Optimal robust control and a separation principle for polytopic time-inhomogeneous Markov jump linear systems

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***Abstract*— Markov-jump linear systems (MJLSs) allow representing linear systems subject to abrupt parameter changes modeled as a Markov chain, and are useful in many application domains. In most real cases the transition probabilities between operational modes of the system cannot be computed exactly and are time-varying. We take into account this aspect by considering MJLSs where the underlying Markov chain is polytopic and time-inhomogeneous, i.e. its transition probability matrix is varying over time with variations that are arbitrary within a polytopic set of stochastic matrices. We address and solve for this class of systems the finite horizon optimal control and filtering problems. In particular, we show that the optimal controller having only partial information on the continuous state can be obtained from two types of coupled Riccati difference equations (CRDEs), one associated to the control problem, and the other one associated to the filtering problem.**

I. INTRODUCTION

Linear systems subject to abrupt parameter changes due, for instance, to environmental disturbances, component failures or repairs, changes in subsystems interconnections, changes in the operation point for a non-linear plant, etc., can be modeled by a set of discrete-time linear systems with modal transition given by a discrete-time finite-state Markov chain. This family of systems is known as discrete-time Markov(ian) jump linear systems (from now on MJLSs, see [1] and references therein for a detailed overview).

MJLS models are useful in many applications, especially for wireless networked control systems (see e.g. [2], [3], [4], [5], [6], [7], [8] and references therein for a general overview), which have a wide spectrum of applications. The wireless communication channels used to convey information between sensors, actuators, and computational units are frequently subject to time-varying fading and interference, which may lead to packet losses. In the wireless networked control system literature the packet dropouts have been modeled either as stochastic or deterministic phenomena [4]. The proposed deterministic models specify packet losses in terms of time averages or in terms of worst case bounds on the number of consecutive dropouts (see e.g. [6]). For what concerns stochastic models, a vast amount of research assumes memoryless packet drops, so that dropouts are realizations of a Bernoulli process ([2], [5], [7]). Other works consider more general correlated (bursty) packet losses and use a transition probability matrix (TPM) of a finite-state (time-homogeneous) Markov chain (see e.g. the finite-state Markov modeling of Rayleigh, Rician and Nakagami fading channels in [9] and references therein) to describe the stochastic process that rules packet dropouts (see [2], [10]). In these works networked control systems with missing packets are modeled as time-homogeneous MJLSs.

However, in most real cases the TPM cannot be computed exactly and is time-varying. The Markov chain models of slow fading channels [9] are derived via measurements on real channels or via numerical reasoning, which always introduces errors. Indeed, a fundamental issue in the design of finite-state Markov channel models is how accurate and reliable the resulting system performance measures are [9]. Moreover, all these models are based on the unrealistic fundamental assumption that underlying parameters are known and static. The study of robustness to such variations becomes naturally important not only in the context of networked control systems, but in any application where the MJLS model is affected by abrupt and unpredictable perturbations on the underlying Markov chain: in [11] it is pointed out that in the vertical take-off landing (VTOL) helicopter system the airspeed variations are ideally modeled as homogeneous Markov process, but because of the external environment (like the wind) the transition probabilities of the jumps are time-varying; in [12] the example of failures and repairs of subsystems is considered, where the transition probabilities deeply depend on system age and working time.

We take into account these aspects by considering MJLSs where the underlying Markov chain is time-inhomogeneous, i.e. the Markov chain has its TPM varying over time with variations that are arbitrary within a polytopic set of stochastic matrices. We denote such model as discrete-time polytopic time-inhomogeneous (from here on, PTI) MJLS.

Given such mathematical model, there are several recent works on (robust) $\mathcal{H}_\infty$ control, filtering and fault detection [11], [12], [13], [14], [15]. These works generally provide sufficient conditions based on linear matrix inequalities (LMIs) and Lyapunov functional approaches for the existence of $\mathcal{H}_\infty$ controllers [13], [14], $\mathcal{H}_\infty$ filters [12], [15] and $\mathcal{H}_\infty/\mathcal{H}_-$ fault detectors [11]. Recently, in [16], we presented the necessary and sufficient conditions for mean square stability of PTI MJLSs, which require to decide whether the joint spectral radius (see [17] and references therein for an overview) of a finite family of matrices is smaller than 1.

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In [18] we provided necessary and sufficient conditions for mean square stability robust to energy-bounded disturbances. In [19] we considered the more general model of PTI switched MJLSs, where discrete inputs are present and the Markov-chain turns into a (time-inhomogeneous) Markov-decision process, and derived the optimal solution of the finite-horizon linear-quadratic regulator (LQR) problem.

Following the same research line, as main contribution of this paper we address and solve the finite horizon optimal control and filtering problems for PTI MJLSs. In particular we show that, as for linear-quadratic-Gaussian (LQG) control in the case with no jumps, for the finite horizon case considered in this paper, the optimal controller can be obtained from two types of coupled Riccati difference equations (CRDEs), one associated to the control problem, and the other one associated to the filtering problem. When the transition probabilities between operation modes are known at each time step, our results coincide with those presented in [1], and when there is only one mode of operation, they coincide with the traditional separation principle for the LQG control of discrete-time linear systems. All the omitted proofs are available in the doctoral dissertation of the first author [20].

The *notation* used throughout is standard. The sets of all positive and nonnegative integers are represented by $\mathbb{N}$ and $\mathbb{N}_0$, respectively. The n-dimensional complex Euclidean space is indicated by $\mathbb{C}^n$, while a set of linear maps between two complex Euclidean spaces $\mathbb{C}^m$ and $\mathbb{C}^n$ is denoted by $\mathbb{C}^{m\times n}$ and is encoded through a set of $m\times n$ complex matrices. The conjugate of a complex matrix M is denoted by $\bar{M}$, while the superscript $*$ indicates the conjugate transpose of a matrix, and ^{T} indicates the transpose. Clearly for a set of real matrices, denoted by $\mathbb{R}^{m\times n}$, $*$ and ^{T} have the same meaning. We indicate with $\mathbb{C}_*^{n\times n}$ the set of Hermitian matrices, and with $\mathbb{C}_+^{n\times n}$ the set of positive semi-definite matrices. The $n\times n$ identity matrix is denoted by $\mathbb{I}_n$, while the null matrix of appropriate size is indicated by 0. Unless otherwise stated, $\|\cdot\|$ will indicate any norm in $\mathbb{C}^n$, and, for $M\in\mathbb{C}^{m\times n}$, $\|M\|$ will denote the induced uniform norm in $\mathbb{C}^{m\times n}$. The linear space made up of all N-dimensional vectors $\mathbf{M}$ of complex matrices $M_i\in\mathbb{C}^{m\times n}$, $i\in\mathcal{N}$, s.t. $\mathbf{M}=[M_1,...,M_N]$ is indicated by $\mathbb{H}^{n,m}$, where $\mathcal{N}\triangleq\{1,\dots,N\}\subset\mathbb{N}$ is a finite set of integers. Similarly, $\mathcal{V}\triangleq\{1,\dots,V\}\subset\mathbb{N}$ and $\mathcal{T}\triangleq\{1,\dots,T\}\subset\mathbb{N}$ indicate other finite sets of integers. For simplicity, we set $\mathbb{H}^n\triangleq\mathbb{H}^{n,n}$. For $\mathbf{M}\in\mathbb{H}^{n,m}$ we write $\mathbf{M}^*=[M_1^*,\dots,M_N^*]\in\mathbb{H}^{m,n}$, and say that $\mathbf{M}\in\mathbb{H}^n$ is Hermitian if $\mathbf{M}=\mathbf{M}^*$. Then,

$$\mathbb{H}^{n*}\triangleq\{\mathbf{M}=[M_1,\dots,M_N]\in\mathbb{H}^n; M_i=M_i^*, i\in\mathcal{N}\},$$
$$\mathbb{H}^{n+}\triangleq\{\mathbf{M}=[M_1,\dots,M_N]\in\mathbb{H}^{n*}; M_i\succeq 0, i\in\mathcal{N}\},$$

where $M_i\succeq 0$ (respectively $M_i\succ 0$) indicates that M_i is positive semi-definite (respectively positive definite). Finally, $\mathbf{E}[\cdot]$ stands for the mathematical expectation of the underlying scalar valued random variables.

II. PROBLEM STATEMENT

Let us consider a probability space $(\Omega,\mathcal{F},\Pr)$, where Ω is the sample space, $\mathcal{F}$ is the σ-algebra of (Borel) measurable events and Pr is the probability measure. Let $\theta:\mathbb{N}_0\times\Omega\to\mathcal{N}$ be a Markov chain defined on the probability space, which takes values in a finite set $\mathcal{N}\triangleq\{1,\dots,N\}$. For every $k\in\mathbb{N}_0$ let us define the transition probability as

$$p_{ij}(k)=\Pr\{\theta(k+1)=j\mid\theta(k)=i\}\geq 0,\sum_{j=1}^{N}p_{ij}(k)=1.$$

The associated TPM $P(k)$ is a stochastic $N\times N$ matrix with entries $p_{ij}(k)$. In this work we assume that $P(k)$ is **unknown** and **time-varying** within a bounded set.

Assumption 1: TPM $P(k)$ is **polytopic**, i.e. $\forall k\in\mathbb{N}_0$

$$P(k)=\sum_{l=1}^{V}\lambda_l(k)P_l,\ \lambda_l(k)\geq 0,\ \sum_{l=1}^{V}\lambda_l(k)=1, \qquad (1)$$

where $\{P_l\}_{l\in\mathcal{V}}\triangleq\mathcal{P}_\mathcal{V}$ is a given finite set of TPMs, which are the vertices of a convex polytope; $\lambda_l(k)$ are unmeasurable.

Remark 1: This assumption is not restrictive, since the polytopic uncertainty model is widely used for robust control of time-homogeneous MJLS (see e.g. [21]) and is considered to be more general than the partly known element model of TPM uncertainties; furthermore, also the interval TPM can be represented as a convex polytope [22].

We denote by $\boldsymbol{p}_{i\bullet}(k)$ the i-th row of the TPM $P(k)$ and by $\boldsymbol{p}_{\bullet j}(k)$ the j-th column of the TPM $P(k)$. It is clear from (1) that $\boldsymbol{p}_{i\bullet}(k)$ and $\boldsymbol{p}_{\bullet j}(k)$ are polytopic sets of stochastic vectors. We slightly abuse our notation by indicating transition probability sequences of length $T\in\mathbb{N}$ as

$$\boldsymbol{p}_{\theta\bullet}\triangleq[\boldsymbol{p}_{\theta(0)\bullet}(0),\dots,\boldsymbol{p}_{\theta(T-1)\bullet}(T-1)],$$
$$\boldsymbol{p}_{\bullet\theta}\triangleq[\boldsymbol{p}_{\bullet\theta(1)}(0),\dots,\boldsymbol{p}_{\bullet\theta(T)}(T-1)].$$

For a set $\mathcal{S}\in\mathcal{F}$, let us define the indicator function $\mathbf{1}_\mathcal{S}$ in the usual way [1, p. 31], that is, $\forall\omega\in\Omega$,

$$\mathbf{1}_\mathcal{S}(\omega)=\begin{cases}1 & \text{if } \omega\in\mathcal{S},\\ 0 & \text{otherwise.}\end{cases} \qquad (2a)$$

Notice that, $\forall i\in\mathcal{N}$,

$$\mathbf{1}_{\{\theta(k)=i\}}(\omega)=1 \text{ if } \theta(k)(\omega)=i, \text{ and } 0 \text{ otherwise;} \qquad (2b)$$

$$\mathbf{E}[\mathbf{1}_{\{\theta(k)=i\}}(\omega)]=\Pr\{\theta(k)=i\}\triangleq\pi_i(k). \qquad (2c)$$

The distribution $\boldsymbol{\pi}(k)\triangleq[\pi_1(k),\dots\pi_N(k)]$ of the random jump variable $\theta(k)$ evolves according to the transition probabilities, i.e.

$$\pi_j(k+1)=\sum_{i=1}^{N}\pi_i(k)p_{ij}(k). \qquad (3)$$

The related discrete-time polytopic time-inhomogeneous (PTI) Markov jump linear system (MJLS $\boldsymbol{\Sigma}$) is described by the following stochastic equations:

$$\begin{cases}x(k+1)=A_{\theta(k)}x(k)+B_{\theta(k)}u(k)+H_{\theta(k)}w(k),\\ y(k)=F_{\theta(k)}x(k)+G_{\theta(k)}w(k),\\ z(k)=C_{\theta(k)}x(k)+D_{\theta(k)}u(k),\\ x(0)=x_0,\ \theta(0)=\theta_0,\ \boldsymbol{\pi}(0)=\boldsymbol{\pi}_0,\end{cases} \qquad (4)$$

where $\forall k\in\mathbb{N}_0$, $x(k)\in\mathbb{C}^n$ is the vector of continuous state variables, $u(k)\in\mathbb{C}^m$ is the control vector, which gathers the control actions applied to the process, $y(k)\in\mathbb{C}^p$ is the vector of measured continuous states, that are available for feedback, while $z(k)\in\mathbb{C}^s$ is the output of the system. The exogenous input $w(k)\in\mathbb{C}^r$ is a **wide sense white noise**, which represents discrepancies between the model and the real process, due for instance to unmodeled dynamics or disturbances. Specifically, we have that $\forall k, l \in \mathbb{N}_0,\ k \neq l$

$$\mathbf{E}[w(k)] = 0,\ \mathbf{E}[w(k)w^*(k)] = \mathbb{I}_r,\ \mathbf{E}[w(k)w^*(l)] = 0. \quad (5)$$

Remark 2: When studying MJLS, it is a standard practice to work with complex fields [1], but one can also work with real-valued ones, by considering complex matrices acting on $\mathbb{C}^{n\times n}$ as real operators acting on $\mathbb{R}^{2n\times 2n}$ [17].

The system matrices $A_{\theta(k)}, B_{\theta(k)}, C_{\theta(k)}, D_{\theta(k)}, F_{\theta(k)}, G_{\theta(k)}$ and $H_{\theta(k)}$ are constant matrices of appropriate sizes, each of which associated with the operational mode $\theta(k) \in \mathcal{N}$, while x_0, θ_0 and $\boldsymbol{\pi}_0$ are initial conditions.

The set $\mathcal{N}$ comprises the operational modes of the system $\mathbf{\Sigma}$, and for each time $k \in \mathbb{N}_0$ and each possible value of the jump variable $\theta(k) = i \in \mathcal{N}$, we denote a system matrix M associated with the i-th mode by $M_i = M_{\{\theta(k)=i\}}$. Thus, we deal with following vectors of matrices: $\mathbf{A} \in \mathbb{H}^n$, $\mathbf{B} \in \mathbb{H}^{m,n}$, $\mathbf{C} \in \mathbb{H}^{n,s}$, $\mathbf{D} \in \mathbb{H}^{m,s}$, $\mathbf{F} \in \mathbb{H}^{n,p}$, $\mathbf{G} \in \mathbb{H}^{r,p}$, and $\mathbf{H} \in \mathbb{H}^{r,n}$.

We assume without loss of generality [1, pp.103-104, Remark 5.1 and p.74, Remark 4.1] that for all $i \in \mathcal{N}$

$$C_i^* D_i = 0, \quad D_i^* D_i \succ 0, \quad H_i G_i^* = 0, \quad G_i G_i^* \succ 0. \quad (6)$$

In this work we deal with the **mode-dependent** quadratic optimal control problem with partial information on the state variable $x(k)$ for the class of dynamical systems described by (4), where the TPM is unknown and time-varying within a convex set, as stated in Assumption 1. Thus, we generalize the results provided in [1]. When there is only one TPM, which is known at each time step, our results coincide with the traditional case presented by Costa et al. [1]. For the ease of notation, we consider the system matrices to be constant. However, when those matrices are time-varying, but known at each time step, the results of our work still hold.

We design a **dynamic Markov jump controller** $\mathcal{K}$ described by the following stochastic equations:

$$\begin{cases} \hat{x}(k+1) = \hat{A}_{\theta(k)}(k)\hat{x}(k) + \hat{B}_{\theta(k)}(k)y(k), \\ \quad u(k) = \hat{C}_{\theta(k)}(k)\hat{x}(k),\ \hat{x}(0) = \hat{x}_0, \end{cases} \quad (7)$$

with $\hat{x}_0$ deterministic, in a way to minimize quadratic functional cost $\mathcal{J}(\theta_0, x_0, u)$ associated to the closed loop system over a **finite time horizon**, for a worst possible sequence of transition probabilities between the operational modes.

Specifically, for $u \triangleq [u(0), \ldots, u(T-1)]$, $T \in \mathbb{N}$, we define

$$\mathcal{J}(\theta_0, x_0) \triangleq \min_u \max_{\boldsymbol{p}_{\bullet\theta}} \sum_{k=0}^{T-1} \mathbf{E}[\|z(k)\|^2] + \mathbf{E}[x^*(T)\mathcal{X}_{\theta(T)}x(T)], \quad (8)$$

with $\mathcal{X} \triangleq [\mathcal{X}_1, \ldots \mathcal{X}_N] \in \mathbb{H}^{n+}$ being a vector of the terminal cost weighting matrices.

So, we cast a finite-horizon robust optimization problem as a $\min$-$\max$ problem of optimizing robust performance, i.e. finding the minimum over the control input and filtering error of the maximum over the transition probability disturbance.

This problem can be presented also from the game-theoretic point of view, where at each time step k the perturbation-player (environment and/or malicious adversary) tries to maximize the cost while the control tries to minimize the cost. Such formulation requires to make explicit the following assumption on the information structure for the controller and the adversary.

Assumption 2: The perturbation-player has no information on the choice of the controller and vice versa.

Remark 3: The state of the controller $\mathcal{K}$ depends only on $\theta(k)$ rather than on the entire past history of the operational modes $\theta(0), \ldots, \theta(k)$, so that the closed loop system is again a Markov jump system. See [1, p.104, Remark 5.2] for comparison with a solution based on a Kalman filter.

As in [1], we have the following information pattern.

Assumption 3: At every time step $k \in \mathbb{N}_0$, the vector of the measured continuous states $y(k)$ and the jump parameters $\theta(k)$ are available to the controller.

The random variables $\{y(t), \theta(t); t = 0, \ldots, k\}$ generate the σ-algebra denoted by $\mathcal{G}_k$. Clearly, for every $k \in \mathbb{N}_0$

$$\mathcal{G}_k \subset \mathcal{G}_{k+1} \subset \mathcal{F}.$$

As in [1, p.133], we assume independence of the noise sequence from the Markov chain and the initial conditions.

Assumption 4: The noise sequence $\{w(k); k \in \mathcal{T}\}$ and the Markov chain $\{\theta(k); k \in \mathcal{T}\}$ are independent sequences, and the initial conditions $(x(0), \theta(0))$ are independent random variables, with $\quad \mathbf{E}[x(0)] = \mu_0,\ \mathbf{E}[x(0)x^*(0)] = \mathcal{Q}_0. \quad (9)$

So, in summary, the finite-horizon robust optimization problem we study is formally defined as follows.

Problem 1: Given a PTI MJLS Σ described by (4) and satisfying Assumptions 1–4, find $\hat{\mathbf{A}}(k) \in \mathbb{H}^n$, $\hat{\mathbf{B}}(k) \in \mathbb{H}^{p,n}$ and $\hat{\mathbf{C}}(k) \in \mathbb{H}^{n,m}$ with elements from (7), such that the control law $u = [u(0), \ldots, u(T-1)]$, $T \in \mathbb{N}$, with $u(k)$ from (7), achieves the optimal cost $\mathcal{J}(\theta_0, x_0)$ defined by (8).

In the next three sections we show that, as for linear-quadratic-Gaussian (LQG) control in the case with no jumps, for the finite horizon case considered in this paper, the optimal controller can be obtained from two types of coupled Riccati difference equations (CRDEs), one associated to the control problem, and the other one associated to the filtering problem. When the transition probabilities between operation modes are known at each time step, our results coincide with those presented in [1], and when there is only one mode of operation, they coincide with the traditional separation principle for the LQG control of discrete-time linear systems.

III. OPTIMAL ROBUST FILTERING

In this section we examine the problem of designing the optimal robust mode-dependent dynamic Markov jump filter, described by the system of stochastic equations (7), for a PTI MJLS Σ from (4), satisfying Assumptions 1–4, s.t. a quadratic cost associated to the filtering error is minimized.

We define the error introduced by any Markov jump filter described by (7) as

$$\hat{e}(k) = x(k) - \hat{x}(k), \quad (10)$$

and denote the sequence of filtering errors over a finite time horizon T as $\hat{e} \triangleq [\hat{e}(1), \ldots, \hat{e}(T)]$.

The cost of robust filtering is described by

$$\hat{\mathcal{J}}(\hat{e}) \triangleq \max_{\boldsymbol{p}_{\bullet\theta}} \hat{\mathcal{J}}(\hat{e}, \boldsymbol{p}_{\bullet\theta}) = \max_{\boldsymbol{p}_{\bullet\theta}} \sum_{k=1}^{T} \mathbf{E}[\|\hat{e}(k)\|^2].$$

We observe that

$$\|\hat{e}(k)\|^2 = \hat{e}^*(k)\hat{e}(k) = \mathrm{tr}[\hat{e}^*(k)\hat{e}(k)] = \mathrm{tr}[\hat{e}(k)\hat{e}^*(k)], \quad (11)$$

where $\mathrm{tr}[\cdot]$ denotes a *trace operator*.

Let us consider a full-order Markov jump filter having a structure similar to the structure of Luenberger observer, i.e.

$$\begin{cases} \tilde{x}(k+1) = A_{\theta(k)}\tilde{x}(k) + B_{\theta(k)}u(k) - L_{\theta(k)}(k)(y(k) - \tilde{y}(k)) \\ \tilde{y}(k) = F_{\theta(k)}\tilde{x}(k), \quad \tilde{x}(0) \triangleq \mathbf{E}[x(0)] = \mu_0, \end{cases} \tag{12}$$

with $\mathbf{L}(k) \triangleq [L_1(k), \ldots, L_N(k)] \in \mathbb{H}^{p,n}$ being a vector of filter gain matrices, each of which related to an operational mode. The associated filtering error for this particular structure of Markov jump filter is denoted by

$$\tilde{e}(k) = x(k) - \tilde{x}(k). \tag{13}$$

In the remaining of this section we show that the considered filter is indeed optimal, i.e. a filter achieving the cost

$$\hat{\mathcal{J}} \triangleq \min_{\hat{e}} \max_{\boldsymbol{p}_{\bullet\theta}} \sum_{k=1}^{T} \mathbf{E}[\|\hat{e}(k)\|^2], \ \forall \hat{e}. \tag{14}$$

From (13), (4) and (12), it follows that

$$\tilde{e}(k+1) = (A_{\theta(k)} + L_{\theta(k)}(k)F_{\theta(k)})\tilde{e}(k) + (H_{\theta(k)} + L_{\theta(k)}(k)G_{\theta(k)})w(k), \tag{15}$$

$$\tilde{e}(0) = x_0 - \mu_0, \quad \mathbf{E}[\tilde{e}(0)] = 0. \tag{16}$$

Lemma 1: The following statements hold $\forall k \in \mathcal{T}, \forall i \in \mathcal{N}$:

$$\mathbf{E}[w(k)x^*(k)\mathbf{1}_{\{\theta(k)=i\}}] = 0, \tag{17}$$

$$\mathbf{E}[w(k)\tilde{x}^*(k)\mathbf{1}_{\{\theta(k)=i\}}] = 0, \tag{18}$$

$$\mathbf{E}[w(k)\tilde{e}^*(k)\mathbf{1}_{\{\theta(k)=i\}}] = 0. \tag{19}$$

We define the vector of the second moment errors associated to each operational mode as $\mathbf{Y}(k) = [Y_1(k), \ldots, Y_N(k)]$, where $k \in \mathcal{T}$, $\mathbf{Y}(k) \in \mathbb{H}^{n+}$, and

$$Y_i(k) \triangleq \mathbf{E}[\tilde{e}(k)\tilde{e}^*(k)\mathbf{1}_{\{\theta(k)=i\}}]. \tag{20}$$

Clearly, $\quad \hat{\mathcal{J}}(\tilde{e}, \boldsymbol{p}_{\bullet\theta}) = \sum_{k=1}^{T}\sum_{i=1}^{N} \mathrm{tr}[Y_i(k)].$

After defining $\quad \hat{\mathcal{J}}(\tilde{e}(k), \boldsymbol{p}_{\bullet i}(k-1)) \triangleq \mathrm{tr}[Y_i(k)], \qquad (21)$

we have that

$$\hat{\mathcal{J}}(\tilde{e}, \boldsymbol{p}_{\bullet\theta}) = \sum_{k=1}^{T}\sum_{i=1}^{N} \hat{\mathcal{J}}(\tilde{e}(k), \boldsymbol{p}_{\bullet i}(k-1)). \tag{22}$$

From (20), (16), Assumption 4, (2c) and (4), it follows that

$$Y_i(0) = \pi_i(0)(\mathcal{Q}_0 - \mu_0\mu_0^*), \tag{23}$$

which is deterministic.

From (20), (15), Assumption 4, (2c), and (19), we have that

$$Y_j(k+1) = \sum_{i=1}^{N} p_{ij}(k)[(A_i + L_i(k)F_i)Y_i(k)(A_i^* + F_i^* L_i^*(k)) + \pi_i(k)(H_i H_i^* + L_i(k)G_i G_i^* L_i^*(k))] \tag{24}$$

Lemma 2: At any time step $k \in \mathbb{N}$ and for any operational mode $j \in \mathcal{N}$, the maximum in transition probabilities of the filtering cost function $\hat{\mathcal{J}}(\tilde{e}(k), \boldsymbol{p}_{\bullet j}(k-1))$ is attained on a vertex of the convex polytope of the column-vectors that define the j-th column of the polytopic TPM, i.e.

$$\hat{\mathcal{J}}_j(\tilde{e}(k)) = \max_{\boldsymbol{p}_{\bullet j}(k-1)} \hat{\mathcal{J}}(\tilde{e}(k), \boldsymbol{p}_{\bullet j}(k-1)) = \max_{\boldsymbol{p}_{\bullet j}^{l}} \hat{\mathcal{J}}(\tilde{e}(k), \boldsymbol{p}_{\bullet j}^{l}). \tag{25}$$

Let us denote by $\boldsymbol{p}_{\bullet j}^{\upsilon(k-1)} \triangleq \arg \max_{\boldsymbol{p}_{\bullet j}(k-1)} \hat{\mathcal{J}}_j(\tilde{e}(k)) \qquad (26)$

the vertex of the convex polytope of the column-vectors (which define the j-th columns of the polytopic TPM) that achieves the maximal filtering cost at time step k for the j-th operational mode. The vector $\boldsymbol{p}_{\bullet j}^{\upsilon(k-1)}$ is obtained during the computation of the robust filtering cost function $\hat{\mathcal{J}}_j(\tilde{e}(k))$. It defines the value of $\pi_j(k)$ via (3) and the value of $Y_j(k)$ via (24), allowing the recursion.

The question of choosing $\mathbf{L}(k)$, in a way to minimize the filtering error, remains open, and it is tackled in the remaining of this section.

Let us compute the value of $\mathbf{E}[\tilde{e}(k)\tilde{x}^*(k)\mathbf{1}_{\{\theta(k)=i\}}]$.

For $k=0$, from (13), (16), (9), and the linearity of the expected value, we have that $\forall i \in \mathcal{N}$ $\mathbf{E}[\tilde{e}(0)\tilde{x}^*(0)\mathbf{1}_{\{\theta(0)=i\}}] = 0$.

Following the mathematical induction technique, we assume that for any $k \in \mathcal{T}$, the structure of $\mathbf{L}(k-1)$ is such that $\forall i \in \mathcal{N}$, $\mathbf{E}[\tilde{e}(k)\tilde{x}^*(k)\mathbf{1}_{\{\theta(k)=i\}}] = 0$, and proceed to find $\mathbf{L}(k)$ such that $\mathbf{E}[\tilde{e}(k+1)\tilde{x}^*(k+1)\mathbf{1}_{\{\theta(k+1)=j\}}] = 0$, $\forall j \in \mathcal{N}$.

From (15), (12), (4), (13), linearity of the expected value, the fact that, $\forall i \in \mathcal{N}$, the matrices A_i, B_i, F_i, G_i, H_i are constant, $L_i(k)$ is determined by our choice, $u(k)$ is deterministic, independence between $w(k)$, $\theta(k)$, and $x(0)$, given by Assumption 4, (5), (19), induction hypothesis, (20), and (6), we have that

$$\mathbf{E}\left[\tilde{e}(k+1)\tilde{x}^*(k+1)\mathbf{1}_{\{\theta(k+1)=j\}}\right] = -\sum_{i=1}^{N} p_{ij}(k)[(A_i + L_i(k)F_i)\, Y_i(k)\, (A_i^* + F_i^* L_i^*(k)) + \pi_i(k)L_i(k)G_i G_i^* L_i^*(k)],$$

which is equal to zero if (and only if, for every $p_{ij}(k) \neq 0$)

$$L_i(k) = \begin{cases} 0, \text{ if } \pi_i(k) = 0; \text{ otherwise} \\ -A_i Y_i(k) F_i^* \left(F_i Y_i(k) F_i^* + \pi_i(k) G_i G_i^*\right)^{-1} \end{cases} \tag{27}$$

which is obtained by observing that $\pi_i(k) = 0$ implies that $Y_i(k) = 0$ (see (2c) and (20)), and thus also $L_i(k) = 0$.

For the notational convenience, we define

$$\mathbb{P}(k) \triangleq \{i \in \mathcal{N}; \pi_i(k) \neq 0\}. \tag{28}$$

From (27), (28), and (26), we have that for

$$Y_j(k+1) = \sum_{i \in \mathbb{P}} p_{ij}^{\upsilon(k)}[A_i Y_i(k) A_i^* + \pi_i(k) H_i H_i^* + L_i(k) F_i Y_i(k) A_i^*], \tag{29}$$

where $p_{ij}^{\upsilon(k)}$ are the elements of $\boldsymbol{p}_{\bullet j}^{\upsilon(k)}$.

The following lemma is useful within a proof that the observer (12), having a filtering gain (27) computed interactively from (23), (29), and (26).

Lemma 3: The following statements hold $\forall k \in \mathcal{T}, \forall i \in \mathcal{N}$:

$$\mathbf{E}[\tilde{e}(k)\hat{x}^*(k)\mathbf{1}_{\{\theta(k)=i\}}] = 0. \tag{30}$$

Lemma 4: Let $\hat{e}(k)$ be the error introduced by any Markov jump filter, as described by (10), and $\mathbf{Y}(k)$ be the solution of the system of coupled Riccati difference equations associated to robust filtering problem at time step k, obtained from (29), (27), and (26). Then, $\forall k \in \mathbb{N}_0$, $\mathbf{E}\left[\|\hat{e}(k)\|^2\right] \geq \sum_{i=1}^{N} \mathrm{tr}\,[Y_i(k)]$.

The main result of this section is straightforward from Lemma 4, (7) and (12).

Theorem 1: An optimal solution for the robust filtering problem posed above is:

$$\hat{A}_i(k) = A_i + L_i(k)F_i + B_i\hat{C}_i(k), \quad \hat{B}_i(k) = -L_i(k), \tag{31}$$

with $L_i(k)$ as in (27), obtained from (29) and (26), $\hat{C}_i(k)$ arbitrary, and the optimal robust cost defined in (14) being

$$\hat{\mathcal{J}} = \sum_{k=1}^{T}\sum_{i=1}^{N} \mathrm{tr}\left[Y_i(k)\right]. \tag{32}$$

IV. OPTIMAL ROBUST CONTROL

We consider in this section the finite horizon quadratic optimal robust control problem for MJLSs when the state variable $x(k)$ and jump variable $\theta(k)$ are available to the controller. The random variables $\{x(t),\theta(t);t=0,\dots,k\}$ generate the σ-algebra denoted by $\mathcal{G}'_k$. For every $k\in\mathbb{N}_0$

$$\mathcal{G}'_k\subset\mathcal{G}'_{k+1}\subset\mathcal{F}$$

In order to be able to apply a separation principle (described in the next section), we consider slightly different PTI MJLS, described by the following stochastic equations:

$$\begin{cases} x(k+1)=A_{\theta(k)}x(k)+B_{\theta(k)}u(k)+R_{\theta(k)}v(k),\\ \quad z(k)=C_{\theta(k)}x(k)+D_{\theta(k)}u(k),\\ \quad x(0)=x_0,\ \theta(0)=\theta_0,\ \boldsymbol{\pi}(0)=\boldsymbol{\pi}_0,\end{cases}\tag{33}$$

with $v=\{v(k);k\in\{0,\dots,T-1\}\}$ being a noise sequence, and $\mathbf{R}=[R_1,\dots,R_N]\in\mathbb{H}^{r,n}$.

Assumption 5: The noise sequence v satisfies $\forall i\in\mathcal{N}$

$$\mathbf{E}[v(k)v^*(k)\mathbf{1}_{\{\theta(k)=i\}}]=\Xi(k),\ \ \mathbf{E}[v(0)x^*(0)\mathbf{1}_{\{\theta(k)=i\}}]=0.\tag{34}$$

As in [1, p.73], we assume that for any measurable functions f and g

$$\mathbf{E}[f(v(k))g(\theta(k+1))|\mathcal{G}'_k]=\mathbf{E}[f(v(k))|\mathcal{G}'_k]\sum_{j=1}^{N}p_{\theta(k)j}(k)g(j)\tag{35}$$

In the next section we will show that this assumption is verified for the controlled system with partial information on continuous state.

The set of admissible controllers, denoted by $\mathcal{U}_T$, is given by the sequence of control laws u such that for each k, $u(k)$ is $\mathcal{G}'_k$-measurable,

$$\mathbf{E}[v(k)x^*(k)\mathbf{1}_{\{\theta(k)=i\}}]=0,\tag{36}$$
$$\mathbf{E}[v(k)u^*(k)\mathbf{1}_{\{\theta(k)=i\}}]=0.\tag{37}$$

The **problem** we examine in this section is to find $u\in\mathcal{U}_T$ which achieves (8). We proceed by using a dynamic programming approach in Bellman's optimization formulation [23]. For lack of space, some of the formal passages are omitted. However, those details can be found in [19], where we present a solution to a similar problem.

The terminal cost is given by

$$\begin{aligned}\mathcal{J}(\theta(T),x(T))&=x^*(T)\mathbf{E}[X_{\theta(T)}|\mathcal{G}'_T]x(T)=\\ &\quad x^*(T)\mathcal{X}_{\theta(T)}x(T),\end{aligned}\tag{38}$$

where $X_{\theta(T)}\triangleq\mathcal{X}_{\theta(T)}$ is a solution to CRDE for the robust control at the terminal time step. We are interested in the explicit form of $\mathbf{X}(k)=[X_1(k),\dots,X_N(k)]\in\mathbb{H}^{n+}$.

A generic cost at time step k is

$$\begin{aligned}&\mathcal{J}(\theta(k),x(k),u(k),\boldsymbol{p}_{\theta(k)\bullet}(k))=\\ &\qquad\mathbf{E}\left[\|z(k)\|^2+\mathcal{J}(\theta(k+1),x(k+1))|\mathcal{G}'_k\right].\end{aligned}\tag{39}$$

The cost-to-go function is, for $\theta(k)=i,\ i\in\mathcal{N}$,

$$\mathcal{J}(i,x(k))=\min_{u(k)}\max_{\boldsymbol{p}_{i\bullet}(k)}\mathcal{J}(i,x(k),u(k),\boldsymbol{p}_{i\bullet}(k)).\tag{40}$$

Observing that from the definitions of the trace operator and of the matrix product, for any matrix M, and column vectors $\mathbf{a}$, $\mathbf{b}$ of appropriate size, we have the following expression:

$$\mathbf{a}^*\mathrm{M}\mathbf{b}=\mathrm{tr}[\mathbf{a}^*\mathrm{M}\mathbf{b}]=\mathrm{tr}[\mathrm{M}\mathbf{b}\mathbf{a}^*].\tag{41}$$

The explicit expression of $\mathbf{E}[X_{\theta(T)}|\mathcal{G}'_{T-1}]$ is given by

$$\mathbf{E}[X_{\theta(T)}|\mathcal{G}'_{T-1}]=\sum_{j=1}^{N}p_{ij}(T-1)X_j(T).\tag{42}$$

The explicit form of CRDE for robust control is obtained recursively from generic cost at time step $k=T-1$. Since $x(T-1)$, $\theta(T-1)$, and any admissible control input $u(k)$ are $\mathcal{G}'_{T-1}$-measurable, A_i, B_i, C_i, D_i and R_i are constant matrices for all $i\in\mathcal{N}$, $\mathbf{X}(T)\in\mathbb{H}^{n+}$, by linearity of the expected value, from (33), (34), (35), (36), (37), (41), and (42), we have for $\theta(T-1)=i$ that

$$\begin{aligned}&\mathcal{J}(i,x(T-1),u(T-1),\boldsymbol{p}_{i\bullet}(T-1))=\\ &x^*(T-1)\left[C_i^*C_i+A_i^*\sum_{j=1}^{N}p_{ij}(T-1)X_j(T)A_i\right]x(T-1)+\\ &2x^*(T-1)A_i^*\sum_{j=1}^{N}p_{ij}(T-1)X_j(T)B_iu(T-1)+\\ &u^*(T-1)\left[D_i^*D_i+B_i^*\sum_{j=1}^{N}p_{ij}(T-1)X_j(T)B_i\right]u(T-1)+\\ &\sum_{i=1}^{N}\mathrm{tr}\left[R_i\Xi_i(T-1)R_i^*\sum_{j=1}^{N}p_{ij}(T-1)X_j(T)\right].\end{aligned}\tag{43}$$

Let us consider $\mathcal{J}(i,x(T-1),u(T-1),\boldsymbol{p}_{i\bullet}(T-1))$ in (43) as a function of only $\boldsymbol{p}_{i\bullet}(T-1)$, which from Assumption 1 is polytopic. It is straightforward verifying that *Jensen's inequality* [24, p. 25, Theorem 4.3] holds. Hence, this generic quadratic cost is a *convex function* in a variable, that belongs to a *polytopic set*. From [24, p. 343, Theorem 32.2] this means that the maximum in transition probabilities of the quadratic cost function is attained on a vertex of the convex polytope of transition probabilities. So, to find this maximum, we need to evaluate $\mathcal{J}(i,x(T-1),u(T-1),\boldsymbol{p}^l_{i\bullet})$ in each (known) vertex of the corresponding row of the polytopic TPM.

Since $u(T-1)$ is unconstrained, we can compute the minimum of (43) in $u(T-1)$ by equaling to 0 its derivative with respect to $u(T-1)$, obtaining that

$$u(T-1)=K_i^l(T-1)x(T-1),\tag{44}$$

where the optimal gain at time $k=T-1$ for i-th operational mode and transition probability vector $\boldsymbol{p}^l_{i\bullet}$ is given by

$$\begin{aligned}&K_i^l(T-1)=\\ &-[D_i^*D_i+B_i^*\sum_{j=1}^{N}p^l_{ij}X_j(T)B_i]^{-1}B_i^*\sum_{j=1}^{N}p^l_{ij}X_j(T)A_i.\end{aligned}\tag{45}$$

We observe that among V vertices of convex polytope of transition probability vectors $\boldsymbol{p}^l_{i\bullet}$, there is one, indicated by $\boldsymbol{p}^{\bar{v}}_{i\bullet}$, for which $\mathcal{J}(i,x(T-1))$ of (40) is attained. We denote by $K_i^{\bar{v}}(T-1)$ the corresponding optimal gain. When the optimal gain is applied, we have that

$$\boldsymbol{p}^{\bar{v}}_{i\bullet}\triangleq\arg\max_{\boldsymbol{p}^l_{i\bullet}}\mathcal{J}(i,x(T-1)).\tag{46}$$

Thus, $\forall l\in\mathcal{V}$

$$\begin{aligned}&\mathcal{J}(i,x(T-1),K_i^{\bar{v}}(T-1)x(T-1),\boldsymbol{p}^{\bar{v}}_{i\bullet})\geq\\ &\mathcal{J}(i,x(T-1),K_i^{\bar{v}}(T-1)x(T-1),\boldsymbol{p}^{l}_{i\bullet}).\end{aligned}\tag{47}$$

This leads us to the following expression of the cost-to-go:

$$\begin{aligned}\mathcal{J}(i,x(T-1))&=x^*(T-1)X_i^{\bar{v}}(T-1)x(T-1)\ +\\ &\sum_{i=1}^{N}\mathrm{tr}\left[R_i\Xi_i(T-1)R_i^*\sum_{j=1}^{N}p^{\bar{v}}_{ij}X_j(T)\right],\end{aligned}\tag{48}$$

where

$$X_i^{\bar{\upsilon}}(T-1) \triangleq C_i^* C_i + A_i^* \sum_{j=1}^{N} p_{ij}^{\bar{\upsilon}} X_j(T) A_i + A_i^* \sum_{j=1}^{N} p_{ij}^{\bar{\upsilon}} X_j(T) B_i K_i^{\bar{\upsilon}}(T-1). \tag{49}$$

We underline that the cost-to-go $\mathcal{J}(i, x(T-1))$ depends on both i and $x(T-1)$, so $\bar{\upsilon}$ can be different for distinct values of i and $x(T-1)$. For any given state $x(T-1)$, $\bar{\upsilon}$ is a compact notation for $\bar{\upsilon}(i, k)$. Without knowing a priori in which state system will be at time step $T-1$, we need to consider all the vertices as possible candidates. The cost-to-go becomes

$$\mathcal{J}(i, x(T-1)) = \max_{\boldsymbol{p}_{i\bullet}^{l}} \; x^*(T-1) X_i^l(T-1) x(T-1) + \sum_{i=1}^{N} \mathrm{tr}\left[R_i \Xi_i(T-1) R_i^* \sum_{j=1}^{N} p_{ij}^{l} X_j(T) \right]. \tag{50}$$

The optimal state-feedback gain for a given operational mode is not unique over the entire state space, but is state dependent; we can have up to L optimal gains at time step $k = T-1$. Given the optimal solution at one time step, we can repeat the described procedure, as presented in [19].

By interacting the procedure, we obtain that the optimal gain has the form as in (45). However, we need to consider all the solutions of CRDEs that can achieve maximum in transition probabilities. See [19] for the hint of the way allowing us to reduce the number of equations to deal with. We observe that when the MJLS is stabilizable, after a transitory period, the finite-horizon optimal state feedback robust solution becomes unique. The length of transitory period depend on how much the joint spectral radius (JSR) associated to the second moment of the state vector through vertices of polytopic TPM is smaller than 1, since from [16] we know that $\mathbf{E}[\|x(k)\|^2] \leq nN\beta\zeta^k \|x_0\|_2^2$, $\zeta \in (\mathrm{JSR}, 1)$ and $\beta \geq 1$. In this way, the state space reachable from the initial state becomes smaller and smaller, with fewer number of possible CDREs. When the MJLS is non-stabilizable, the behavior is opposite.

V. Separation Principle

With the solutions of optimal robust filtering and control problems presented in the previous two sections, the separation principle can be easily derived following the line of reasoning presented in [1, pp.132–136]. The optimal solution is given by: $\hat{A}_i(k) = A_i + L_i(k) F_i + B_i K_i^{l(k)}(k)$,

$$\hat{B}_i(k) = -L_i(k), \qquad \hat{C}_i(k) = K_i^{l(k)}(k).$$

VI. Conclusions

This paper is part of a research line with the final aim of deriving novel fundamental results for the class of MJLSs, where the transition probability matrix of the underlying Markov chain is varying over time with variations that are arbitrary within a polytopic set of stochastic matrices. In particular, in this paper we address and solve the finite horizon optimal control and filtering problems. The next step is to extend our results to the infinite time horizon case, and to tackle problems of fault detection and isolation in the general framework of security of cyber-physical systems.

References

[1] O. L. V. Costa, M. D. Fragoso, and R. P. Marques, *Discrete-time Markov jump linear systems*. Springer-Verlag, London, 2005.

[2] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S. S. Sastry, "Foundations of control and estimation over lossy networks," *Proc. of the IEEE*, vol. 95, no. 1, pp. 163–187, Jan. 2007.

[3] M. Tabbara, D. Nesic, and A. R. Teel, "Stability of wireless and wireline networked control systems," *IEEE Trans. Autom. Control*, vol. 52, no. 9, pp. 1615–1630, 2007.

[4] J. P. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of Recent Results in Networked Control Systems," *Proc. of the IEEE*, vol. 95, no. 1, pp. 138–162, Jan. 2007.

[5] V. Gupta, A. F. Dana, J. P. Hespanha, R. M. Murray, and B. Hassibi, "Data transmission over networks for estimation and control," *IEEE Trans. Autom. Control*, vol. 54, no. 8, pp. 1807–1819, 2009.

[6] W. M. H. Heemels, A. R. Teel, N. van de Wouw, and D. Nesic, "Networked control systems with communication constraints: Tradeoffs between transmission intervals, delays and performance," *IEEE Trans. Autom. Control*, vol. 55, no. 8, pp. 1781–1796, 2010.

[7] M. Pajic, S. Sundaram, G. J. Pappas, and R. Mangharam, "The wireless control network: a new approach for control over networks," *IEEE Trans. Autom. Control*, vol. 56, no. 10, pp. 2305–2318, Oct. 2011.

[8] R. Alur, A. D'Innocenzo, K. H. Johansson, G. J. Pappas, and G. Weiss, "Compositional Modeling and Analysis of Multi-Hop Control Networks," *IEEE Trans. Autom. Control*, vol. 56, no. 10, pp. 2345–2357, 2011.

[9] P. Sadeghi, R. A. Kennedy, P. B. Rapajic, and R. Shams, "Finite-state Markov modeling of fading channels - a survey of principles and applications," *IEEE Signal Processing Mag.*, vol. 25, no. 5, pp. 57–80, Sept. 2008.

[10] A. P. Gonçalves, A. R. Fioravanti, and J. C. Geromel, "Markov jump linear systems and filtering through network transmitted measurements," *Signal Process.*, vol. 90, no. 10, pp. 2842–2850, 2010.

[11] Y. Long and G.-H. Yang, "Fault detection for a class of nonhomogeneous Markov jump systems based on delta operator approach," *Proc. of the Institution of Mech. Engineers, Part I: J. of Syst. and Control Eng.*, vol. 227, no. 1, pp. 129–141, 2013.

[12] Y. Yin, P. Shi, F. Liu, and K. L. Teo, "Filtering for discrete-time nonhomogeneous Markov jump systems with uncertainties," *Inform. Sciences*, vol. 259, pp. 118–127, 2014.

[13] ——, "Observer-based $\mathcal{H}_\infty$ control on nonhomogeneous Markov jump systems with nonlinear input," *Int. J. of Robust and Nonlinear Control*, vol. 24, no. 13, pp. 1903–1924, 2014.

[14] H. Zhang, R. Yang, H. Yan, and F. Yang, "$\mathcal{H}_\infty$ consensus of event-based multi-agent systems with switching topology," *Inform. Sciences*, vol. 370-371, pp. 623–635, 2016.

[15] Y. Ding, H. Liu, and J. Cheng, "$\mathcal{H}_\infty$ filtering for a class of discrete-time singular Markovian jump systems with time-varying delays," *ISA Trans.*, vol. 53, no. 4, pp. 1054–1060, 2014.

[16] Y. Zacchia Lun, A. D'Innocenzo, and M. D. Di Benedetto, "On stability of time-inhomogeneous Markov jump linear systems," in *IEEE 55th Conf. on Decision and Control*, Dec. 2016, pp. 5527–5532.

[17] R. Jungers, *The joint spectral radius: theory and applications*, ser. Lecture Notes in Control and Inform. Sci. Springer-Verlag Berlin Heidelberg, 2009, vol. 385.

[18] Y. Zacchia Lun, A. D'Innocenzo, and M. D. Di Benedetto, "Robust stability of time-inhomogeneous Markov jump linear systems," in *Proc. of the 20th IFAC World Congress*, July 2017, pp. 3473–3478.

[19] ——, "Robust LQR for time-inhomogeneous Markov jump switched linear systems," in *Proc. of the 20th IFAC World Congress*, July 2017, pp. 2235–2240.

[20] Y. Zacchia Lun, "Stability and optimal control of polytopic time-inhomogeneous Markov jump linear systems," Ph.D. dissertation, Gran Sasso Science Institute (GSSI), L'Aquila, Italy, 2017.

[21] A. P. Gonçalves, A. R. Fioravanti, and J. C. Geromel, "Filtering of discrete-time markov jump linear systems with uncertain transition probabilities," *Int. J. of Robust and Nonlinear Control*, vol. 21, no. 6, pp. 613–624, 2011.

[22] D. J. Hartfiel, *Markov Set-Chains*, ser. Lecture Notes in Math. Springer-Verlag Berlin Heidelberg, 1998, vol. 1695.

[23] D. P. Bertsekas, *Dynamic programming and optimal control*. Athena Scientific, Belmont, MA, 1995, vol. I & II.

[24] R. T. Rockafellar, *Convex analysis*. Princeton Univ. Press, 1997.