Regularization of Bellman equations for infinite-horizon probabilistic properties*

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ABSTRACT

This work studies Bellman integral equations arising in infinite-horizon probabilistic verification problems for discrete time homogeneous Markov processes over general state spaces. The problems of interest are expressed via specifications such as probabilistic reachability, invariance, reach-avoid and mean exit time. The contribution shows that the uniqueness of the solutions of the corresponding Bellman equations depends on the presence of absorbing sets within the state space. Furthermore, the work puts forward methods to modify the integral equations to obtain unique solutions for them, techniques to compute such solutions with explicit bounds on the approximation error, and conditions to characterize the possible presence of absorbing sets over the state space.

Keywords

Markov processes, probabilistic properties, Bellman equations, absorbing sets.

1. INTRODUCTION

Bellman recursions and related Bellman fixpoint equations play a prominent role in various problems in optimal control, verification, operations research and economics.

In the context of probabilistic verification, Bellman recursions are used to express properties formulated in PCTL, a modal logic that accommodates for probabilistic requirements [3] that can be efficiently computed via model checkers [7, 9]. Verification of PCTL specifications has been widely studied over stochastic processes with countable spaces, but only in part addressed for processes on continuous (uncountable) spaces [1]. In the latter case, and with regards to finite time horizon specifications, [1] has provided explicit bounds on the error of computation via Bellman recursions using a state space discretization approach. On the other hand, such results are missing for the important instance of infinite time horizon properties. In this challenging case, specifications are shown to be characterized in two possible ways: either as limits of Bellman recursions, or as solution of Bellman integral equations. From the second perspective, [11] has provided necessary conditions for Bellman equation to have a unique solution. However, due to the conservatism of these conditions, a large class of processes is left uncovered.

The focus of this paper is on understanding when the solution of a given Bellman equation is unique, and on how to compute it with explicit convergence rates or with explicit bounds on the approximation error. The contribution shows that the uniqueness of the solution of the considered class of integral equations depends on the presence of absorbing sets within the state space. In particular, the absence of absorbing sets allows us not only to establish uniqueness, but also to put forward techniques to compute these solutions. On the other hand, in the presence of absorbing sets this work proposes explicit modifications of the Bellman equations, which allow for unique solutions and for approximations methods with explicit error bounds and convergence rates. To the best of authors' knowledge, there exist no general methods to compute infinite time horizon probabilistic properties with explicit bounds on the error. Hence, the goal of the paper is to develop such methods, while leaving computational improvements and scalability issues to later work.

Given the central role played by absorbing sets for this class of problems, this paper also investigates in what instances or under what conditions it is possible to ensure the absence of such sets, or conversely to either characterize them or to approximately compute them.

The contribution is structured as follows. Section 2 introduces the notations and explains the problem under investigation. As reference properties, the contribution looks at probabilistic reachability, invariance, reach-avoid and mean exit time (see Section 2.3). Section 2.2 introduces a case study, employed to clarify and give examples for the discussed notions and presented results. Section 3 deals with uniqueness issues, regularizations, and computations of a number of different properties. Furthermore, Section 4 studies the characterization of absorbing sets.

2. PRELIMINARIES

2.1 Notations and basic concepts

We consider a homogeneous Markov process X in discrete time. The state space $\mathscr{X}$ is a Borel space and its Borel σ-algebra

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is denoted as $\mathscr{B}(\mathscr{X})$[6]. The process X is characterized by its transition kernel $T(A|x)$, which is such that $x \mapsto T(A|x)$ is a measurable function for any $A \in \mathscr{B}(\mathscr{X})$, and such that $T(\cdot|x)$ is a probability measure on $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$, for all $x \in \mathscr{X}$.

The sample space is a space of trajectories $\Omega = \mathscr{X}^{\mathbb{N}_0}$ endowed with a product σ-algebra $\mathscr{F}$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. X_k is the value of X at the time k. With P_x we denote the probability measure induced by T, such that $\mathsf{P}_x\{X_0 = x\} = 1$, and with E_x the corresponding expectation.

We say that a random variable τ is a stopping time if for all $n \geq 0$ it holds that $\{\tau \leq n\} \in \mathscr{F}_n$. For a set $B \in \mathscr{B}(\mathscr{X})$ we define the first hitting time $\tau_B = \inf\{n \geq 0 : X_n \in B\}$ and the first exit time $\varsigma_B = \tau_{B^c}$, both of which are clearly stopping times.

The space $\mathbb{B}(\mathscr{X})$ of all measurable functions $f : \mathscr{X} \to \mathbb{R}$ that are bounded on $\mathscr{X}$ is a Banach space with a norm $\|f\| = \sup_{x\in\mathscr{X}} |f(x)|$ [5]. For any $B \in \mathscr{B}(\mathscr{X})$ its indicator function $1_B(x)$ is in $\mathbb{B}(\mathscr{X})$ with $\|1_B\| = 1$.

If $\mathscr{J}$ is an operator acting on $\mathbb{B}(\mathscr{X})$, then we define the norm of $\mathscr{J}$ as follows:

$$\|\mathscr{J}\| = \sup_{f\in\mathbb{B}(\mathscr{X})} \frac{\|\mathscr{J}f\|}{\|f\|}.$$

If for an operator $\mathscr{J}$ it holds that $\|\mathscr{J}f_1 - \mathscr{J}f_2\| \leq \alpha\|f_1 - f_2\|$ for all $f_1, f_2 \in \mathbb{B}(\mathscr{X})$ and some $\alpha < 1$, then we say that $\mathscr{J}$ is a contraction (with rate α). In addition, with $\mathscr{C}(B)$ we denote the class of real-valued functions defined on $\mathbb{B}(\mathscr{X})$ that are continuous on $B \in \mathscr{B}(\mathscr{X})$ and we put $\mathbb{L}(\mathscr{X})$ to be the space of all measurable functions $f : \mathscr{X} \to (-\infty, \infty]$.

Since $\mathscr{X}$ is a subset of a Polish space, it is metrizable. We select a generic metric and denote it by ρ. Finally, for any subset $A \subseteq \mathscr{X}$ we define by $\bar{A}$ the closure of A, by A° the interior of A, and by ∂A the boundary of A. For any $a, b \in \mathbb{R}$ we put $a \wedge b = \min\{a, b\}$.

In this work we are focused on transition kernels T that admit a Lipschitz continuous density.

Assumption 1. *Assume that there exists a σ-finite Borel measure μ on $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$ such that $\mu(B) < \infty$ for any compact set B. In addition, let there exist a Lipschitz continuous function $\xi : \mathscr{X} \times \mathscr{X} \to \mathbb{R}_{\geq 0}$ for any $A \in \mathscr{B}(\mathscr{X})$, satisfying*

$$T(A|x) = \int_A \xi(x, y)\mu(dy). \tag{2.1}$$

Function ξ is the density of the transition kernel T with respect to the measure μ. By $\beta \in (0, \infty)$ we denote the Lipschitz constant of ξ. Note that a large class of processes admits Assumption 1, for instance Markov Chains [10] and dtSHS [1].

Remark 1 (On Assumption 1). *By [6, Example C.6, p. 176] Assumption 1 is sufficient for X to admit the strong Feller property [6, Appendix C, p. 174] hence the results of [14] are applicable.*

2.2 Case study

In order to illustrate the results of this work, we use a Markov process X as a benchmark to do verification of its infinite-horizon properties. The state space of X is hybrid, namely given by $\mathscr{X} = \bigcup_{i=1}^{4} \{l_i\} \times [0, 1]$, see Figure 1. $L = \{l_i\}_{i=1}^4$ is a set of discrete locations, each of which is associated to the continuous interval $[0, 1]$. For any $x \in \mathscr{X}$ we write $x = (l_x, c_x)$, where $l_x \in L$ and $c_x \in [0, 1]$. The metric on $\mathscr{X}$ is given as

$$\rho(l_x, c_x, l_y, c_y) = \begin{cases} 1, & \text{if } l_x \neq l_y; \\ |c_x - c_y|, & \text{if } l_x = l_y. \end{cases}$$

Such a metric endows $\mathscr{X}$ with the Borel σ-algebra $\mathscr{B}(\mathscr{X})$. Clearly, any $A \in \mathscr{B}(\mathscr{X})$ admits a unique representation $A = \bigcup_{i=1}^{4} \{l_i\} \times A^i$ where $A^i \in \mathscr{B}([0, 1])$ for $1 \leq i \leq 4$. The measure μ on $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$ is such that $\mu(A) = \sum_{i=1}^{4} \lambda(A^i)$, where λ is the Lebesgue measure over $[0, 1]$.

In the following, the probability density function of the normal distribution is denoted by

$$f_{\mathscr{N}}(x, y, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-x)^2}{2\sigma^2}},$$

and we also define $\Phi(x, y, \sigma) = \int_0^y f_{\mathscr{N}}(x, s, \sigma)\, ds$.

At each time step the process X moves according to the following rule: first, it selects a discrete location according to a discrete distribution that depends on the current state; then, it resets over the continuous domain $[0, 1]$ according to a continuous probabilistic law that depends also on the new location. These semantics can be precisely characterized by the transition kernel T as:

$$\xi(l_x, c_x, l_y, c_y) = \sum_{i,j=1}^{4} \delta_{ij}(l_x, l_y) p_{ij}(c_x) \xi_{ij}(c_x, c_y).$$

Here $\delta_{ij}(l_x, l_y) = 1$ only if $l_x = i, l_y = j$, and zero otherwise. The functions p_{ij} and ξ_{ij} are location-dependent and different from zero only if (see Figure 1)

$$(i, j) \in \{(1, 1), (2, 3), (2, 4), (3, 3), (3, 4), (4, 2), (4, 4)\}.$$

The quantity $p_{11} \equiv 1$, since from location l_1 jumps to other locations are not allowed. The corresponding continuous density is a truncated normal with $\xi_{11}(c_x, c_y) = \frac{f_{\mathscr{N}}(c_x, c_y, 0.3)}{\Phi(c_x, 1, 0.3)}$.

From location l_2 discrete jumps are allowed to any other, except l_3. The discrete jump distribution is given by functions $p_{21}(c_x) = \frac{c_x^2}{2}, p_{24}(c_x) = \frac{c_x}{2}$ and $p_{22}(c_x) = 1 - \frac{1}{2}(c_x + c_x^2)$. The continuous reset distribution over the new locations is simply uniform: $\xi_{21} = \xi_{24} \equiv 1$, and is truncated normal if the location does not change: $\xi_{22}(c_x, c_y) = \frac{f_{\mathscr{N}}(c_x, c_y, 0.4)}{\Phi(c_x, 1, 0.4)}$.

For the last two locations we have $p_{33}(c_x) = c_x^2, p_{44}(c_x) = c_x$ and $p_{34}(c_x) = 1 - c_x^2, p_{42}(c_x) = 1 - c_x$. Finally,

$$\xi_{33}(c_x, c_y) = \xi_{44}(c_x, c_y) = \frac{f_{\mathscr{N}}(c_x, c_y, 0.5)}{\Phi(c_x, 1, 0.5)}$$

and

$$\xi_{34}(c_x, c_y) = \xi_{42}(c_x, c_y) \equiv 1.$$

2.3 Specifications and related operators

This work is focused on operators and corresponding fixpoint equations arising in the theory of discrete-time Markov processes. As in [4] we are especially interested in the study of properties related to the first exit time from some set $A \in \mathscr{B}(\mathscr{X})$. Most of them can be expressed through the distribution of ς_A.

Denote $p_n(x; A) = \mathsf{P}_x\{\varsigma_A = n\}$ and $p(x; A) = \mathsf{P}_x\{\varsigma_A = \infty\}$. The *reachability* value function is the cumulative distribution function for ς_A and is given by $v_n(x; A^c) = \mathsf{P}_x\{\varsigma_A \leq n\}$.

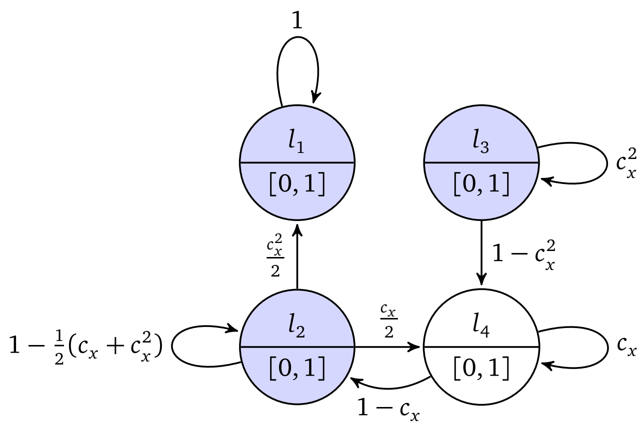


Figure 1: Case study: gray nodes denote the safe set.

The *invariance* value function is the tail probability for ς_A, i.e. $u_n(x;A) = \mathsf{P}_x\{\varsigma_A > n\} = 1 - v_n(x;A^c)$. It is worth mentioning that for the infinite-horizon invariance $u(x;A) := \lim_{n\to\infty} u_n(x;A) = p(x;A)$ and clearly, $u(x;A) = 1 - v(x;A^c)$.

Besides the distribution of ς_A, one important characteristic is its mean $t(x;A) = \mathsf{E}_x[\varsigma_A]$: we will show how the properties of this function depend on the invariance value function $u(x;A)$.

Finally, we consider the reach-avoid value function denoted as $w_n(x;A,B) = \mathsf{P}_x\{\tau_B < \varsigma_A \wedge n\}$ for any $A,B \in \mathscr{B}(\mathscr{X}), n \geq 0$, and the infinite-horizon version $w(x;A,B) = \mathsf{P}_x\{\tau_B < \varsigma_A \wedge \infty\} = \lim_{n\to\infty} w_n(x;A,B)$[13].

To compute the value of these functions an operator approach is used. Two basic operators, namely the linear *transition* $\mathscr{P}$ and the *invariance* $\mathscr{I}_A$, are defined as follows:

$$\mathscr{P} f(x) = \mathsf{E}_x[f(X_1)] = \int_{\mathscr{X}} f(y) T(dy|x).$$

$$\mathscr{I}_A f(x) = 1_A(x) \mathscr{P} f(x).$$

Clearly, it holds that $\|\mathscr{P}\| = 1$ and that in general $\|\mathscr{I}_A\| \leq 1$. By *Bellman equation* we refer to an equation of the form

$$f(x) = g(x) + \mathscr{I}_A f(x), \tag{2.2}$$

with $A \in \mathscr{B}(\mathscr{X})$ and $g \in \mathbb{B}(\mathscr{X})$. The choice of g determines the related problem of probabilistic verification, namely

$$f(x) = \mathscr{I}_A f(x) \tag{2.3}$$

for the invariance problem and

$$f(x) = 1_B(x) + \mathscr{I}_{A\setminus B} f(x) \tag{2.4}$$

for the reach-avoid one. Let us introduce the shorthand

$$\mathscr{R}_{A,B} f(x) := 1_B(x) + \mathscr{I}_{A\setminus B} f(x).$$

A solution for equations (2.3) or (2.4) always exists in $\mathbb{B}(\mathscr{X})$ and is given by the value functions $u(x;A)$ and $w(x;A,B)$ respectively, which are constructed as follows:

$$\begin{cases} u_{n+1}(x;A) &= \mathscr{I}_A u_n(x;A) \\ u_0(x;A) &= 1_A(x); \end{cases}$$

and

$$\begin{cases} w_{n+1}(x;A,B) &= \mathscr{R}_{A,B} w_n(x;A,B) \\ w_0(x;A,B) &= 1_B(x), \end{cases}$$

with $u(x;A) = \lim_{n\to\infty} u_n(x;A)$ and $w(x;A,B) = \lim_{n\to\infty} w_n(x;A,B)$ pointwise and monotonically [11, 14].

For the mean exit time we have $g(x) = 1_A(x)$, thus the Bellman equation is given by

$$f(x) = 1_A(x) + \mathscr{I}_A f(x) \tag{2.5}$$

as in [10], and the solution exists in $\mathbb{L}(\mathscr{X})$ but may not exist in $\mathbb{B}(\mathscr{X})$. Such a solution is given by the value function

$$t(x;A) = \sum_{n=1}^{\infty} n p_n(x;A) + p(x;A) \cdot \infty \tag{2.6}$$

where $p(x;A) \cdot \infty = \infty$ if $p(x;A) > 0$, and is equal to 0 otherwise.

Finally, we introduce the operator $\mathscr{H}_A$ defined for any function $f \in \mathbb{B}(\mathscr{X})$ and set $A \in \mathscr{B}(\mathscr{X})$ by

$$\mathscr{H}_A f(x) = \mathscr{P} 1_A(x) f(x),$$

which is to be used throughout the work.

3. UNIQUE SOLUTIONS OF BELLMAN EQUATIONS

3.1 Simple and absorbing sets

The Bellman equation in (2.2) is a linear equation, so whenever it is solved on $\mathbb{B}(\mathscr{X})$ the uniqueness of its solution holds if and only if the solution is unique for its homogeneous version (2.3), which happens to be the equation for the invariance value function. Due to this reason we first study the properties of equation (2.3).

Proposition 1. *For any $n \geq 0$ it holds that $u_n(x;A) \in \mathscr{C}(A)$.*

Proof. The proof immediately follows from Remark 1. □

Lemma 1. *Equation (2.3) has a unique solution if and only if $u(x;A) = 0$ for all $x \in \mathscr{X}$.*

Proof. Equation (2.3) is linear and homogeneous, so if it has a unique solution then it is the zero solution. The other direction was proved in [11, Proposition 9]. □

Based on the results above, we have to verify the triviality of the invariance problem: namely, if its value function is equal to zero everywhere which is clearly equivalent to being equal to zero just on A. This instance can be characterized by the presence of absorbing sets, which are defined in the following.

Definition 1 (Absorbing and simple sets). *A non-empty set $A' \in \mathscr{B}(\mathscr{X})$ is called absorbing if for all $x \in A'$ it holds that*

$$T(A'|x) = 1.$$

Given a set $A \in \mathscr{B}(\mathscr{X})$, the set A' is the largest absorbing subset of A if for any absorbing set $A'' \subseteq A$ it holds that $A'' \subseteq A'$. A set $A \in \mathscr{B}(\mathscr{X})$ is called simple if it does not contain any absorbing set.

3.2 Simplicity and uniqueness of solution of Bellman equations

Lemma 2. *[14, Theorem 3] Let A be a compact set, then it is simple if and only if $u(x;A) = 0$ for all $x \in \mathscr{X}$.*

If A is compact, then it is simple if and only if equation (2.3) has a unique (zero) solution. This fact leads to the following questions:

1. how to verify if a given compact set A is simple?
2. how to solve equation (2.3) if the compact set A is not simple?

The rest of this Section provides an answer to these questions. Given a set $A \in \mathscr{B}(\mathscr{X})$ define for any $n \geq 0$

$$A_n = \{x \in A : u_n(x;A) = 1\}, \qquad A_0 = A.$$

LEMMA 3. *[14, Lemma 1] For all $n \geq 0$ it holds that $A_{n+1} \subseteq A_n$ and*

$$A_{n+1} = \{x : T(A_n|x) = 1\}. \tag{3.1}$$

Since $A_{n+1} \subseteq A_n$ is a non-increasing sequence of sets, the limit $A_\infty := \bigcap_{n=0}^{\infty} A_n$ always exists, though it may be empty. Moreover, by [14, Theorem 3] if A is not simple, then A_∞ is its largest absorbing subset.

REMARK 2. *The sequence A_n can be effectively used to verify the simplicity of a set A as well as to overapproximate its largest absorbing subset.*

Let us introduce the quantity

$$m(A) = \inf\{m \geq 0 : \sup_{x \in \mathscr{X}} u_m(x,A) < 1\}$$

and $\alpha(A) = \sup_{x \in \mathscr{X}} u_{m(A)}(x,A)$.

LEMMA 4. *A compact set A is simple if and only if $m(A) < \infty$.*

PROOF. If $m(A) < \infty$, then by [14, Theorem 2] the solution of the invariance problem is trivial: $u(x;A) = 0$ for all $x \in \mathscr{X}$. Thus A is simple. Let us suppose now that A is simple but $m(A) = \infty$. Consider a function $u_n(x;A), n \geq 0$. Since by Proposition 1 it is a continuous function on a compact set A with $\sup_{x \in \mathscr{X}} u_n(x,A) = 1$, it holds that A_n is a non-empty compact set. Thus, A_∞ is also non-empty, which contradicts the simplicity of A. □

THEOREM 1. *A compact set A is simple if and only if $\mathscr{I}_A^m$ is a contraction for some $m > 0$. In this case for any $g \in \mathscr{B}(\mathscr{X})$ Equation (2.2) admits a solution $f(x) \in \mathscr{B}(\mathscr{X})$, which is unique in $\mathbb{B}(\mathscr{X})$ and given by $f(x) = \lim_{n\to\infty} f_n(x)$ with any $f_0 \in \mathbb{B}(\mathscr{X})$ and such that*

$$f_{n+1}(x) = \mathscr{J}^{m(A)+1} f_n(x).$$

Here $\mathscr{J}$ is given as $\mathscr{J}h(x) = g(x) + \mathscr{I}_A h(x)$ for any $h \in \mathbb{B}(\mathscr{X})$. Moreover, the following bound holds for all $n \geq 0$

$$\|f - f_n\| \leq \frac{\alpha^n(A)}{1 - \alpha(A)} \|\mathscr{J}^{m(A)+1} f_0 - f_0\|.$$

PROOF. If $\mathscr{I}_A^m$ is a contraction, then the solution of (2.3) is unique, hence A is simple. On the other hand, if A is simple then $m(A) < \infty$ and

$$\mathscr{I}_A^{m(A)+1} f(x) \leq \mathscr{I}_A^{m(A)} \|f\| 1_A(x) \leq \alpha(A)\|f\|,$$

which proves that $\mathscr{I}_A^{m(A)+1}$ is contractive with rate $\alpha(A)$. This fact implies that for any $g \in \mathscr{B}(\mathscr{X})$ the operator $\mathscr{J}^{m(A)+1}$ is a contraction on $\mathbb{B}(\mathscr{X})$ with rate $\alpha(A)$, and the rest of the proof immediately follows from the Contraction Mapping Theorem [8]. □

REMARK 3. *The finite horizon value function f_n can be computed with explicit bounds on the error, for example with the space discretization algorithm in [1].*
The paper [11] showed a special instance of this theorem, namely that $m(A) = 1$ implies contractivity and uniqueness.

Below we also need the following corollary.

COROLLARY 1. *If A is compact and simple then $\mathscr{H}_A^{m(A)+1}$ is a contraction on $\mathbb{B}(\mathscr{X})$.*

PROOF. For all $x \in A$ it holds that $(\mathscr{H}_A^n 1)(x) = u_n(x;A)$ so $\sup_{x\in A}(\mathscr{H}_A^{m(A)} 1)(x) = \alpha(A) < 1$. Now, for an arbitrary $f \in \mathbb{B}(\mathscr{X})$:

$$\begin{aligned}\left\|\mathscr{H}_A^{m(A)+1} f\right\| &\leq \int_A \left\|\mathscr{H}_A^{m(A)} f(x)\right\| \xi(x,y)\mu(dy) \\ &\leq \int_A \alpha(A)\|f\|\xi(x,y)\mu(dy) \leq \alpha(A)\|f\|.\end{aligned}$$

□

3.3 Modified Bellman equations over sets that are not simple

We employ Theorem 1 to provide an answer to the second question above: how to solve (2.3) if the compact set A is not simple. We put forward a modification of the original Bellman equation that has a unique solution given by the function $u(x;A) = \mathsf{P}_x\{\varsigma_A = \infty\}$.

We start by discussing some additional facts about simplicity and absorbance.

LEMMA 5. *Let A and A' be respectively simple and absorbing sets, then for any $C \in \mathscr{B}(\mathscr{X})$ such that $\mu(C) = 0$ it holds that $A \cup C$ is simple and $A' \setminus C$ is absorbing. In particular, $\mu(A') > 0$.*

PROOF. Consider an arbitrary set $C \in \mathscr{B}(\mathscr{X})$ of zero measure. If A' is absorbing then for any $x \in A' \setminus C$ it holds that

$$T(A'\setminus C|x) = \int_{A'\setminus C} \xi(x,y)\mu(dy) = \int_{A'} \xi(x,y)\mu(dy) = T(A'|x) = 1,$$

so the absorbance of $A' \setminus C$ is proved. Now let the set A be simple. Suppose that $A \cup C$ is not simple, so there is an absorbing set $A' \subseteq A \cup C$. This means that $A' \setminus C \subseteq A$ is an absorbing set, which contradicts the simplicity of A. □

LEMMA 6. *If $\mu(A) < \infty$ then $u(x;A)$ is Lipschitz on A.*

PROOF. Recall that $u(x;A)$ is a solution of (2.3). Now, for any $x', x'' \in A$ we have

$$\begin{aligned}|u(x';A) - u(x'';A)| &\leq \int_A |\xi(x',y) - \xi(x'',y)|\mu(dy) \\ &\leq \beta\mu(A)\rho(x',x'').\end{aligned}$$

□

Note that if A is compact and hence of finite measure, the sequence of continuous functions $u_n(x;A)$ converges pointwise and monotonically to a continuous function $u(x;A)$, thus by Dini's theorem [12] the convergence is uniform on A.

THEOREM 2 (MODIFIED BELLMAN EQUATION). *Let $A \in \mathscr{B}(\mathscr{X})$ be compact, $A' \subset A$ be the largest absorbing subset of A, and $\mu(\partial A') = 0$. Then $u(x;A)$ is the unique solution of the equation*

$$u(x;A) = 1_A(x) + \mathscr{I}_{A\setminus A'} u(x;A). \tag{3.2}$$

PROOF. Let us start from the original Bellman equation (2.3) and raise trivial restrictions on its solution:

$$u(x;A) = \begin{cases} 1, \text{ if } x \in A', \\ 0, \text{ if } x \in A^c. \end{cases}$$

Using this information, we obtain from (2.3):

$$u(x;A) = 1_A(x)T(A'|x) + 1_A(x)\int\limits_{A\setminus A'} u(y;A)\xi(x,y)\mu(dy).$$

If $x \in A'$, then $\int\limits_{A\setminus A'} u(y;A)\xi(x,y)\mu(dy) = 0$, hence

$$u(x;A) = 1_A(x)T(A'|x) + 1_{A\setminus A'}(x)\int\limits_{A\setminus A'} u(y;A)\xi(x,y)\mu(dy),$$

or equivalently $u(x;A) = 1_{A'}(x) + \mathscr{I}_{A\setminus A'}u(x;A)$. We are left to show that (3.2) has a unique solution. First, $A \setminus A'$ is simple and since $\mu(\partial A') = 0$ the set $A^* = \overline{A\setminus A'}$ is simple by Lemma 5. Moreover, it is a closed subset of a compact set A, so A^* is compact. By Theorem 1 there exists a finite $m < \infty$ such that $\mathscr{I}^m_{A^*}$ is a contraction, so the same holds for $\mathscr{I}^m_{A\setminus A'}$. This implies that (3.2) has a unique solution in $\mathbb{B}(\mathscr{X})$. $\square$

REMARK 4. *The condition $\mu(\partial A') = 0$ is used to avoid some degenerate cases – for instance, when A' is a Smith-Volterra-Cantor set in $\mathbb{R}$, which has a boundary of finite Lebesgue measure.*
A computational note: since $\mathscr{I}_{A\setminus A'}$ in (3.2) is a contraction on $\mathbb{B}(\mathscr{X})$, the invariance value function can be found by the iterative procedure – with associated bounds – given in Theorem 1.

For the reach-avoid problem a similar result easily follows.

COROLLARY 2. *Let $A, B \in \mathscr{B}(\mathscr{X})$ be such that $\mu(A\setminus B) < \infty$, then $w(x;A,B) \in \mathrm{Lip}(A\setminus B)$. If in addition $A\setminus B$ is compact and A' is the largest absorbing subset of $A\setminus B$ and is such that $\mu(\partial A') = 0$, then*

$$w(x;A,B) = w(x;A^*,B) \tag{3.3}$$

where $A^* = A \setminus (A')^\circ$.

3.4 Approximate solution of modified Bellman equations

To derive the modified Bellman equations we have explicitly assumed to have knowledge of the largest absorbing subset A' of a set A. Here we relax this assumption and employ only an over- and under-approximation of the set A': let us consider sets C, D, such that $C \subseteq A' \subseteq D$ and so that D is open. Based on the proof of Theorem 2 it is clear that for any set A it holds that

$$u(x;A) = 1_A(x)T(A'|x) + 1_A(x)\int\limits_{A\setminus A'} u(y;A)\xi(x,y)\mu(dy). \tag{3.4}$$

Let us introduce an approximation $\tilde{u}$ of u, as the solution of the following integral equation:

$$\tilde{u}(x) = 1_A(x)T(D|x) + 1_A(x)\int\limits_{A\setminus D} \tilde{u}(y)\xi(x,y)\mu(dy). \tag{3.5}$$

Next we show that $\|\tilde{u} - u\| \le M\mu(D\setminus C)$, for a finite constant M, and then provide a method to calculate $\tilde{u}$ with any desired accuracy.

THEOREM 3. *Let A be a compact set. The following bound holds:*

$$\|\tilde{u} - u\| \le \frac{\xi_2(\xi_1^{m+1} - 1)}{(\xi_1 - 1)(1-\alpha)}\mu(D\setminus C), \tag{3.6}$$

where $m = m(A\setminus D), \alpha = \alpha(A\setminus D)$. Here ξ_1, ξ_2 are given by $\xi_1 = \sup\{\xi(x,y)|x \in A\setminus C, y \in A\setminus D\}$, $\xi_2 = \sup\{\xi(x,y)|x \in A\setminus C, y \in D\setminus C\}$.

PROOF. Let us define a function $h(x) = \tilde{u}(x) - u(x;A)$. Since $h(x) = 0, \forall x \in A^c$, let us focus on the case $x \in A$. From (3.4) and (3.5) it follows that

$$h(x) = g(x) + \mathscr{H}_{A\setminus D}h(x), \tag{3.7}$$

where $g(x) = \int\limits_{D\setminus A'} (1 - u(y;A))\xi(x,y)\mu(dy)$.

From Corollary 1 we have that $\|\mathscr{H}^{m+1}_{A\setminus D}f\| \le \alpha\|f\|$ for any $f \in \mathbb{B}(\mathscr{X})$. Now, (3.7) is equivalent to

$$h(x) = \sum_{k=0}^{m}\mathscr{H}^k_{A\setminus D}g(x) + \mathscr{H}^{m+1}_{A\setminus D}h(x),$$

hence $(1-\alpha)\|h(x)\| \le \sum\limits_{k=0}^{m}\|\mathscr{H}^k_{A\setminus D}g(x)\|$. Since $\|\mathscr{H}^k_{A\setminus D}g(x)\| \le \xi_1^k\xi_2\mu(D\setminus C)$, we obtain (3.6). $\square$

Having derived bounds on $\|\tilde{u} - u\|$, we need a procedure to calculate $\tilde{u}(x)$ for all $x \in A$. To do this we define the following sequence: $\tilde{u}_0(x) = 1_A(x)T(D|x)$ and

$$\tilde{u}_{n+1}(x) = 1_A(x)T(D|x) + 1_A(x)\mathscr{H}_{A\setminus D}\tilde{u}_n(x).$$

PROPOSITION 2. *Given $m = m(A\setminus D)$ and $\alpha = \alpha(A\setminus D)$, for all $n > 0$ the following bound holds:*

$$\|\tilde{u} - \tilde{u}_n\| \le \frac{m+2}{1-\alpha}\alpha^{\left\lfloor \frac{n}{m+2}\right\rfloor}.$$

PROOF. Define a function $\Delta_n(x) = \tilde{u}_{n+1}(x) - \tilde{u}_n(x)$, then

$$\Delta_n(x) = 1_A(x)\mathscr{H}_{A\setminus D}\Delta_{n-1}(x)$$

and $\Delta_0(x) = 1_A(x)\mathscr{H}_{A\setminus D}T(D|x)$. Since $\mathscr{H}^{m+1}_{A\setminus D}$ is a contraction on $\mathbb{B}(\mathscr{X})$ we have that

$$\|\Delta_{k(m+2)}(x)\| \le \alpha^k$$

for all $k \ge 0$. The desired inequality follows from

$$\|\tilde{u}(x) - \tilde{u}_n(x)\| \le \sum_{k=n}^{\infty}\|\Delta_k(x)\|.$$

$\square$

3.5 Probabilistic invariance: an example

Let us refer to the case study of Section 2.2 to elucidate the application of the methods we have developed for the study of the infinite-horizon probabilistic invariance. Let the invariant set $A = \bigcup\limits_{i=1}^{3}\{l_i\}\times[0,1]$ – in Figure 1, this is labeled with gray nodes. Let us start by finding the largest absorbing subset A' of A. To do it we calculate the sets A_n as suggested in Lemma 3:

$$\begin{aligned} &A_0 = A \\ &A_1 = (\{l_1\}\times[0,1]) \cup (\{l_2\}\times\{0\}) \\ &A_2 = A_3 = \{l_1\}\times[0,1], \end{aligned}$$

which leads to conclude that $A' = A_\infty = \{l_1\}\times[0,1]$. Let us denote $u^i(c_x) = u(l_i, c_x; A)$. It is clear that $u^4(c_x) = 0$ and

$u^1(c_x) = 1$, for any $c_x \in [0,1]$. Let us now consider the other two locations. If $l_x = l_3$ then $T(A'|x) = 0$ and $T(\{l_2\} \times [0,1]|x) = 0$ so the modified Bellman equation (3.2) reduces to

$$u^3(c_x) = \int_0^1 u^3(c_y) p_{33}(c_x) \xi_{33}(c_x, c_y) \lambda(dc_y)$$
$$= \int_0^1 u^3(c_y) c_x^2 \frac{f_{\mathcal{N}}(c_x, c_y, 0.5)}{\Phi(c_x, 1, 0.5)} \lambda(dc_y),$$

which has a unique zero solution. Let us now focus on $l_x = l_2$, where $T(A'|x) = \frac{c_x}{2}$, so we obtain

$$u^2(c_x) = \frac{c_x}{2} + \int_0^1 u^2(c_y) \left(1 - \frac{1}{2}(c_x + c_x^2)\right) \frac{f_{\mathcal{N}}(c_x, c_y, 0.4)}{\Phi(c_x, 1, 0.4)} \lambda(dc_y),$$

which is a Fredholm equation of the second kind, and has a unique solution that can be computed by applying well-developed numerical methods [2]. The overall results are summarized in Figure 3.5, which displays the value functions $u^i(c_x), c_x \in [0,1], i \in L$.

3.6 Distribution and mean of first exit time

In the current section we focus on the mean value of the first exit time ς_A, for a compact set A. Recall that $p_n(x;A) = \mathsf{P}_x\{\varsigma_A = n\}$ and $p(x;A) = \mathsf{P}_x\{\varsigma_A = \infty\}$. From this definition it is clear that

$$\begin{cases} p_0(x;A) & = 1_{A^c}(x) \\ p_n(x;A) & = u_{n-1}(x;A) - u_n(x;A), \quad n \in \mathbb{N} \\ p(x;A) & = u(x;A), \end{cases} \tag{3.8}$$

thus $p_{n+1}(x;A) = \mathcal{I}_A p_n(x;A)$, for all $n \geq 0$. Since A is compact and $\xi \in \mathrm{Lip}(\mathcal{X})$, one can apply a space discretization procedure [1] to compute an approximate value of p_n, with an arbitrarily small bound on the error.

The problem of the mean value for the exit time $t(x;A) = \mathsf{E}_x[\varsigma_A]$ is slightly different from what we considered above, since in general $t \in \mathbb{L}(\mathcal{X})$ (rather than in $\mathbb{B}(\mathcal{X})$). Recall that from (2.6) and from the equality $p(x;A) = u(x;A)$ we know that $t(x;A) = \infty$ if $x \in A^+$, where

$$A^+ := \{x \in A : u(x;A) > 0\}. \tag{3.9}$$

We now show that if A is compact, then this set contains the only points where $t(x;A)$ takes an infinite value.

Theorem 4. *Let A be compact and A^+ is defined by (3.9). Then*

$$t(x;A) = \begin{cases} \infty, & \text{for } x \in A^+ \\ h(x), & \text{for } x \in A \setminus A^+ \\ 0, & \text{for } x \in A^c, \end{cases} \tag{3.10}$$

where $h(x)$ is the unique solution of the following equation:

$$h(x) = 1_{A\setminus A^+}(x) + \mathcal{I}_{A\setminus A^+} h(x). \tag{3.11}$$

In particular, if A is simple then $t(x;A)$ is bounded and the unique solution of (2.5).

Proof. We have already proved that $t(x;A) = \infty$ for $x \in A^+$ and $t(x;A) = 0$ for $x \in A^c$. We use this information to rewrite (2.5) as:

$$t(x;A) = 1_A(x) + 1_A(x) \left(\int_{A\setminus A^+} t(y;A) \xi(x,y) \mu(dy) + T(A^+|x) \cdot \infty \right).$$

Note now that if $x \in A \setminus A^+$, then $T(A^+|x) = 0$. Indeed,

$$0 = u(x;A) = \int_A u(y;A) \xi(x,y) \mu(dy),$$

and since the integrand is non-negative, it is equal to 0 μ-a.e. From this it follows that

$$t(x;A) = 1 + \int_{A\setminus A^+} t(y;A) \xi(x,y) \mu(dy),$$

for all $x \in A \setminus A^+$. So

$$t(x;A) = 1_{A^+}(x) \cdot \infty + 1_{A\setminus A^+}(x) h(x),$$

where $h(x) = 1 + \mathcal{H}_{A\setminus A^+} h(x)$ for all $x \in A \setminus A^+$.

Let us prove now that $A\setminus A^+$ is compact and simple. First of all, since A is compact, $u(x;A) \in \mathcal{C}(A)$ from Lemma 6. This means that A^+ is open in A and hence $A \setminus A^+$ is compact. Next, for the largest absorbing subset $A' \subseteq A$ it clearly holds that $A' \subseteq A^+$.

From Corollary 1 we conclude that $\mathcal{H}^m$ is a contraction on $\mathbb{B}(\mathcal{X})$ with rate $\alpha = \alpha(A \setminus A^+)$, where $m = m(A \setminus A^+) + 1$. This leads to the uniqueness of h, which now can be found with any bound on the error as in Theorem 1, so (3.10) is proved.

Finally, if the set A is simple, $A^+ = \emptyset$, so the last assertion of the theorem immediately follows from (3.10). □

3.7 Mean of first exit time: an example

We illustrate obtained results by providing an example, where we characterize the value function $t(x;A)$ for the Markov process of the case study in Section 2.2 and the set $A = \bigcup_{i=1}^{3} \{l_i\} \times [0,1]$ as in Section 3.5.

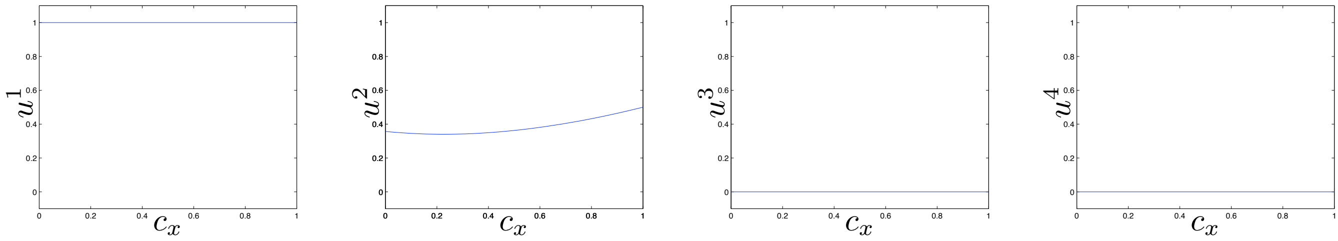
From the solution of the infinite-horizon probabilistic invariance problem, we can determine the set $A^+ = \bigcup_{i=1,2} \{l_i\} \times [0,1]$. Thus $A \setminus A^+ = \{l_3\} \times [0,1]$. From Theorem 4 we conclude that

$$t(x;A) = \begin{cases} \infty, & \text{for } l_x \in \{l_1, l_2\} \\ h(x), & \text{for } l_x = l_3 \\ 0, & \text{for } l_x = l_4, \end{cases}$$

where $h(x)$ is a unique solution of (3.11). Since we are only interested in its values for $l_x = l_3$ we denote (similar as before) $t^3(c_x) := t(l_3, c_x)$ then $t^3(c_x)$ is the unique solution of the following Fredholm equation of the second kind:

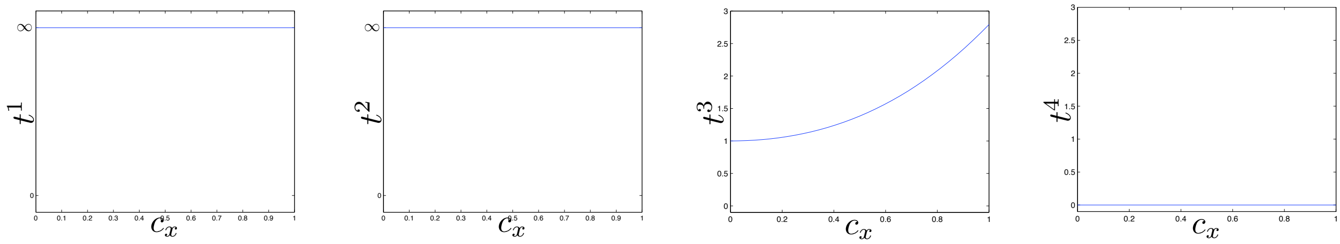
$$t^3(c_x) = 1 + \int_0^1 t^3(c_y) p_{33}(c_x) \xi_{33}(c_x, c_y) \lambda(dc_y)$$
$$= \int_0^1 t^3(c_y) c_x^2 \frac{f_{\mathcal{N}}(c_x, c_y, 0.5)}{\Phi(c_x, 1, 0.5)} \lambda(dc_y).$$

The solution is computed according to [2], and represented in Figure 3.7. Note that for the point $x = (l_3, 0)$ the value function $t(x;A) = 1$. Indeed, for this point it holds that $T(A^c|x) = 1$, hence starting from this point the process will leave the set A almost surely in one step.



(a) Location l_1 (b) Location l_2 (c) Location l_3 (d) Location l_4

Figure 2: Solution of the infinite-horizon probabilistic invariance problem discussed in Section 3.5.



(a) Location l_1 (b) Location l_2 (c) Location l_3 (d) Location l_4

Figure 3: Solution of the mean of first exit time problem discussed in Section 3.7.

4. COMPUTATION OF ABSORBING SETS

4.1 Characterization of absorbing sets

In the previous chapters we have emphasized the role that absorbing sets play in the computation of the value function of infinite-horizon specifications of Markov processes. This leads to the need to characterize the largest absorbing set A' of a given compact set A, or alternatively to establish the simplicity of A. These are in general difficult goals: this section focuses on approximation techniques to tackle this class of problems, and focuses on models that allow the verification of simplicity of A or to find its largest absorbing subset A'.

A general procedure to characterize the largest absorbing set is suggested in Lemma 3: it computes the sequence $(A_n)_{n\geq 0}$. This sequence is such that if $A_m = A_{m+1}$ for some $m \geq 0$, then $A_\infty = A_m$. Note that in this case either A_∞ is empty – hence A is simple – or non-empty, which leads to $A' = A_\infty$. We have explicitly applied this procedure in the example of Section 3.5, which has resulted in an analytical characterization of the sets A_n.

This general approach presents two issues. The first arises when the sets A_n can be only over-approximated numerically – also, due to over-approximation errors, it may be hard to check the set equality $A_m = A_{m+1}$ for some m. Secondly, even if sets A_n can be computed analytically, it may be hard to characterize the limit $A_\infty = \lim_{n\to\infty} A_n$. To mitigate these limitations, we have proposed to use an open neighborhood (over-approximation) of A_n as a candidate for the set D in Theorem 3.

Absorbing sets can be found analytically when a Markov process X is expressed by a recursive formula, such as

$$X_{n+1} = F(X_n, \eta_n),$$

where η_n are iid random variables. This happens for instance in the case of stochastic difference equations with linear drift and diffusion terms, which admit the origin as a absorbing point. However, experience has shown that for many such models admitting analytical characterization of an absorbing set, finding a measure μ, such that T is absolutely continuous with respect to μ and its density is Lipshitz continuous, is often problematic. We do not further pursue this characterization in the present contribution.

Finally, there are instances when both the verification of simplicity of A and the computation of A' up to admissible precision are decidable procedures. The simplest one is represented by Markov Chains but this case can be generalized to a larger class of Markov processes. We focus on such processes in Section 4.2 – before, we give a characterization of absorbing sets that we later use to find such procedures.

Let $s(x) = \{y \in \mathscr{X} : \xi(x,y) > 0\}$, so clearly $s(x) \in \mathscr{B}(\mathscr{X})$. The following proposition gives a characterization of an absorbing set in terms of sets $s(x)$.

Proposition 3. *Let $A' \in \mathscr{B}(\mathscr{X})$, then A' is absorbing if and only if $\mu(s(x) \setminus A') = 0$ for all $x \in A'$.*

Proof. Notice that A' is absorbing if and only if $\int_{A'} \xi(x,y)\mu(dy) = 1$ for any $x \in A'$, and further that $\int_{A\cap s^c(x)} \xi(x,y)\mu(dy) = 0$ for any $A \in \mathscr{B}(\mathscr{X})$. Thus $\int_{A} \xi(x,y)\mu(dy) = \int_{A\cap s(x)} \xi(x,y)\mu(dy)$ and hence

$$\int_{s(x)} \xi(x,y)\mu(dy) = 1$$

for all $x \in \mathscr{X}$. If $x \in A'$ then

$$0 = \int\limits_{s(x)} \xi(x,y)\mu(dy) - \int\limits_{A'} \xi(x,y)\mu(dy)$$
$$= \int\limits_{s(x)} \xi(x,y)\mu(dy) - \int\limits_{A' \cap s(x)} \xi(x,y)\mu(dy),$$

so $\int\limits_{s(x)\setminus A'} \xi(x,y)\mu(dy) = 0$ and hence $\mu(s(x)\setminus A') = 0$ since $\xi \geq 0$.

For the opposite direction, assuming that $\mu(s(x)\setminus A') = 0$ for all $x \in A'$, we have:

$$\int\limits_{A'} \xi(x,y)\mu(dy) = \int\limits_{s(x)\cap A'} \xi(x,y)\mu(dy)$$
$$= 1 - \int\limits_{s(x)\setminus A'} \xi(x,y)\mu(dy) = 1,$$

then A' is absorbing. □

4.2 Densities with a hybrid structure

We describe a class of processes for which the characterization of absorbing subsets is a solvable problem. We show how to find the largest absorbing subset A' of a given set A up to a set of a measure zero, which is sufficient for the modified Bellman equations introduced in Section 3 and thus allow applying the developed theory to solve infinite-horizon problems.

Consider densities with a hybrid structure, namely made up of a discrete part – similar to Markov Chains – and a continuous part. More precisely, we suppose that there exists a disjoint collection of sets (cells) $\mathfrak{Q} = (q_i)_{i=1}^N$ such that

1. q_i is open, $\mu(q_i) > 0$ and $\mu(\partial q_i) = 0$ for all $1 \leq i \leq N$,
2. $\overline{Q} = \mathscr{X}$ where $Q = \bigcup_{i=1}^{N} q_i$,
3. for any $1 \leq i, j \leq N$
   - either $\xi(x,y) > 0$ for all $x \in q_i$ and $y \in q_j$ (we write then $i \to j$),
   - or $\xi(x,y) = 0$ for all $x \in q_i$ and $y \in q_j$.

Some comments on these assumptions are in order:

- the described structure is a generalization of a finite-state Markov Chain, since for the finite state space $\mathscr{X}$ one can select $q_i = x_i$ and all assumptions are satisfied;
- another instance of this model is given by discrete-time Stochastic Hybrid Systems [1], where $\mathfrak{Q}$ can be derived from the set of locations (modes) and each set q_i corresponds to the associated continuous domain. The assumptions above indicate that whenever there is a positive transition probability between locations $l \to l'$, then the corresponding stochastic kernel has a full support. For instance, for the process described in Section 2.2, we can select $q_i = l_i \times (0,1)$, for all $1 \leq i \leq 4$;
- furthermore, these assumptions are satisfied for the case when $\mathscr{X} = \mathbb{R}^n$ and the support of ξ is made up of a finite union of hypercubes in $\mathbb{R}^n \times \mathbb{R}^n$ aligned with the coordinate axes.

Let us introduce an adjacency matrix $\mathbb{Q}$ of dimension $N \times N$, with the entry on the i-th row and j-th column equal to 1 if $i \to j$ and equal to 0 otherwise. $\mathbb{Q}$ plays a role in solving the problem of verification of simplicity, as well as in finding the largest absorbing subset of a given set. To formally describe both procedures we first introduce the following concepts, denoting

- for $x \in Q$ a *discrete closure* as $\lceil x \rceil = \{j : x \in q_j\}$ and for $A \subseteq \mathscr{X}$
$$\lceil A \rceil = \bigcup_{x \in A \cap Q} \lceil x \rceil;$$
- for $x \in Q^c$ a *discrete neighborhood* as $\mathfrak{n}(x) = \{j : x \in \overline{q_j}\}$;
- for $A \subset \mathscr{X}$ a *discrete interior* as
$$\lfloor A \rfloor = \{j : \mu(q_j \setminus A) = 0\}$$
  and put
$$Q_A = \bigcup_{i \in \lfloor A \rfloor} \overline{q_i}; \tag{4.1}$$
- for $x \in Q$ a *discrete support* as $\mathfrak{s}(x) = \{j : \lceil x \rceil \to j\}$ and for $A \subseteq \mathscr{X}$
$$\mathfrak{s}(A) = \bigcup_{x \in A \cap Q} \mathfrak{s}(x);$$
- if $B \subseteq \mathscr{X}$ admits representation $B = \bigcup_{i \in \lceil B \rceil} q_i$ then we call it a *multicell*.

For all $x \in Q$, the unique cell that point x belongs to is given by $q_{\lceil x \rceil}$, so $\lceil A \rceil$ is the set of indexes of the cells that set A overlaps with. $\mathfrak{n}(x)$ denotes the set of indexes of the cells that the boundary $x \in Q^c$ belongs to. A cell q_i is a subset of A – with the possible exception of a set of measure zero – if and only if $i \in \lfloor A \rfloor$. Finally, given a point $x \in Q$, $T(q_i|x) > 0$ if and only if $i \in \mathfrak{s}(x)$. Clearly, $\mathfrak{s}(x)$ is not empty for any $x \in Q$ and $\mathfrak{n}(x)$ is not empty for any $x \in Q^c$. Also, $\mathfrak{s}(x)$ depends only on $\lceil x \rceil$ for $x \in Q$.

EXAMPLE 1. *Let us consider the process described in Section 2.2, define $q_i = \{l_i\} \times (0,1), 1 \leq i \leq 4$, and select point $x = (l_2, \frac{1}{2})$. For this point, $\lceil x \rceil = \{2\}$ since $x \in l_2$, and $\mathfrak{s}(x) = \{1,2,4\}$ since x admits a non-zero probability to transition to locations l_1, l_2, and l_4. A second point $y = (l_1, 1)$ is such that $y \notin Q$ and $\mathfrak{n}(y) = q_1$.*

Additionally, the set $B = \bigcup_{i=1}^{3} q_i = \bigcup_{i=1}^{3} \{l_1\} \times (0,1)$ is a multicell. Note that the invariant set $A = \bigcup_{i=1}^{3} \{l_1\} \times [0,1] = \overline{B}$, which is the closure of the multicell B and which is not a multicell. For set A it can be seen that $\lceil A \rceil = \{1,2,3\}$, that $\lfloor A \rfloor = \{1,2,3\}$, and that the discrete interior will not change if we eliminate any single point from A, e.g. $\left\lfloor A \setminus \{(l_1, \frac{1}{2})\} \right\rfloor = \{1,2,3\}$. Finally, given two sets A and q_2, their discrete support are $\mathfrak{s}(A) = \{1,2,3,4\}$ and $\mathfrak{s}(q_2) = \{1,2,4\}$ respectively.

The plan is to tackle the verification of simplicity first for multicells, then to extend the procedure to arbitrary sets. In order to employ the absorbance characterization provided in Proposition 3, it is important to establish a connection between sets $s(x)$ and $\mathfrak{s}(x)$. (We indeed show in the following that for an absorbing set its discrete support is a subset of the discrete interior.)

LEMMA 7. *For any $x \in Q$, $q_i \subset s(x)$ if $i \in \mathfrak{s}(x)$, and*

$$\bigcup_{i \in \mathfrak{s}(x)} \overline{q_i} = \overline{s(x)}.$$

If $x \in Q^c$ then $\overline{s(x)} \subseteq \overline{s(x')}$ for any $x' \in q_i$ and $i \in \mathfrak{n}(x)$.

PROOF. For $x \in Q$ if $i \in \mathfrak{s}(x)$ and $y \in q_i$ then $\xi(x,y) > 0$. Hence, $q_i \in s(x)$ for $i \in \mathfrak{s}(x)$. Suppose now that $y \in \overline{s(x)}$. Then in each neighborhood of y there exists $y' \in Q$ such that $\xi(x,y') > 0$, so there exists $i \in \mathfrak{s}(x)$ such that $y \in \overline{q_i}$. Note that $\overline{s(x)}$ for $x \in Q$ depends now only on $\lceil x \rceil$.

Suppose now that $x \in Q^c$ and for some $y \in \mathscr{X}$ we have $\xi(x,y) > 0$. Due to continuity of ξ, for any $i \in \mathfrak{n}(x)$ there exists $x' \in q_i$ such that $\xi(x',y) > 0$, so $s(x) \subseteq s(x')$. Since $\overline{s(x')}$ depends only on $\lceil x' \rceil = i$, we have proven the last statement of the proposition. $\square$

LEMMA 8. *If A' is an absorbing set then $\mathfrak{s}(A') \subseteq \lfloor A' \rfloor$. In particular, $\lfloor A' \rfloor \neq \emptyset$. Any set A such that $\lfloor A \rfloor = \emptyset$ is simple.*

PROOF. Note that $A' \cap Q$ is non-empty since $\mu(A') > 0$. Take $x \in A' \cap Q$ then

$$0 = \mu(s(x) \setminus A') = \mu\left(\bigcup_{i \in \mathfrak{s}(x)} \overline{q_i} \setminus A'\right) \geq \max_{i \in \mathfrak{s}(x)} \mu(q_i \setminus A'),$$

so $\mathfrak{s}(x) \in \lfloor A' \rfloor$ for all $x \in A' \cap Q$ and $\mathfrak{s}(A') \subseteq \lfloor A' \rfloor$. Since for all $x \in Q$ it holds that $\mathfrak{s}(x) \neq \emptyset$ then $\lfloor A' \rfloor$ contains at least one element. Finally, if $\lfloor A \rfloor = \emptyset$ then any subset of A has an empty discrete interior, so A is simple. $\square$

EXAMPLE 2. *For the process in Section 2.2 consider the set $A' = \{l_1\} \times [0,1]$. It holds that $\mathfrak{s}(A') = \{1\} = \lfloor A' \rfloor$. The preceding lemma confirms that it is absorbing, as we already showed in Section 3.5.*

The next proposition provides the solution to the problem of verification of simplicity for a multicell B, and is based on operations over the adjacency matrix $\mathbb{Q}$.

PROPOSITION 4. *Let $C \subset \lceil B \rceil$ be the largest index set such that if $i \in C$ and $i \to j$, then $j \in C$. If C is empty then B is simple. If C is not empty then the multicell*

$$C' := \bigcup_{i \in C} q_i$$

is an absorbing subset of B and $\mu(B' \setminus C') = 0$, where B' is the largest absorbing subset of B.

PROOF. First, since B is a multicell, $C' \subseteq B$. Next, if $x \in C'$ then $x \in Q$ and $\lceil x \rceil \in C$. By construction of C it follows that $\mathfrak{s}(x) \subset C$ for all $x \in C'$, so $\mu(s(x) \setminus C') = 0$ for all $x \in C'$ and this set is absorbing.

Consider now $x \in B'$. Clearly, $q_{\lceil x \rceil} \subseteq B'$ since $\mathfrak{s}(x) = \mathfrak{s}(x')$ for all $x' \in q_{\lceil x \rceil}$, thus $\mu(s(x) \setminus B') = 0$ implies $\mu(s(x') \setminus B') = 0$. As a consequence, $\lceil B' \rceil = \lfloor B' \rfloor$.

If for some $i \in \lceil B' \rceil$ exists j such that $i \to j$ but $j \notin \lceil B' \rceil$ then for $x \in q_i \subseteq B'$ we have $\mu(s(x) \setminus B') \geq \mu(q_j) > 0$, so for all $i \in \lceil B' \rceil$ and j such that $i \to j$ it holds that $j \in \lceil B' \rceil$. Hence, $\lceil B' \rceil \subseteq C$ which means that $\mu(C' \setminus B') = 0$. $\square$

Set C' can be computed with a standard procedure over $\mathbb{Q}$ in $\mathcal{O}(N^2)$ [3]. We have shown how to verify the simplicity of a multicell, now we extend the result to an arbitrary set.

PROPOSITION 5. *Let $A \in \mathscr{B}(\mathscr{X})$, then A is not simple if and only if there exists a multicell B such that $A \cap B$ is absorbing and $\lceil B \rceil \subseteq \lfloor A \rfloor$. Moreover, $\mathfrak{s}(A') \subseteq \lfloor A \rfloor$.*

PROOF. Obviously, if there exists such a set B then A has an absorbing subset $A \cap B$ and hence it is not simple. Suppose now that A is not simple and A' is the largest absorbing subset of A. Consider a multicell

$$B = \bigcup_{i \in \mathfrak{s}(A')} q_i.$$

Since A' is absorbing, for all $x \in A'$ it holds that $\mu(s(x) \setminus A') = 0$ hence for any $i \in \mathfrak{s}(A')$ we obtain that $\mu(q_i \setminus A') = 0$ and $A' \cap q_i \neq \emptyset$, which leads to $\mathfrak{s}(q_i) \subseteq \mathfrak{s}(A')$ and $\mathfrak{s}(B) \subseteq \mathfrak{s}(A')$.

On the other hand, if $x \in B$ then

$$s(x) \subseteq \bigcup_{i \in \mathfrak{s}(x)} \overline{q_i} \subseteq \bigcup_{i \in \mathfrak{s}(B)} \overline{q_i}$$

and that

$$\mu(s(x) \setminus B) \leq \mu\left(\bigcup_{i \in \mathfrak{s}(B)} \overline{q_i} \setminus \bigcup_{i \in \mathfrak{s}(A')} q_i\right) \leq \mu\left(\bigcup_{i \in \mathfrak{s}(B)} \partial q_i\right) = 0,$$

which proves that B is absorbing.

Note that $\lceil B \rceil = \mathfrak{s}(A')$ from the construction of B. We showed that $\mu(q_i \setminus A') = 0$ for any $i \in \mathfrak{s}(A')$, so $\lceil B \rceil \subseteq \lfloor A \rfloor$. $\square$

Note that we have reduced the simplicity verification problem to that for multicells, which we know how to solve. The final step is to find the relation between the largest absorbing sets of A and Q_A. This leads us to an algorithm for finding such a set for an arbitrary A.

PROPOSITION 6. *If A' is the largest absorbing subset of A then $Q' = A' \cap Q_A$ is the largest absorbing subset of Q_A. Moreover, for $x \in A \cap Q$ condition $\mathfrak{s}(x) \subseteq \lceil Q' \rceil$ is satisfied if and only if $x \in A'$.*

PROOF. For $x \in Q' \cap Q$ then $x \in A' \cap Q$ and hence

$$\begin{aligned}\mu(s(x) \setminus Q') &= \mu\left(s(x) \setminus (A' \cap Q_A)\right) \\ &\leq \mu(s(x) \setminus A') + \mu\left(s(x) \setminus Q_A\right) = 0.\end{aligned}$$

For $x \in Q' \setminus Q$ by Lemma 7 there exists $x' \in Q' \cap Q$ such that $\overline{s(x)} \subseteq \overline{s(x')}$ and hence $\mu(s(x) \setminus Q') = 0$ for all $x \in Q'$, so Q is absorbing.

If $Q'' \subseteq Q_A$ is absorbing then $Q'' \cup A'$ is absorbing and hence $Q'' \subseteq A'$, so $Q'' \subseteq Q'$, which proves the statement of the proposition.

Note that we proved that for any $x \in A' \cap Q$ it holds that $\mu(s(x) \setminus Q') = 0$. Suppose that there exists $x \in A' \cap Q$ and $i \in \mathfrak{s}(x) \setminus \lceil Q' \rceil$, then

$$\mu(s(x) \setminus Q') \geq \mu\left(\bigcup_{j \in \mathfrak{s}(x)} \overline{q_i} \setminus \bigcup_{j \in \lceil Q' \rceil} \overline{q_i}\right) \geq \mu(\overline{q_i}) > 0,$$

hence for all $x \in A' \cap Q$ it holds that $\mathfrak{s}(x) \subseteq \lceil Q' \rceil$. On the other hand, if for $x \in A \cap Q$ it holds that $\mathfrak{s}(x) \subseteq \lceil Q' \rceil$, then by Lemma 7 $\mu(s(x) \setminus Q') = 0$ and $x \in Q' \subseteq A'$. $\square$

The construction of the largest absorbing subset of A, up to a set of measure zero, develops along the following two steps:

1. given a set A, find the largest absorbing subset Q' of Q_A, up to a set of measure zero, by applying Proposition 4. To find Q' one should find the largest absorbing set of Q_A° with

the method given in Proposition 4. This set differs from Q' only up to set of measure zero. If Q_A is simple, then A is simple by Proposition 5;

2. consider $x \in A \cap Q$. By Proposition 6 one can have $x \in A'$ if and only if $\mathfrak{s}(x) \subseteq \lceil Q' \rceil$. On the other hand, if $x \in A'$ then $x' \in A'$ for all $x' \in q_{\lceil x \rceil} \cap A$ since $\mathfrak{s}(x) = \mathfrak{s}(x')$. This means that it is sufficient to consider just one representative point x_i from each set $q_i \cap A$ where $i \in \lceil A \rceil$ and $q_i \cap A \subseteq A'$ if and only if $\mathfrak{s}(x_i) \subseteq \lceil Q' \rceil$. Now the only points which may still be left out are in Q^c, however $\mu(Q^c) = 0$.

EXAMPLE 3. *Consider the Markov process given in Section 2.2 and let us apply the developed algorithm. Recall that we set* $q_i = \{l_i\} \times (0,1)$*, so the adjacency matrix has the form*

$$\mathbb{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

For the safe set A we know that $Q_A = \bigcup_{i=1}^{3} \overline{q}_i$*, so according to the algorithm we first look for the largest absorbing subset Q' of Q_A. This is done according to Proposition 4, which finds the largest index set $C \subseteq \lceil Q_A \rceil$ such that $i \in C$ and $i \to j$ implies $j \in C$.*

Recall that $\lceil Q_A \rceil = \{1, 2, 3\}$*. Let us find C: if $2 \in C$ or $3 \in C$ then it is necessary that $4 \in C$ because $2 \to 4$ and $3 \to 4$. Due to this fact it is only possible to have $1 \in C$. Since $1 \to 1$ only, we conclude that $C = \{1\}$ and $C' = q_1$.*

To build A' we start off with C'. We consider $i \in \lceil A \rceil = \{1, 2, 3\}$ *and add $q_i \cap A$ to A' if for all $x_i \in q_i$ it holds that $\mathfrak{s}(x_i) \subseteq \lceil Q' \rceil$. Clearly this holds only for $i = 1$, hence $q_1 \subseteq A'$, $\mu(A' \setminus q_1) = 0$. Note that this outcome corresponds with the $A' = \{l_1\} \times [0, 1]$ that was found in Section 3.5.*

5. CONCLUSIONS AND FUTURE WORK

This work has discussed issues related to the solution of integral Bellman equations and has showed that absorbing sets play a prominent role both in the analysis of such problems and in the computation of a solution with explicit bounds on the error. The contribution has shown that, in the case of compact simple sets, the solution is unique and can be found in a finite number of steps with any precision. On the other hand, in the presence of a set that is not simple, the knowledge of its largest absorbing set is again crucial for finding the solution of Bellman equation. The work has also highlighted the differences in the solution of invariance, reach-avoid, and mean exit time problems.

Both the verification of simplicity and the characterization of the largest absorbing subset of a given set are difficult problems in general, though in some cases they admit solvable procedures, as it was shown in Section 4. The authors are interested in generalizing these solvable procedures. Furthermore, the authors are also interested in extensions to other infinite time horizon properties, as well as to the continuous time case.

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