is $\{S_2, B_2, C_2\}$. In this initial work we consider the full information case. In particular, the distribution μ_ξ of the asset return is common knowledge. In the absence of moral hazard and adverse selection (in this initial model the bank 2's action is perfectly observable, and there is no information asymmetry), we are interested in the first-best solution to the contract design problem of bank 1.

C. Regulatory Requirements

Banks are subject to a *Reserve Requirement* (RR), imposed by the regulator, which forces them to keep at least a fraction $\eta \in (0, 1)$ of their deposits in cash, so that bank i has to satisfy $C_i \geq \eta D_i$. In addition, banks must fulfill a *Capital Requirement* (CR), which puts a lower bound $\zeta \in (0, 1)$ on the ratio of equity, E_i, to risky assets[2]. In our case, the only risky assets are the loan L (for bank 1) and stocks S_2 (for bank 2), hence the CRs read as $L \leq (Q_1 - D_1)/\zeta$ and $S_2 \leq (Q_2 - D_2)/\zeta$. The balance equations

$$C_1 + B_1 + L = Q_1 \tag{1a}$$

$$C_2 + B_2 + S_2 = Q_2 + L \tag{1b}$$

require equality of assets and liabilities. Since in our model cash and risk-free asset do not differ in terms of liquidity, it is clear that the RR constraint is binding, i.e., $C_i = \eta D_i$.

D. Individual Rationality Constraints

Recall that $(Q_2 - D_2)/\zeta$ is the maximal investment in stocks allowed by CR, and that $Q_2 - \eta D_2$ are the available initial funds of bank 2. The *liquidity demand* of bank 2 is

$$\alpha_2 := (Q_2 - D_2)/\zeta - (Q_2 - \eta D_2)$$

If $\alpha_2 < 0$, the amount of cash available to bank 2 is more than the amount it can invest in stock. In this case both banks

[1] Since our model does not encompass different maturities, we do not have to distinguish between insolvency and illiquidity.

[2] In practice different assets are weighted according to their risk level [3].

have excess liquidity and no loan will be made. Thus from now on we assume that $\alpha_2 > 0$. In case of positive liquidity demand bank 2 has an incentive to pay interest $r \in [\mathsf{r}^{\mathsf{b}}, \mathbb{E}[\xi]]$ on the loan if the resulting expected increase in terminal equity exceeds that of its outside option. Since $\mathbb{E}[\xi] > \mathsf{r}^{\mathsf{b}}$ and bank 2 is risk-neutral, its optimal decision is to invest in stocks as much as allowed by CR, and use the rest to buy risk-free bonds B_2. By (1b), the optimal risky investment is

$$S_2 = \min(Q_2 - \eta D_2 + L,\, (Q_2 - D_2)/\zeta)$$

If no loan is made, i.e. $L = 0$, the expected increments of equity for banks 1 and 2 are $\Delta E_1 = (Q_1 - \eta D_1)\mathsf{r}^{\mathsf{b}}$ and $\Delta E_2 = (Q_2 - \eta D_2)\mathbb{E}[\xi]$, respectively. Hence, for a contract (L, r) to be accepted by bank 2 it must satisfy the following *Individual Rationality* (IR) constraint:

$$r \leq \begin{cases} \mathbb{E}[\xi] & \text{if } L \leq \alpha_2 \\ \mathsf{r}^{\mathsf{b}} + \frac{\alpha_2}{L}(\mathbb{E}[\xi] - \mathsf{r}^{\mathsf{b}}) & \text{if } L > \alpha_2 \end{cases} \tag{2}$$

Here the condition for $L > \alpha_2$ ensures that the loss bank 2 incurs by investing money borrowed at rate r into bonds returning $\mathsf{r}^{\mathsf{b}} < r$ is less than what it expects to earn from investing α_2 into stocks with expected return $\mathbb{E}[\xi] > r$.

E. The Optimal Contract Design Problem

For bank 1 to determine the optimal contract, it must understand the underlying credit risk. Bank 2 will not default if its investment decisions result in non-negative terminal equity $E_2 = Q_2 - D_2 + B_2\mathsf{r}^{\mathsf{b}} + S_2\xi$. If bank 2 accepts a loan (L, r), its *survival probability* under its optimal investment is $P_s(L, r) = \mathbb{P}(\xi \geq \vartheta(L, r)) = 1 - F_\xi(\vartheta(L, r))$, where

$$\vartheta(L, r) = \begin{cases} -\frac{Q_2 - D_2 - rL}{Q_2 - \eta D_2 + L} & \text{if } L \leq \alpha_2 \\ -\frac{Q_2 - D_2 - rL + (L - \alpha_2)\mathsf{r}^{\mathsf{b}}}{Q_2 - \eta D_2 + \alpha_2} & \text{if } L > \alpha_2 \end{cases} \tag{3}$$

The expected increase in bank 1's equity is

$$\Delta E_1(L, r) := \mathbb{E}[E_1 - E_i^0] = B_1\mathsf{r}^{\mathsf{b}} + L\,\mathsf{r}^{\mathsf{eff}}(L, r) \tag{4}$$

where the *effective lending rate* $\mathsf{r}^{\mathsf{eff}}$ given by

$$\mathsf{r}^{\mathsf{eff}}(L, r) := (1 + r)P_s(L, r) - 1 \tag{5}$$

describes the expected interest that bank 1 receives on the loan L when taking into account the default probability of bank 2. Note that $\mathsf{r}^{\mathsf{eff}} = r$ if $P_s = 1$ and $\mathsf{r}^{\mathsf{eff}} = -1$ if $P_s = 0$. Intuitively, the lending bank has to trade off a potentially high return (large L and r^{b}) with an increase in the probability that the borrowing bank defaults.

Using (1a) we can eliminate B_1 from (4) and obtain the following optimal contract design problem for bank 1:

$$\max_{L,r}\ (Q_1 - \eta D_1)\mathsf{r}^{\mathsf{b}} + L(\mathsf{r}^{\mathsf{eff}}(L, r) - \mathsf{r}^{\mathsf{b}}) \tag{6a}$$

$$\text{s.t.}\quad 0 \leq L \leq \min\{(Q_1 - D_1)/\zeta, Q_1 - \eta D_1\} \tag{6b}$$

$$r \leq \begin{cases} \mathbb{E}[\xi] & \text{if } L \leq \alpha_2 \\ \mathsf{r}^{\mathsf{b}} + \frac{\alpha_2}{L}(\mathbb{E}[\xi] - \mathsf{r}^{\mathsf{b}}) & \text{if } L > \alpha_2 \end{cases} \tag{6c}$$

Remark 1: Observe that both L and r affect P_s in a complex way via the distribution of the asset return ξ. In particular, the utility function of bank 1 does not satisfy common assumptions of the Principal-Agent literature [8] such as quasi-linearity.

We make the following standard assumption:

Assumption 1 (Existence of density): The distribution μ_ξ of the asset ξ is absolutely continuous w.r.t the Lebesgue measure and has density $f_\xi : \mathbb{R} \to \mathbb{R}_+$.

Let $u : \mathbb{R}^2 \to \mathbb{R}$ denote the objective function in (6a) and consider the hazard ratio $h_\xi := \frac{f_\xi}{1 - F_\xi}$ of ξ. Further, define

$$\theta(r) := \zeta\,(r\tfrac{\alpha_2}{Q_2 - D_2} - 1), \qquad \underline{\mathrm{r}} := -\tfrac{Q_2 - D_2}{Q_2 - \eta D_2}$$

Proposition 1: Suppose that for some $L \in [0, M]$

$$h_\xi(x) \leq \begin{cases} (x + \kappa(L))^{-1} \quad \forall\, x \in [\theta(\underline{\mathrm{r}}), \theta(\mathbb{E}[\xi])] & \text{if } L \leq \alpha \\ (x + \tilde{\kappa}(L))^{-1} \quad \forall\, x > \theta(\mathbb{E}[\xi]) & \text{if } L > \alpha_2 \end{cases} \tag{7}$$

where

$$\kappa(L) = \frac{Q_2 - D_2 + L}{Q_2 - \eta D_2 + L}$$

$$\tilde{\kappa}(L) = \frac{Q_2 - D_2 + (L - \alpha_2)\mathsf{r}^{\mathsf{b}}}{Q_2 - \eta D_2 + \alpha_2} + \frac{L}{Q_2 - \eta D_2 + L}$$

Then u is non-decreasing in r for the given L. In particular, if (7) holds $\forall\, L \in [0, M]$, then the IR constraint (6c) is tight.

The condition of Proposition 1 is satisfied by most distributions F_ξ of interest to our model. Intuitively speaking, in this case the effect of the size L of the loan on the default probability of the borrowing bank is stronger than that of the rate r. Hence, a lending bank that had to choose between a large loan L under a small rate r and a small loan $\hat{L}$ under a large rate $\hat{r}$, with $Lr = \hat{L}\hat{r}$, would prefer the latter.

Remark 2: It should not be surprising that the IR constraints are tight for a large class of distributions of ξ. Indeed, in standard principal-agent problems with quasi-linear utilities this is true in general under full information [8].

Proposition 2: Suppose that $f_\xi \in \mathcal{C}^1[-1, \infty)$ and that

$$\frac{d}{dx}\log f_\xi(x) < \frac{1}{2}\Big(\frac{Q_2 - D_2}{Q_2 - \eta D_2} + x\Big) \quad \forall\, x \in [\theta(\underline{\mathrm{r}}), \theta(\mathbb{E}[\xi])] \tag{8}$$

Then u is strictly concave in L for $L \leq \alpha_2$. Similarly, if

$$\frac{d}{dx}\log f_\xi(x) > -\frac{2}{x + \theta(r)} \quad \forall\, x > \theta(\mathbb{E}[\xi]) \tag{9}$$

then u is strictly concave in L for $\alpha_2 \leq L \leq M$.

Since u need not be differentiable at $L = \alpha_2$, (8) and (9) together do not imply that u is concave in L on $[0, M]$. However, if (8) and (9) hold, then finding the maximizer reduces to comparing the maximizers of u restricted to $0 \leq L \leq \alpha_2$ and $\alpha_2 \leq L \leq M$. We have found that for reasonable parameter choices (e.g. normally distributed ξ) conditions (7) and (8) were both satisfied with maximizer $L^* \leq \alpha_2$. The following Proposition provides a sufficient condition for the maximizer of problem (6) to be unique:

Proposition 3: Suppose that (7) holds for all $L \in [0, M]$, and that $f_\xi \in \mathcal{C}^1([-1, \infty))$ satisfies (8). Suppose further that

$$F_\xi(\theta(\mathbb{E}[\xi])) \leq \frac{1 + r^b}{1 + \mathbb{E}[\xi]} \tag{10}$$

Then (6) has a unique maximizer for which $r = \mathbb{E}[\xi]$.

If the assumptions of Proposition 3 hold true, one could in principle find the maximizer using calculus. However, in all but the simplest cases this problem will not admit an analytical solution. Instead, we use a gradient-based optimization algorithm, for which convergence to the global optimum is guaranteed by strict convexity of the problem.

III. A Mean-Field Equilibrium Model for Interbank Lending

In the previous section we have introduced a contract-based interbank lending model with one borrowing and one lending banks. Such model could be used as a building block for a larger network of bilateral interbank trading, leading to the game between multiple banks, so that we could define a Nash equilibrium w.r.t. the contract offerings. However, the full information case suffers from a number of issues, including the non-existence of non-trivial equilibria[3]. More importantly, it is questionable how realistic the full information assumption is: in reality a lending bank faces uncertainty regarding the borrowing alternatives of its potential borrower. For these reasons, in this section we relax the full information assumption and introduce an equilibrium model for interbank lending based on the concept of a mean field equilibrium.

Since information about bilateral interbank loans is rarely publicly available, reference rates such as the LIBOR [2] play an important role in financial markets. Such reference rates are often computed as a weighted average of either offered rates (e.g. LIBOR, EURIBOR) or actual rates in transfers made overnight (EONIA). In practice, after assessing the creditworthiness of a potential borrower, a lending bank will make an offer based on its beliefs regarding the rates the borrower could obtain from other banks. In non-crisis scenarios this rate is usually close to the reference rate, and hence the latter provides a reasonable estimate of the borrowing alternatives.

In the following we assume that lending banks only have beliefs regarding borrowing alternatives of their potential borrowers. These beliefs are modeled by a conditional probability distribution, parameterized by the global reference rate $\mathsf{r}^{\mathsf{ref}}$ and such observable characteristics of borrowers as creditworthiness. In turn, $\mathsf{r}^{\mathsf{ref}}$ is defined as a function of the individual bilateral contracts. We are interested in situations in which the offered contracts, chosen optimally with respect to the banks' respective beliefs, and the aggregate reference rate $\mathsf{r}^{\mathsf{ref}}$ are consistent. This consistency condition leads us to the definition of an equilibrium $\mathsf{r}^{\mathsf{ref}}$, which is determined endogenously through the parameters of the model.

A. Two-Bank Model with Borrowing Alternatives

We first consider as a building block an extension of the two-bank model from Section II. Suppose that in addition to bank 1, bank 2 also has the alternative to borrow from other sources, which at this point we do not model explicitly. Let ρ denote the borrowing rate that bank 2 can achieve on the lending market, so that bank 2 will only accept a contract from bank 1 if the specified rate r is at most ρ. We make the following simplifying assumption:

Assumption 2: The achievable borrowing rate ρ is independent of the size of the loan.

Assumption 2 is quite restrictive: in practice the interest on a loan may well depend on the size of the loan. If a functional parametrization for this relation were common knowledge, one could relax Assumption 2 and incorporate this into the formulation. However, while interesting, this extension goes beyond the scope of the current work.

Given the reference rate $\mathsf{r}^{\mathsf{ref}}$, the belief of bank 1 about the borrowing alternatives of bank 2 is given by a distribution $\mu_\rho(\cdot\,|\mathsf{r}^{\mathsf{ref}})$ whose CDF we denote by $F_\rho(r|\mathsf{r}^{\mathsf{ref}})$. In particular, $1-F_\rho(r|\mathsf{r}^{\mathsf{ref}})$ is the belief of bank 1 that bank 2 will accept a loan (satisfying IR) offered at rate r given reference rate $\mathsf{r}^{\mathsf{ref}}$.

We need two additional assumptions: the first one is a technical one that concerns the continuity of μ_ρ, while the second one asserts that the belief about the other bank's achievable rate "increases" with the reference rate: we formalize this using the notion of first order stochastic dominance (FOSD).

Assumption 3 (Continuous density): For each $\mathsf{r}^{\mathsf{ref}}$ the distribution $\mu_\rho(\cdot\,|\mathsf{r}^{\mathsf{ref}})$ is absolutely continuous w.r.t. the Lebesgue measure; the density function f_ρ is such that $f_\rho(r|\cdot) \in \mathcal{C}([\mathsf{r}^{\mathsf{b}}, \mathbb{E}[\xi]])$ for all $r \in [\mathsf{r}^{\mathsf{b}}, \mathbb{E}[\xi]]$.

Assumption 4 (FOSD): For each $\hat{\mathsf{r}}^{\mathsf{ref}} \geq \mathsf{r}^{\mathsf{ref}}$, $\mu_\rho(\cdot\,|\hat{\mathsf{r}}^{\mathsf{ref}})$ first order stochastically dominates $\mu_\rho(\cdot\,|\mathsf{r}^{\mathsf{ref}})$, i.e.

$$1 - F_\rho(r|\hat{\mathsf{r}}^{\mathsf{ref}}) \geq 1 - F_\rho(r|\mathsf{r}^{\mathsf{ref}}), \quad \forall\, r \in \mathbb{R}. \tag{11}$$

We retain the assumption that the lending bank has complete bargaining power, and that it may only offer a single contract (L, r). Intuitively, bank 1 faces a tradeoff between receiving a high premium when r is high, and a higher probability of bank 2 accepting the contract when r is low (that is, the additional borrowing alternatives increase competition). In particular, contrary to the basic problem discussed in Section II, borrowing banks will generally receive a profit that is strictly higher than their outside option.

We suppose that the lending market is liquid, which is usually the case in non-crisis scenarios, and assume that the borrowing bank can fulfill its liquidity demand, i.e., take on a total of α_2 in loans, at a rate ρ representing its borrowing alternatives. As a result, the lending bank expects the borrowing bank to survive with probability

$$\hat{P}_s(\mathsf{r}^{\mathsf{ref}}) = \int P_s(\alpha_2, r)\, \mu_\rho(\mathrm{d}r|\mathsf{r}^{\mathsf{ref}}). \tag{12}$$

In contrast to the case without borrowing alternatives, the lending bank assumes that bank 2 will always fulfill its liquidity demand. As a result, the survival probability $\hat{P}_s(\mathsf{r}^{\mathsf{ref}})$ is independent of the size L of the loan given by bank 1. If there is only a single borrowing bank (bank 2), bank 1 will therefore offer all its available liquidity to bank 2, provided that the effective rate $(1+r)\hat{P}_s(\mathsf{r}^{\mathsf{ref}}) - 1$ exceeds r^{b}.[4]

[3] Since in our basic model they cannot share profits, lending banks always have an incentive to underbid each others offerings, so in equilibrium each bank i would offer a rate r_i such that $\mathsf{r}_i^{\mathsf{eff}} = \mathbb{E}[\xi_i]$.

[4] If we suppose that banks must honor (if accepted) any loan offers made, then in case of multiple borrowing banks, banks with excess liquidity would need to decide how to optimally distribute their loans across borrowers.

The optimization problem of the lending bank is now in the variable r, with objective function u given by:

$$u(r|\mathsf{r}^{\mathsf{ref}}) = (1-F_\rho(r|\mathsf{r}^{\mathsf{ref}}))\big((1+r)\hat{P}_s(\mathsf{r}^{\mathsf{ref}}) - (1+\mathsf{r}^{\mathsf{b}})\big) \quad (13)$$

Proposition 4: Under Assumption 3, $u \in \mathcal{C}([\mathsf{r}^{\mathsf{b}}, \mathbb{E}[\xi]]^2)$. Furthermore, under Assumption 4, $\hat{P}_s$ is non-increasing.

Since $u(\cdot\,|\mathsf{r}^{\mathsf{ref}})$ is continuous on a compact interval, the set of its maximizers is compact and non-empty. Denote by $\varphi(\mathsf{r}^{\mathsf{ref}}) := \max(\arg\max_r u(r|\mathsf{r}^{\mathsf{ref}}))$ its largest element. Intuitively, one expects the optimal rate φ to grow with $\mathsf{r}^{\mathsf{ref}}$, as in this case the banks will have worse borrowing alternatives. The following lemma gives sufficient conditions for such monotonicity. Let $h_\rho(\cdot\,, \mathsf{r}^{\mathsf{ref}}) := f_\rho(\cdot\,|\mathsf{r}^{\mathsf{ref}})/(1-F_\rho(\cdot\,|\mathsf{r}^{\mathsf{ref}}))$ denote the hazard ratio of $\mu_\rho(\cdot\,|\mathsf{r}^{\mathsf{ref}})$.

Lemma 1 (Sufficient Condition for Monotonicity): Suppose $f_\rho(r|\cdot) \in \mathcal{C}^1([\mathsf{r}^{\mathsf{b}}, \mathbb{E}[\xi]])$ for all $r \in [\mathsf{r}^{\mathsf{b}}, \mathbb{E}[\xi]]$ and

$$\partial_{\mathsf{r}^{\mathsf{ref}}} h_\rho(r, \mathsf{r}^{\mathsf{ref}}) \leq 0 \quad (14)$$

for all $r, \mathsf{r}^{\mathsf{ref}} \in [\mathsf{r}^{\mathsf{b}}, \mathbb{E}[\xi]]$. Then $\varphi(\mathsf{r}^{\mathsf{ref}})$ is non-decreasing.

The next result provides sufficient conditions for the continuity of the map φ.

Lemma 2 (Sufficient Condition for Continuity): Suppose that $f_\rho(\cdot\,|\mathsf{r}^{\mathsf{ref}}) \in \mathcal{C}^2([\mathsf{r}^{\mathsf{b}}, \mathbb{E}[\xi]])$ for all $\mathsf{r}^{\mathsf{ref}} \in [\mathsf{r}^{\mathsf{b}}, \mathbb{E}[\xi]]$, and

$$\partial_r(\log f_\rho(r|\mathsf{r}^{\mathsf{ref}})) > -\frac{2\hat{P}_s(\mathsf{r}^{\mathsf{ref}})}{(1+r)\hat{P}_s(\mathsf{r}^{\mathsf{ref}}) - (1+r^b)} \quad (15)$$

for all $r, \mathsf{r}^{\mathsf{ref}} \in [\mathsf{r}^{\mathsf{b}}, \mathbb{E}[\xi]]$ whenever $f_\rho(r|\mathsf{r}^{\mathsf{ref}}) > 0$. Then $u(\cdot\,|\mathsf{r}^{\mathsf{ref}})$ has a unique maximizer $\varphi(\mathsf{r}^{\mathsf{ref}})$, which is a continuous function of $\mathsf{r}^{\mathsf{ref}}$.

B. Mean-Field Equilibrium for Multiple Lending Banks

In this section we define and analyze a mean-field equilibrium for the interbank lending problem involving multiple lending banks $i \in \{1, \ldots, N\}$ and a single representative borrowing bank, as illustrated in Figure 2. While so far the belief of a bank about borrowing alternatives had been specified for some exogenously given reference rate $\mathsf{r}^{\mathsf{ref}}$, we are now interested in the problem in which beliefs depend on the actual rates r_i offered by other banks.

For any fixed reference rate $\mathsf{r}^{\mathsf{ref}}$, each bank i with a liquidity surplus solves the optimal contract design problem as per Section III-A under its respective beliefs $\mu^i_\rho(\cdot\,|\mathsf{r}^{\mathsf{ref}})$ about the borrowing alternatives. In an equilibrium, the reference rate will itself be determined from the individual banks' decisions. We assume that $\mathsf{r}^{\mathsf{ref}}$ is computed as a weighted average of the individual bilateral rates[5] r_i:

Assumption 5: There exist weights $\lambda_1, \ldots, \lambda_N \geq 0$ with $\sum_i \lambda_i = 1$ such that $\mathsf{r}^{\mathsf{ref}} = \sum_i \lambda_i r^l_i$

The weights λ_i in Assumption 5 correspond to the "importance" of a bank in the financial system, as represented for example by its market share.

As has become evident from the LIBOR scandal [11], certain large banks in the past had indeed been aware of how their individual actions affected the reference rate. In particular, collusion between banks has been suspected to have played an important role in fixing the LIBOR rate. However, under new, stricter regulations put in place [16] that preclude the report of unverified rates it is reasonable to assume that individual banks cannot significantly affect $\mathsf{r}^{\mathsf{ref}}$ without reducing their profits. We therefore make the following simplifying assumption[6]:

Assumption 6 (Bounded Rationality): Banks do not take the effect of their contract offerings on the reference rate $\mathsf{r}^{\mathsf{ref}}$ into account. That is, from the point of view of a single bank i the reference rate $\mathsf{r}^{\mathsf{ref}}$ is independent of r_i.

The equilibrium concept we will define in the following can be seen as a Mean-Field Equilibrium (MFE) [9], [7] under the assumption that banks are bounded in their rationality. In the literature, the term "oblivious equilibrium" [15] has been used for similar models.

Definition 1 (Mean Field Equilibrium): Let $\varphi_i(\mathsf{r}^{\mathsf{ref}})$ denote the maximizer of the optimization problem for bank i in the presence of borrowing alternatives for a given reference rate $\mathsf{r}^{\mathsf{ref}}$. Further, consider the *aggregation function*

$$\psi(\mathsf{r}^{\mathsf{ref}}) = \textstyle\sum_i \lambda_i \varphi_i(\mathsf{r}^{\mathsf{ref}}) \quad (16)$$

Then $\mathsf{r}^{\mathsf{ref}}$ is a mean field equilibrium of the interbank lending problem (MFE $\mathsf{r}^{\mathsf{ref}}$) if $\mathsf{r}^{\mathsf{ref}} = \psi(\mathsf{r}^{\mathsf{ref}})$.

The following theorem provides sufficient conditions for the existence of the MFE $\mathsf{r}^{\mathsf{ref}}$.

Theorem 1: If $\partial_{\mathsf{r}^{\mathsf{ref}}} h_{\rho_i}(r|\mathsf{r}^{\mathsf{ref}}) \leq 0$ for all $r, \mathsf{r}^{\mathsf{ref}} \in [\mathsf{r}^{\mathsf{b}}, \mathbb{E}[\xi]]$ and $i \in \{1, \ldots, N\}$, then there exists a MFE $\mathsf{r}^{\mathsf{ref}}$.

Theorem 1 establishes existence of the MFE using set-theoretic lattice arguments. While it is not possible to guarantee uniqueness under the assumptions in place, it does follows from the Tarski fixed point theorem [13] that there exists a least and a largest MFE.

Rather than via monotonicity, existence of a MFE can also be obtained based on the continuity of the map ψ:

Theorem 2: If for all $r, \mathsf{r}^{\mathsf{ref}} \in [\mathsf{r}^{\mathsf{b}}, \mathbb{E}[\xi]]$ and $i \in \{1, \ldots, N\}$, (15) holds with $f_\rho = f^i_\rho$, then there exists a MFE $\mathsf{r}^{\mathsf{ref}}$.

Since ψ is continuous, the set of its fixed points is compact, and hence admits for the least and the largest elements similarly to the case when ψ is monotonic. Importantly, the continuity of ψ also allows using a variety of root-finding methods from the literature, as finding a fixed point of ψ is equivalent to solving $\psi(\mathsf{r}^{\mathsf{ref}}) - \mathsf{r}^{\mathsf{ref}} = 0$.

IV. Numerical Example

In this section we provide a numerical example that illustrates the properties of the MFE for interbank lending. All computations have been performed using a fixed point iteration on the aggregation function ψ, using IPOPT [14].

Parameters of our example are $\eta = 0.15$, $\zeta = 0.25$ and $\mathsf{r}^{\mathsf{b}} = 0.02$. We consider $N = 10$ lending banks with initial conditions Q_i and D_i generated uniformly at random on $[15, 100]$ and $[0, Q_i]$, respectively. We assume that the asset

[5] In particular, we assume that the reference rate is determined only from binding offers. That is, banks cannot report unverified rates and volumes, a possibility that contributed to the LIBOR scandal [11].

[6] Assumption 6 is easier to justify in competitive lending markets with a large number of lending banks. In small markets with only few large banks the situation may be different.

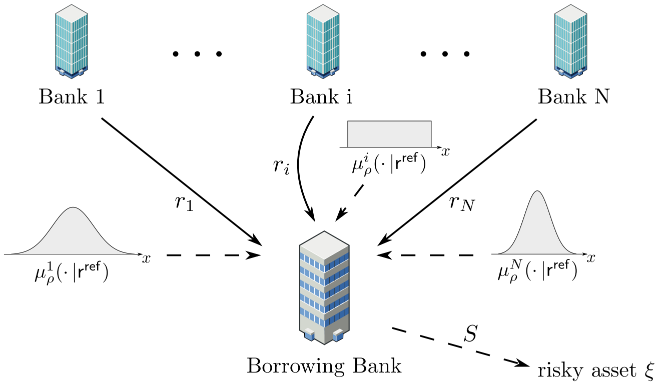

Fig. 2. Setting with Multiple Lending Banks

return ξ is distributed according to a truncated Gaussian on $[-1, \infty)$ with mean $\mathbb{E}[\xi]$ and standard deviation σ_ξ. Similarly, the beliefs $\mu_\rho^i(\cdot\,|\mathsf{r}^{\mathsf{ref}})$ are truncated Gaussians on $[\mathsf{r}^{\mathsf{b}}, \mathbb{E}[\xi]]$ with mean $\mathsf{r}^{\mathsf{ref}}$ and standard deviation σ_i.

Intuitively, one would expect the distribution of the asset return ξ to have a strong influence on the MFE $\mathsf{r}^{\mathsf{ref}}$. In particular, because of the increased credit risk, a "good" asset with high mean $\mathbb{E}[\xi]$ and low variance σ_ξ^2 should result in lower equilibrium rates $\mathsf{r}^{\mathsf{ref}}$ than a "bad asset" with low mean and high variance. The results of our simulations shown in Figure 3 corroborate this intuition[7].

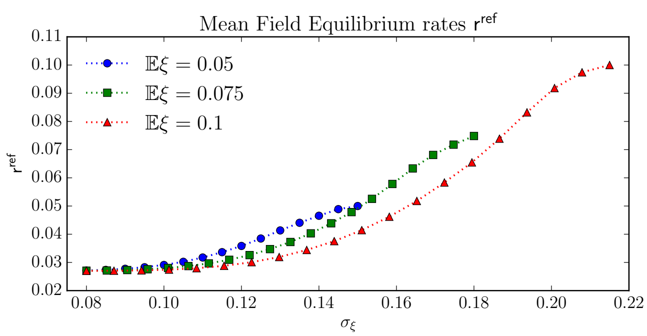

Fig. 3. Equilibrium reference rate $\mathsf{r}^{\mathsf{ref}}$ as a function of the asset return standard deviation σ_ξ for different means $\mathbb{E}[\xi]$

V. Conclusion and Future Work

We studied the formation of short-term interest rates in a simple model of interbank lending which is driven by heterogeneous liquidity requirements. Based on the analysis of an optimal contract design problem for bilateral loans we defined a Mean Field Equilibrium, in which banks with limited information act optimally with respect to their beliefs derived from a publicly available common reference rate. Our methodology allows to analyze how interbank lending rates form endogenously, and how they depend on various model parameters.

[7]Figure 3 shows the equilibrium rates $\mathsf{r}^{\mathsf{ref}}$ only until they reach the respective mean $\mathbb{E}[\xi]$ of the asset return. In our model there is only a single asset with a return distribution μ_ξ that is common knowledge. Hence if this asset is "sufficiently bad" then under any non-degenerate belief distribution that is consistent with the available information, all banks choose to offer $r_i = \mathbb{E}[\xi]$, which yields as payoff their outside option (as they expect it their offer to be rejected by the borrowing bank).

Several generalizations of this initial model are of interest. Among those is an extension to financial networks in which interactions are influenced by the underlying network structure, e.g., by trust established through long-term relationships between banks. Another interesting direction is to incorporate dynamics in a multi-period model, and investigate the role that assets and liabilities with different maturities play. However, the resulting dynamic problem becomes significantly more difficult.

We expect that the resulting models can provide valuable insights into how systemic risk in financial networks arises based on microscopic interactions of banks that are modeled as profit-maximizing agents with bounded rationality.

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