Bisimilar Symbolic Models for Stochastic Control Systems without State-Space Discretization *

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ABSTRACT

In the past few years different techniques have been developed for constructively deriving symbolic abstractions of (stochastic) control systems. The obtained symbolic models allow us to leverage the apparatus of finite-state reactive synthesis towards the problem of designing hybrid controllers enforcing rich logic specifications over the concrete models. Unfortunately, most of the existing techniques severely suffer from the curse of dimensionality due to the need to discretize state and input sets. In this paper we provide a symbolic abstraction technique for incrementally stable stochastic control systems, which only requires discretizing input sets. We show that for every incrementally stable stochastic control system, and for every given positive precision ε, the discretization of exclusively the input set allows constructing a symbolic model which is ε-approximate bisimilar (in moments) to the original stochastic control system. The details of the proposed technique are elucidated by synthesizing a control policy for a 6-dimensional linear stochastic control system satisfying some logic specifications, which would not be tractable using existing approaches based on state-space discretization.

1. INTRODUCTION

In the last decade several abstraction techniques have been developed, providing symbolic models for (stochastic) control systems obtained by replacing aggregates or collections of states of such systems by symbols. When a system with a finite number of states is obtained, one can use mature methodologies available in the literature [9] to leverage fixed-point computations in order to synthesize hybrid controllers enforcing rich complex specifications over the original system. Examples of such specifications include properties expressed as formulas in linear temporal logic (LTL) or as automata on infinite strings.

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The construction of symbolic models has been studied extensively for *continuous-time* non-probabilistic systems. This includes results on the construction of approximately bisimilar symbolic abstractions for incrementally stable control systems [8, 12], switched systems [4], and control systems with disturbances [13], as well as the construction of sound abstractions based on the convexity of reachable sets [14] and for unstable control systems [20]. Recently, there have been some results on the construction of symbolic models for *continuous-time* stochastic systems, including the construction of approximately bisimilar (in moments) symbolic models for incrementally stable stochastic control systems [18, 19] and for stochastic switched systems [16], as well as sound abstractions for unstable stochastic control systems [17]. The results in [6] propose abstraction notions for continuous-time stochastic hybrid systems, but with a different purpose: while we are interested in the construction of bisimilar abstractions that are finite, the work in [6] uses the notion of bisimulation to relate continuous (and thus infinite) stochastic hybrid systems. Note that all the abstraction techniques provided in [4, 8, 12, 13, 14, 16, 17, 18, 19, 20] are based on the discretization of state and input sets. Therefore, they suffer severely from *the curse of dimensionality* due to the need to grid both sets, which is especially tolling for models with high-dimensional state spaces.

In this paper, we construct approximately bisimilar symbolic models for incrementally stable continuous-time stochastic control systems, where only the input set requires to be discretized. This work is inspired by the recently proposed result in [2] for discrete-time non-probabilistic switched systems, in which mode sequences of a given length are considered as symbolic states. Since in the new approach we do not discretize the state space, this is potentially more efficient than the one proposed in [18, 19] when dealing with higher dimensional continuous-time stochastic systems. We provide a simple criterion that helps choosing the the most suitable among two approaches (in terms of the sizes of the symbolic models) for a given stochastic control system. Another advantage of the technique proposed here is that it allows us to construct symbolic models with probabilistic output values, resulting in less conservative symbolic abstractions than those proposed in [16, 17, 18, 19] that allow for non-probabilistic output values exclusively. We then explain how the proposed symbolic models with probabilistic output values can be used for synthesizing hybrid controllers enforcing logic specifications. The effectiveness of the proposed results is illustrated by synthesizing a control policy for a simple 6-dimensional linear stochastic control system against an LTL specification, which is not amenable to be

dealt with the approaches proposed in [18, 19]. Due to space constraints, most of the proofs of the main results are omitted from this manuscript.

2. STOCHASTIC CONTROL SYSTEMS

2.1 Notation

The identity map on a set A is denoted by 1_A. The symbols $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}_0^+$ denote the set of natural, nonnegative integer, integer, real, positive, and nonnegative real numbers, respectively. The symbols I_n, 0_n, and $0_{n\times m}$ denote the identity matrix, zero vector, and zero matrix in $\mathbb{R}^{n\times n}$, $\mathbb{R}^n$, and $\mathbb{R}^{n\times m}$, respectively. Given a vector $x \in \mathbb{R}^n$, we denote by x_i the i–th element of x, and by $\|x\|$ the infinity norm of x, namely, $\|x\| = \max\{|x_1|, |x_2|, ..., |x_n|\}$, where $|x_i|$ denotes the absolute value of x_i. Given a matrix $P = \{p_{ij}\} \in \mathbb{R}^{n\times n}$, we denote by $\mathrm{Tr}(P) = \sum_{i=1}^n p_{ii}$ the trace of P. We denote by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ the minimum and maximum eigenvalues of symmetric matrix A, respectively. The diagonal set $\Delta \subset \mathbb{R}^n \times \mathbb{R}^n$ is defined as: $\Delta = \{(x, x) \mid x \in \mathbb{R}^n\}$.

The closed ball centered at $x \in \mathbb{R}^m$ with radius λ is defined by $\mathcal{B}_\lambda(x) = \{y \in \mathbb{R}^m \mid \|x - y\| \le \lambda\}$. A set $B \subseteq \mathbb{R}^m$ is called a *box* if $B = \prod_{i=1}^m [c_i, d_i]$, where $c_i, d_i \in \mathbb{R}$ with $c_i < d_i$ for each $i \in \{1, \dots, m\}$. The *span* of a box B is defined as $span(B) = \min\{|d_i - c_i| \mid i = 1, \dots, m\}$. For a box $B \subseteq \mathbb{R}^m$ and $\mu \le span(B)$, define the μ-approximation $[B]_\mu = [\mathbb{R}^m]_\mu \cap B$, where $[\mathbb{R}^m]_\mu = \{a \in \mathbb{R}^m \mid a_i = k_i\mu, k_i \in \mathbb{Z}, i = 1, \dots, m\}$. Note that $[B]_\mu \neq \varnothing$ for any $\mu \le span(B)$. Geometrically, for any $\mu \in \mathbb{R}^+$ with $\mu \le span(B)$ and $\lambda \ge \mu$, the collection of sets $\{\mathcal{B}_\lambda(p)\}_{p\in[B]_\mu}$ is a finite covering of B, i.e. $B \subseteq \bigcup_{p\in[B]_\mu} \mathcal{B}_\lambda(p)$. We extend the notions of *span* and *approximation* to finite unions of boxes as follows. Let $A = \bigcup_{j=1}^M A_j$, where each A_j is a box. Define $span(A) = \min\{span(A_j) \mid j = 1, \dots, M\}$, and for any $\mu \le span(A)$, define $[A]_\mu = \bigcup_{j=1}^M [A_j]_\mu$.

Given a measurable function $f : \mathbb{R}_0^+ \to \mathbb{R}^n$, the (essential) supremum of f is denoted by $\|f\|_\infty := (\mathrm{ess})\sup\{\|f(t)\|, t \ge 0\}$. A continuous function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ is said to belong to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the map $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed nonzero r, the map $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$. We identify a relation $R \subseteq A \times B$ with the map $R : A \to 2^B$ defined by $b \in R(a)$ iff $(a, b) \in R$. Given a relation $R \subseteq A \times B$, R^{-1} denotes the inverse relation defined by $R^{-1} = \{(b, a) \in B \times A : (a, b) \in R\}$.

2.2 Stochastic control systems

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_s)_{s\ge0}$ satisfying the usual conditions of completeness and right continuity [7, p. 48]. Let $(W_s)_{s\ge0}$ be a p-dimensional $\mathbb{F}$-adapted Brownian motion.

Definition 2.1. *A stochastic control system is a tuple* $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f, \sigma)$, *where*

- $\mathbb{R}^n$ *is the state space;*
- $\mathsf{U} \subseteq \mathbb{R}^m$ *is a compact input set;*
- $\mathcal{U}$ *is a subset of the set of all measurable functions of time from* $\mathbb{R}_0^+$ *to* U*;*
- $f : \mathbb{R}^n \times \mathsf{U} \to \mathbb{R}^n$ *satisfies the following Lipschitz assumption: there exist constants* $L_x, L_u \in \mathbb{R}^+$ *such that:* $\|f(x, u) - f(x', u')\| \le L_x\|x - x'\| + L_u\|u - u'\|$ *for all* $x, x' \in \mathbb{R}^n$ *and all* $u, u' \in \mathsf{U}$*;*
- $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times p}$ *satisfies the following Lipschitz assumption: there exists a constant* $Z \in \mathbb{R}^+$ *such that:* $\|\sigma(x) - \sigma(x')\| \le Z\|x - x'\|$ *for all* $x, x' \in \mathbb{R}^n$.

A continuous-time stochastic process $\xi : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}^n$ is said to be a *solution process* of Σ if there exists $\upsilon \in \mathcal{U}$ satisfying the following stochastic differential equation (SDE):

$$\mathrm{d}\,\xi = f(\xi, \upsilon)\,\mathrm{d}\,t + \sigma(\xi)\,\mathrm{d}\,W_t, \tag{2.1}$$

$\mathbb{P}$-almost surely ($\mathbb{P}$-a.s.), where f is known as the drift and σ as the diffusion. We also write $\xi_{a\upsilon}(t)$ to denote the value of the solution process at time $t \in \mathbb{R}_0^+$ under the input curve υ from initial condition $\xi_{a\upsilon}(0) = a$ $\mathbb{P}$-a.s., in which a is a random variable that is measurable in $\mathcal{F}_0$. Let us emphasize that the solution process is unambiguously determined, since the assumptions on f and σ ensure its existence and uniqueness [10, Theorem 5.2.1, p. 68].

3. INCREMENTAL STABILITY

We recall a stability notion for stochastic control systems, introduced in [19], on which the main results presented in this work rely.

Definition 3.1. *A stochastic control system* Σ *is incrementally input-to-state stable in the* qth *moment (*δ*-ISS-M*$_q$*), where* $q \ge 1$*, if there exist a* $\mathcal{KL}$ *function* β *and a* $\mathcal{K}_\infty$ *function* γ *such that for any* $t \in \mathbb{R}_0^+$*, any* $\mathbb{R}^n$*-valued random variables* a *and* a' *that are measurable in* $\mathcal{F}_0$*, and any* $\upsilon, \upsilon' \in \mathcal{U}$*, the following condition is satisfied:*

$$\mathbb{E}\left[\|\xi_{a\upsilon}(t) - \xi_{a'\upsilon'}(t)\|^q\right] \le \beta\left(\mathbb{E}\left[\|a - a'\|^q\right], t\right) + \gamma\left(\|\upsilon - \upsilon'\|_\infty\right). \tag{3.1}$$

As showed in [19], one can describe δ-ISS-M$_q$ in terms of the existence of so-called *incremental Lyapunov functions*, as defined next.

Definition 3.2. *Consider a stochastic control system* Σ *and a continuous function* $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ *that is twice continuously differentiable on* $\{\mathbb{R}^n \times \mathbb{R}^n\}\backslash\Delta$*. The function* V *is called an incremental input-to-state stability in the* qth *moment (*δ*-ISS-M*$_q$*) Lyapunov function for* Σ*, where* $q \ge 1$*, if there exist* $\mathcal{K}_\infty$ *functions* $\underline{\alpha}$*,* $\overline{\alpha}$*,* ρ*, and a constant* $\kappa \in \mathbb{R}^+$*, such that*

(i) $\underline{\alpha}$ *(resp.* $\overline{\alpha}$*) is a convex (resp. concave) function;*

(ii) for any $x, x' \in \mathbb{R}^n$*,*
$\underline{\alpha}\left(\|x - x'\|^q\right) \le V(x, x') \le \overline{\alpha}\left(\|x - x'\|^q\right)$*;*

(iii) for any $x, x' \in \mathbb{R}^n$*,* $x \neq x'$*, and for any* $u, u' \in \mathsf{U}$*,*

$$\begin{aligned}\mathcal{L}^{u,u'}V(x, x') &:= [\partial_x V \;\; \partial_{x'} V]\begin{bmatrix} f(x, u) \\ f(x', u')\end{bmatrix} \\ &\quad + \frac{1}{2}Tr\left(\begin{bmatrix}\sigma(x) \\ \sigma(x')\end{bmatrix}\begin{bmatrix}\sigma^T(x) & \sigma^T(x')\end{bmatrix}\begin{bmatrix}\partial_{x,x}V & \partial_{x,x'}V \\ \partial_{x',x}V & \partial_{x',x'}V\end{bmatrix}\right) \\ &\le -\kappa V(x, x') + \rho(\|u - u'\|),\end{aligned}$$

where $\mathcal{L}^{u,u'}$ is the infinitesimal generator associated to the process $V(\xi,\xi')$ where ξ and ξ' are solution processes of the SDE (2.1) [10, Section 7.3]. The symbols ∂_x and $\partial_{x,x'}$ denote first- and second-order partial derivatives with respect to x and (x,x'), respectively.

Although condition (ii) in the above definition implies that the growth rate of functions $\overline{\alpha}$ and $\underline{\alpha}$ is linear, this condition does not restrict the behavior of $\overline{\alpha}$ and $\underline{\alpha}$ to only linear functions on a compact subset of $\mathbb{R}^n$. The following theorem, borrowed from [19], describes δ-ISS-M$_q$ in terms of the existence of δ-ISS-M$_q$ Lyapunov functions.

THEOREM 3.3. *A stochastic control system Σ is δ-ISS-M$_q$ if it admits a δ-ISS-M$_q$ Lyapunov function.*

One can resort to available software tools, such as SOSTOOLS [11], to search for appropriate δ-ISS-M$_q$ Lyapunov functions for polynomial type Σ. We refer the interested readers to the results in [19], providing special instances where these functions can be easily computed. For example, for linear stochastic control systems Σ (that is, for systems with linear drift and diffusion terms), one can search for appropriate δ-ISS-M$_q$ Lyapunov functions by solving a linear matrix inequality (LMI).

3.1 Noisy and noise-free trajectories

In order to introduce the symbolic models in Subsection 5.2 (Theorems 5.6 and 5.7) for a stochastic control system, we need the following technical result, borrowed from [19], which provides an upper bound on the distance (in the qth moment) between the solution process of Σ and the solution of the corresponding non-probabilistic control system obtained by disregarding the diffusion term σ. From now on, we use the notation $\overline{\xi}_{xv}$ to denote the solution of the ordinary differential equation (ODE) $\dot{\overline{\xi}}_{xv} = f\left(\overline{\xi}_{xv}, v\right)$ starting from the non-probabilistic initial condition x and under the input curve v.

LEMMA 3.4. *Consider a stochastic control system Σ such that $f(0_n,0_m)=0_n$ and $\sigma(0_n)=0_{n\times p}$. Suppose that $q\geq 2$ and there exists a δ-ISS-M$_q$ Lyapunov function V for Σ such that its Hessian is a positive semidefinite matrix in $\mathbb{R}^{2n\times 2n}$ and $\partial_{x,x}V(x,x')\leq P$, for any $x,x'\in\mathbb{R}^n$, and some positive semidefinite matrix $P\in\mathbb{R}^{n\times n}$. Then for any $x\in\mathbb{R}^n$ and any $v\in\mathcal{U}$, we have*

$$\mathbb{E}\left[\left\|\xi_{xv}(t)-\overline{\xi}_{xv}(t)\right\|^q\right]\leq h_x(\sigma,t), \tag{3.2}$$

where

$$h_x(\sigma,t)=\underline{\alpha}^{-1}\Bigg(\frac{1}{2}\left\|\sqrt{P}\right\|^2 n\min\{n,p\}Z^2\mathsf{e}^{-\kappa t} \cdot\int_0^t\left(\beta\left(\|x\|^q,s\right)+\gamma\left(\sup_{u\in\mathsf{U}}\{\|u\|\}\right)\right)^{\frac{2}{q}}\mathsf{d}s\Bigg).$$

It can be readily seen that the nonnegative valued function h_x tends to zero as $t\to 0$, $t\to+\infty$, or as $Z\to 0$, where Z is the Lipschitz constant for the diffusion, introduced in Definition 2.1. The interested readers are referred to [19], which provides results in line with that of Lemma 3.4 for (linear) stochastic control systems Σ admitting a specific type of δ-ISS-M$_q$ Lyapunov functions.

4. SYSTEMS AND APPROXIMATE EQUIVALENCE RELATIONS

4.1 Systems

We employ the notion of system, introduced in [15], to describe both stochastic control systems as well as their symbolic models.

DEFINITION 4.1. *A system S is a tuple $S=(X,X_0,U,\longrightarrow,Y,H)$, where X is a set of states (possibly infinite), $X_0\subseteq X$ is a set of initial states (possibly infinite), U is a set of inputs (possibly infinite), $\longrightarrow\subseteq X\times U\times X$ is a transition relation, Y is a set of outputs, and $H:X\to Y$ is an output map.*

A transition $(x,u,x')\in\longrightarrow$ is also denoted by $x\xrightarrow{u}x'$. For a transition $x\xrightarrow{u}x'$, state x' is called a u-successor, or simply a successor, of state x. We denote by $\mathbf{Post}_u(x)$ the set of all u-successors of a state x. For technical reasons, we assume that for any $x\in X$, there exists some u-successor of x, for some $u\in U$ — let us remark that this is always the case for the considered systems later in this paper.

System S is said to be

- *metric*, if the output set Y is equipped with a metric $\mathbf{d}:Y\times Y\to\mathbb{R}_0^+$;
- *finite* (or *symbolic*), if X and U are finite sets;
- *deterministic*, if for any state $x\in X$ and any input $u\in U$, $|\mathbf{Post}_u(x)|\leq 1$.

For a system $S=(X,X_0,U,\longrightarrow,Y,H)$ and given any initial state $x_0\in X_0$, a finite state run generated from x_0 is a finite sequence of transitions:

$$x_0\xrightarrow{u_0}x_1\xrightarrow{u_1}\cdots\xrightarrow{u_{n-2}}x_{n-1}\xrightarrow{u_{n-1}}x_n, \tag{4.1}$$

such that $x_i\xrightarrow{u_i}x_{i+1}$ for all $0\leq i<n$. A finite state run can be directly extended to an infinite state run as well. A finite output run is a sequence $\{y_0,y_1,\ldots,y_n\}$ such that there exists a finite state run of the form (4.1) with $y_i=H(x_i)$, for $i=0,\ldots,n$. A finite output run can also be directly extended to an infinite output run as well.

4.2 Relations among systems

We recall the notion of approximate (bi)simulation relation, introduced in [3], which is useful when analyzing or synthesizing controllers for deterministic systems.

DEFINITION 4.2. *Let $S_a=(X_a,X_{a0},U_a,\xrightarrow[a]{},Y_a,H_a)$ and $S_b=(X_b,X_{b0},U_b,\xrightarrow[b]{},Y_b,H_b)$ be metric systems with the same output sets $Y_a=Y_b$ and metric $\mathbf{d}$. For $\varepsilon\in\mathbb{R}_0^+$, a relation $R\subseteq X_a\times X_b$ is said to be an ε-approximate simulation relation from S_a to S_b if, for all $(x_a,x_b)\in R$, the following two conditions are satisfied:*

(i) $\mathbf{d}(H_a(x_a),H_b(x_b))\leq\varepsilon$;

(ii) $x_a\xrightarrow[a]{u_a}x_a'$ in S_a implies the existence of $x_b\xrightarrow[b]{u_b}x_b'$ in S_b satisfying $(x_a',x_b')\in R$.

A relation $R\subseteq X_a\times X_b$ is said to be an ε-approximate bisimulation relation between S_a and S_b if R is an ε-approximate simulation relation from S_a to S_b and R^{-1} is an ε-approximate simulation relation from S_b to S_a.

System S_a is ε-approximately simulated by S_b, or S_b ε-approximately simulates S_a, denoted by $S_a \preceq^{\varepsilon}_{\mathcal{S}} S_b$, if there exists an ε-approximate simulation relation R from S_a to S_b such that:

- *for every $x_{a0} \in X_{a0}$, there exists $x_{b0} \in X_{b0}$ with $(x_{a0}, x_{b0}) \in R$.*

System S_a is ε-approximately bisimilar to S_b, denoted by $S_a \cong^{\varepsilon}_{\mathcal{S}} S_b$, if there exists an ε-approximate bisimulation relation R between S_a and S_b such that:

- *for every $x_{a0} \in X_{a0}$, there exists $x_{b0} \in X_{b0}$ with $(x_{a0}, x_{b0}) \in R$;*
- *for every $x_{b0} \in X_{b0}$, there exists $x_{a0} \in X_{a0}$ with $(x_{a0}, x_{b0}) \in R$.*

5. SYMBOLIC MODELS FOR STOCHASTIC CONTROL SYSTEMS

5.1 Describing stochastic control systems as metric systems

In order to show the main results of the paper, we use the notion of system to abstractly represent a stochastic control system: given a stochastic control system Σ, we define an associated metric system $S(\Sigma) = (X, X_0, U, \longrightarrow, Y, H)$, where:

- X is the set of all $\mathbb{R}^n$-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
- X_0 is a subset of the set of $\mathbb{R}^n$-valued random variables that are measurable over $\mathcal{F}_0$;
- $U = \mathcal{U}$;
- $x \xrightarrow{\upsilon} x'$ if x and x' are measurable in $\mathcal{F}_t$ and $\mathcal{F}_{t+\tau}$, respectively, for some $t \in \mathbb{R}^+_0$ and $\tau \in \mathbb{R}^+$, and there exists a solution process $\xi : \Omega \times \mathbb{R}^+_0 \to \mathbb{R}^n$ of Σ satisfying $\xi(t) = x$ and $\xi_{x\upsilon}(\tau) = x'$ $\mathbb{P}$-a.s.;
- $Y = X$;
- $H = 1_X$.

We assume that the output set Y is equipped with the metric $\mathbf{d}(y, y') = \left(\mathbb{E}\left[\|y - y'\|^q\right]\right)^{\frac{1}{q}}$, for any $y, y' \in Y$ and some $q \geq 1$. Let us remark that the set of states and inputs of $S(\Sigma)$ are uncountable and that $S(\Sigma)$ is a deterministic system in the sense of Definition 4.1, since (cf. Subsection 2.2) the solution process of Σ is uniquely determined.

As usual, since the concrete system $S(\Sigma)$ is infinite and does not allow for the direct control synthesis over itself, we are interested in finding a finite abstract system that is (bi)similar to the original concrete one. In order to talk about approximate (bi)simulation relations between two metric systems, such systems have to share the same output set (cf. Definition 4.2). The latter clearly determines the output behavior of the model that needs to be used to compare the concrete and the abstract models. Obviously, the system $S(\Sigma)$ inherits a classical trace-based semantics [15], and the only subtle point in our case is that the outputs of $S(\Sigma)$ (and those of any approximately (bi)similar one) are random variables. This fact is especially important due to the metric $\mathbf{d}$ with which the output set is endowed: for any non-probabilistic point one can always find a non-degenerate random variable which is as close as desired to the original point in the metric $\mathbf{d}$.

To elucidate the discussion in the previous paragraph, let us consider the following example. Let $A \subset \mathbb{R}^n$ be the set (of non-probabilistic points) whose safety we are interested in, so we formulate the problem as satisfying the LTL formula[1] $\Box A$. Suppose that over the abstract system we are able to synthesize a control strategy that makes an output run of the abstraction satisfy $\Box A$. Although the run would in general be consisting of random variables y, the fact that $y \in A$ means that y has a Dirac probability distribution centered at y, that is $y \in Y$ is a degenerate random variable that can be identified with a point in $A \subset \mathbb{R}^n \subset Y$: note that since any non-probabilistic point can be regarded as a random variable with a Dirac probability distribution centered at that point, $\mathbb{R}^n$ can be embedded in Y, which we denote as $\mathbb{R}^n \subset Y$ with a slight abuse of notation. As a result, satisfying $\Box A$ precisely means that the output run of the abstraction indeed stays in the set $A \subset \mathbb{R}^n$ forever. On the other hand, suppose that the original system is ε-approximate bisimilar to the abstraction. If we want to interpret the result $\Box A$ obtained over the abstraction, we can guarantee that the corresponding output run of the original system satisfies $\Box A_\varepsilon$, that is any output y of the run of the original system is within ε $\mathbf{d}$-distance from the set A: $\mathbf{d}(y, A) = \inf_{a \in A} \mathbf{d}(y, a) \leq \varepsilon$. Note that although the original set $A \subset Y$ is a subset of $\mathbb{R}^n \subset Y$, its ε-inflation $A_\varepsilon = \{y \in Y : \mathbf{d}(y, A) \leq \varepsilon\}$ is not a subset of $\mathbb{R}^n$ anymore and hence contains non-degenerate random variables. In particular, $A_\varepsilon \neq \{y \in \mathbb{R}^n : \inf_{a \in A} \|y - a\| \leq \varepsilon\}$ and is in fact bigger than the latter set of non-probabilistic points. As a result, although satisfying $\Box A_\varepsilon$ does not necessarily mean that a trajectory of Σ always stays within some non-probabilistic set, it means that the associated random variables always belong to A_ε and hence are close to the non-probabilistic set A with respect to the qth moment metric.

We are now able to provide two versions of finite abstractions: one whose outputs are always non-probabilistic points – that is degenerate random variables, elements of $\mathbb{R}^n \subset Y$, and one whose outputs can be non-degenerate random variables. Recall, however, that in both cases the output set is still the whole Y and the semantics is the same as for the original system $S(\Sigma)$.

5.2 Main results

This subsection contains the main contributions of the paper. We show that for any δ-ISS-M$_q$ stochastic control system Σ, and for any precision level $\varepsilon \in \mathbb{R}^+$, we can construct a finite system that is ε-approximate bisimilar to Σ. The results in this subsection rely on additional assumptions on the model Σ that are described next. We restrict our attention to stochastic control systems Σ with input sets U that are assumed to be finite unions of boxes (cf. Subsection 2.1). We further restrict our attention to sampled-data stochastic control systems, where input curves belong to set $\mathcal{U}_\tau$ which contains only curves that are constant over intervals of length $\tau \in \mathbb{R}^+$, i.e.

$$\mathcal{U}_\tau = \Big\{\upsilon \in \mathcal{U} \mid \upsilon(t) = \upsilon((k-1)\tau), t \in [(k-1)\tau, k\tau[, k \in \mathbb{N}\Big\}.$$

Let us denote by $S_\tau(\Sigma)$ a sub-system of $S(\Sigma)$ obtained by selecting those transitions of $S(\Sigma)$ corresponding to solution processes of duration τ and to control inputs in $\mathcal{U}_\tau$. This can be seen as the time discretization of Σ. More precisely,

[1]We refer the interested readers to [1] for the detailed definition of the safety property.

given a stochastic control system Σ, we define the associated metric system $S_\tau(\Sigma) = \left(X_\tau, X_{\tau 0}, U_\tau, \underset{\tau}{\longrightarrow}, Y_\tau, H_\tau\right)$, where $X_\tau = X$, $X_{\tau 0} = X_0$, $U_\tau = \mathcal{U}_\tau$, $Y_\tau = Y$, $H_\tau = H$, and

- $x_\tau \xrightarrow[\tau]{\upsilon_\tau} x'_\tau$ if x_τ and x'_τ are measurable, respectively, in $\mathcal{F}_{k\tau}$ and $\mathcal{F}_{(k+1)\tau}$ for some $k \in \mathbb{N}_0$, and there exists a solution process $\xi : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}^n$ of Σ satisfying $\xi(k\tau) = x_\tau$ and $\xi_{x_\tau \upsilon_\tau}(\tau) = x'_\tau$ $\mathbb{P}$-a.s..

Notice that a finite state run $x_0 \xrightarrow[\tau]{\upsilon_0} x_1 \xrightarrow[\tau]{\upsilon_1} \cdots \xrightarrow[\tau]{\upsilon_{N-1}} x_N$ of $S_\tau(\Sigma)$, where $\upsilon_{i-1} \in \mathcal{U}_\tau$ and $x_i = \xi_{x_{i-1}\upsilon_{i-1}}(\tau)$ $\mathbb{P}$-a.s. for $i = 1, \ldots, N$, captures the solution process of Σ at times $t = 0, \tau, \ldots, N\tau$, started from the initial condition x_0 and resulting from a control input υ obtained by the concatenation of the input curves υ_{i-1} (i.e. $\upsilon(t) = \upsilon_{i-1}(t)$ for any $t \in [(i-1)\tau, i\,\tau[$), for $i = 1, \ldots, N$.

Let us proceed introducing two fully symbolic systems for the concrete model Σ. Consider a stochastic control system Σ and a tuple $\mathsf{q} = (\tau, \mu, N, x_s)$ of parameters, where τ is the sampling time, μ is the input set quantization, $N \in \mathbb{N}$ is a *temporal horizon*, and $x_s \in \mathbb{R}^n$ is a *source state*. Given Σ and q, consider the following systems:

$$S_{\mathsf{q}}(\Sigma) = (X_{\mathsf{q}}, X_{\mathsf{q}0}, U_{\mathsf{q}}, \underset{\mathsf{q}}{\longrightarrow}, Y_{\mathsf{q}}, H_{\mathsf{q}}),$$
$$\overline{S}_{\mathsf{q}}(\Sigma) = (X_{\mathsf{q}}, X_{\mathsf{q}0}, U_{\mathsf{q}}, \underset{\mathsf{q}}{\longrightarrow}, Y_{\mathsf{q}}, \overline{H}_{\mathsf{q}}),$$

consisting of:

- $X_{\mathsf{q}} = \left\{ (u_1, \ldots, u_N) \in \overbrace{[\mathsf{U}]_\mu \times \cdots \times [\mathsf{U}]_\mu}^{N \text{ times}} \right\}$;
- $X_{\mathsf{q}0} = X_{\mathsf{q}}$;
- $U_{\mathsf{q}} = [\mathsf{U}]_\mu$;
- $x_{\mathsf{q}} \xrightarrow[\mathsf{q}]{u_{\mathsf{q}}} x'_{\mathsf{q}}$, where $x_{\mathsf{q}} = (u_1, u_2, \ldots, u_N)$, if and only if $x'_{\mathsf{q}} = (u_2, \ldots, u_N, u_{\mathsf{q}})$;
- Y_{q} is the set of all $\mathbb{R}^n$-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
- $H_{\mathsf{q}}(x_{\mathsf{q}}) = \xi_{x_s x_{\mathsf{q}}}(N\tau)$ $\left(\overline{H}_{\mathsf{q}}(x_{\mathsf{q}}) = \overline{\xi}_{x_s x_{\mathsf{q}}}(N\tau)\right)$.

Note that the transition relation in $S_{\mathsf{q}}(\Sigma)$ admits a very compact representation in the form of a shift operator. We have abused notation by identifying $u_{\mathsf{q}} \in [\mathsf{U}]_\mu$ with the constant input curve with domain $[0, \tau[$ and value u_{q} and identifying $x_{\mathsf{q}} \in [\mathsf{U}]_\mu^N$ with the concatenation of N control inputs $u_i \in [\mathsf{U}]_\mu$ (i.e. $x_{\mathsf{q}}(t) = u_i$ for any $t \in [(i-1)\tau, i\tau[$) for $i = 1, \ldots, N$. Notice that the proposed abstraction $S_{\mathsf{q}}(\Sigma)$ (resp. $\overline{S}_{\mathsf{q}}(\Sigma)$) is a deterministic system in the sense of Definition 4.1. Note that H_{q} and $\overline{H}_{\mathsf{q}}$ are mappings from a non-probabilistic point x_{q} to the random variable $\xi_{x_s x_{\mathsf{q}}}(N\tau)$ and to the one with a Dirac probability distribution centered at $\overline{\xi}_{x_s x_{\mathsf{q}}}(N\tau)$, respectively.

The control synthesis for $\overline{S}_{\mathsf{q}}(\Sigma)$ is simple as the outputs are non-probabilistic points. For $S_{\mathsf{q}}(\Sigma)$ it is perhaps less intuitive. Hence, we discuss it in more details later in Subsection 5.3.

An example of abstraction $S_{\mathsf{q}}(\Sigma)$ with $N = 2$ and $U_{\mathsf{q}} = \{0, 1\}$ is depicted in Figure 1, where the initial states are shown as targets of sourceless arrows. Note that $S_{\mathsf{q}}(\Sigma)$ only has four possible states: $X_{\mathsf{q}} = \{(0,0), (0,1), (1,0), (1,1)\}$.

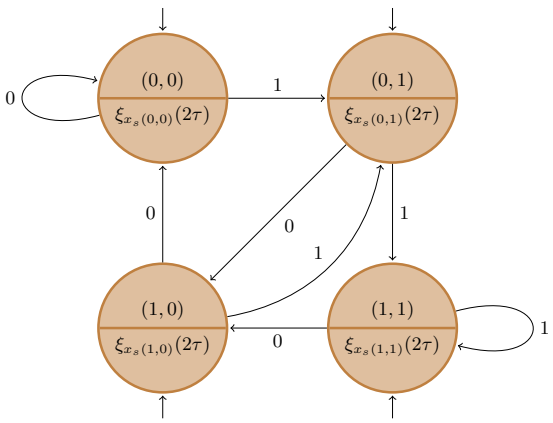

Figure 1: Example of abstraction $S_{\mathsf{q}}(\Sigma)$ with $N = 2$ and $U_{\mathsf{q}} = \{0, 1\}$. The lower part of the states are labeled with their output values.

If one chooses $N = 3$, then $S_{\mathsf{q}}(\Sigma)$ will have eight possible states.

In order to obtain some of the main results of this work, we raise an assumption on the δ-ISS-M$_q$ Lyapunov function V as follows:

$$|V(x, y) - V(x, z)| \leq \widehat{\gamma}(\|y - z\|), \tag{5.1}$$

for any $x, y, z \in \mathbb{R}^n$, and some $\mathcal{K}_\infty$ and concave function $\widehat{\gamma}$. As long as one is interested to work in a compact subset of $\mathbb{R}^n$, the function $\widehat{\gamma}$ in (5.1) can be readily computed. Indeed, for all $x, y, z \in \mathsf{D}$, where $\mathsf{D} \subset \mathbb{R}^n$ is compact, one can readily apply the mean value theorem to the function $y \to V(x, y)$ to get

$$|V(x, y) - V(x, z)| \leq \widehat{\gamma}\left(\|y - z\|\right),$$

$$\text{where } \widehat{\gamma}(r) = \left(\max_{x,y \in \mathsf{D} \setminus \Delta} \left\| \frac{\partial V(x, y)}{\partial y} \right\| \right) r.$$

In particular, for the δ-ISS-M$_1$ Lyapunov function $V(x, x') := \sqrt{(x - x')^T P (x - x')}$, for some positive definite matrix $P \in \mathbb{R}^{n \times n}$ and for all $x, x' \in \mathbb{R}^n$, one obtains $\widehat{\gamma}(r) = \frac{\lambda_{\max}(P)}{\sqrt{\lambda_{\min}(P)}} r$ [15, Proposition 10.5], which satisfies (5.1) globally on $\mathbb{R}^n$.

Before providing the main results of the paper, we need the following technical claims.

LEMMA 5.1. *Consider a stochastic control system Σ, admitting a δ-ISS-M$_q$ Lyapunov function V, and consider its corresponding symbolic model $\overline{S}_{\mathsf{q}}(\Sigma)$. We have:*

$$\eta \leq \left(\underline{\alpha}^{-1} \left(\mathsf{e}^{-\kappa N \tau} \max_{u_{\mathsf{q}} \in U_{\mathsf{q}}} V\left(\overline{\xi}_{x_s u_{\mathsf{q}}}(\tau), x_s\right) \right) \right)^{1/q}, \tag{5.2}$$

where

$$\eta := \max_{\substack{u_{\mathsf{q}} \in U_{\mathsf{q}}, x_{\mathsf{q}} \in X_{\mathsf{q}} \\ x'_{\mathsf{q}} \in \mathbf{Post}_{u_{\mathsf{q}}}(x_{\mathsf{q}})}} \left\| \overline{\xi}_{\overline{H}_{\mathsf{q}}(x_{\mathsf{q}}) u_{\mathsf{q}}}(\tau) - \overline{H}_{\mathsf{q}}\left(x'_{\mathsf{q}}\right) \right\|. \tag{5.3}$$

The proof of Lemma 5.1 is provided in the Appendix. The next lemma provides similar result as the one in Lemma 5.1, but without explicitly using any Lyapunov function.

LEMMA 5.2. *Consider a δ-ISS-M_q stochastic control system Σ and its corresponding symbolic model $\overline{S}_{\mathsf{q}}(\Sigma)$. We have:*

$$\eta \leq \left(\beta\left(\max_{u_{\mathsf{q}}\in U_{\mathsf{q}}} \left\|\overline{\xi}_{x_s u_{\mathsf{q}}}(\tau) - x_s\right\|^q, N\tau\right)\right)^{1/q}, \tag{5.4}$$

where η is given in (5.3).

The proof of Lemma 5.2 is provided in the Appendix. The next two lemmas provide similar results as the ones of Lemmas 5.1 and 5.2, but by using the symbolic model $S_{\mathsf{q}}(\Sigma)$ rather than $\overline{S}_{\mathsf{q}}(\Sigma)$.

LEMMA 5.3. *Consider a stochastic control system Σ, admitting a δ-ISS-M_q Lyapunov function V, and consider its corresponding symbolic model $S_{\mathsf{q}}(\Sigma)$. One has:*

$$\widehat{\eta} \leq \left(\underline{\alpha}^{-1}\left(\mathsf{e}^{-\kappa N\tau}\max_{u_{\mathsf{q}}\in U_{\mathsf{q}}} \mathbb{E}\left[V\left(\xi_{x_s u_{\mathsf{q}}}(\tau), x_s\right)\right]\right)\right)^{1/q}, \tag{5.5}$$

where

$$\widehat{\eta} := \max_{\substack{u_{\mathsf{q}}\in U_{\mathsf{q}}, x_{\mathsf{q}}\in X_{\mathsf{q}}\\ x'_{\mathsf{q}}\in \mathbf{Post}_{u_{\mathsf{q}}}(x_{\mathsf{q}})}} \mathbb{E}\left[\left\|\xi_{H_{\mathsf{q}}(x_{\mathsf{q}})u_{\mathsf{q}}}(\tau) - H_{\mathsf{q}}\left(x'_{\mathsf{q}}\right)\right\|\right]. \tag{5.6}$$

PROOF. The proof is similar to the one of Lemma 5.1 and can be shown by using convexity of $\underline{\alpha}$ and Jensen inequality [10]. □

LEMMA 5.4. *Consider a δ-ISS-M_q stochastic control system Σ and its corresponding symbolic model $S_{\mathsf{q}}(\Sigma)$. We have:*

$$\widehat{\eta} \leq \left(\beta\left(\max_{u_{\mathsf{q}}\in U_{\mathsf{q}}} \mathbb{E}\left[\left\|\xi_{x_s u_{\mathsf{q}}}(\tau) - x_s\right\|^q\right], N\tau\right)\right)^{1/q}, \tag{5.7}$$

where $\widehat{\eta}$ is given in (5.6).

PROOF. The proof is similar to the one of Lemma 5.2 and can be shown by using Jensen inequality [10]. □

REMARK 5.5. *It can be readily verified that by choosing N sufficiently large, η and $\widehat{\eta}$ can be made arbitrarily small. One can even try to reduce the upper bound of η in* (5.2) *by choosing the source point x_s as the following:*

$$x_s = arg\min_{x\in\mathbb{R}^n}\max_{u_{\mathsf{q}}\in U_{\mathsf{q}}} V\left(\overline{\xi}_{x u_{\mathsf{q}}}(\tau), x\right). \tag{5.8}$$

We can now present the first main result of the paper, which relates the existence of a δ-ISS-M_q Lyapunov function to the construction of a symbolic model.

THEOREM 5.6. *Consider a stochastic control system Σ with $f(0_n, 0_m) = 0_n$ and $\sigma(0_n) = 0_{n\times p}$, admitting a δ-ISS-M_q Lyapunov function V, of the form of the one explained in Lemma 3.4, such that* (5.1) *holds for some concave $\widehat{\gamma}\in\mathcal{K}_\infty$. Let η be given by* (5.3). *For any $\varepsilon\in\mathbb{R}^+$ and any tuple $\mathsf{q} = (\tau, \mu, N, x_s)$ of parameters satisfying $\mu \leq span(\mathsf{U})$ and*

$$\mathsf{e}^{-\kappa\tau}\underline{\alpha}\left(\varepsilon^q\right) + \frac{1}{\mathsf{e}\kappa}\rho(\mu) + \widehat{\gamma}\left(\left(h_{x_s}(\sigma, (N+1)\tau)\right)^{\frac{1}{q}} + \eta\right) \leq \underline{\alpha}\left(\varepsilon^q\right), \tag{5.9}$$

the relation

$$R = \left\{(x_\tau, x_{\mathsf{q}}) \in X_\tau \times X_{\mathsf{q}} \mid \mathbb{E}\left[V\left(x_\tau, \overline{H}_{\mathsf{q}}(x_{\mathsf{q}})\right)\right] \leq \underline{\alpha}\left(\varepsilon^q\right)\right\}$$

is an ε-approximate bisimulation relation between $\overline{S}_{\mathsf{q}}(\Sigma)$ and $S_\tau(\Sigma)$.

By choosing N sufficiently large, one can enforce $h_{x_s}(\sigma, (N+1)\tau)$ and η to be sufficiently small. Hence, it can be readily seen that for a given precision ε, there always exists a sufficiently small value of μ and a large value of N, such that the condition in (5.9) is satisfied.

In order to mitigate the conservativeness caused by using Lyapunov functions, the next theorem provides a result that is similar to the one of Theorem 5.6, which is however not obtained by explicit use of δ-ISS-M_q Lyapunov functions, but by using functions β and γ as in (3.1).

THEOREM 5.7. *Consider a δ-ISS-M_q stochastic control system Σ, satisfying the result of Lemma 3.4. Let η be given by* (5.3). *For any $\varepsilon\in\mathbb{R}^+$, and any tuple $\mathsf{q} = (\tau, \mu, N, x_s)$ of parameters satisfying $\mu \leq span(\mathsf{U})$ and*

$$\left(\beta\left(\varepsilon^q, \tau\right) + \gamma(\mu)\right)^{\frac{1}{q}} + \left(h_{x_s}(\sigma, (N+1)\tau)\right)^{\frac{1}{q}} + \eta \leq \varepsilon, \tag{5.10}$$

the relation

$$R = \left\{(x_\tau, x_{\mathsf{q}}) \in X_\tau \times X_{\mathsf{q}} \mid \left(\mathbb{E}\left[\left\|x_\tau - \overline{H}_{\mathsf{q}}(x_{\mathsf{q}})\right\|^q\right]\right)^{\frac{1}{q}} \leq \varepsilon\right\}$$

is an ε-approximate bisimulation relation between $\overline{S}_{\mathsf{q}}(\Sigma)$ and $S_\tau(\Sigma)$.

By choosing N sufficiently large, one can force $h_{x_s}(\sigma, (N+1)\tau)$ and η to be sufficiently small. Hence, it can be readily seen that for a given precision ε, there always exist a sufficiently large value of τ and N and small value of μ such that the condition in (5.10) is satisfied. However, unlike the result in Theorem 5.6, notice that here for a given fixed sampling time τ, one may not find any values of N and μ satisfying (5.10) because the quantity $\left(\beta\left(\varepsilon^q, \tau\right)\right)^{\frac{1}{q}}$ may be larger than ε. The symbolic model $\overline{S}_{\mathsf{q}}(\Sigma)$, computed by using the parameter q provided in Theorem 5.7 whenever existing, is likely to have fewer states than the model computed by using the parameter q provided in Theorem 5.6. Similar observation has been verified in the first example in [19].

The next theorems provide results that are similar to those of Theorems 5.6 and 5.7, but by using the symbolic model $S_{\mathsf{q}}(\Sigma)$.

THEOREM 5.8. *Consider a stochastic control system Σ, admitting a δ-ISS-M_q Lyapunov function V such that* (5.1) *holds for some concave $\widehat{\gamma}\in\mathcal{K}_\infty$. Let $\widehat{\eta}$ be given by* (5.6). *For any $\varepsilon\in\mathbb{R}^+$ and any tuple $\mathsf{q} = (\tau, \mu, N, x_s)$ of parameters satisfying $\mu \leq span(\mathsf{U})$ and*

$$\mathsf{e}^{-\kappa\tau}\underline{\alpha}\left(\varepsilon^q\right) + \frac{1}{\mathsf{e}\kappa}\rho(\mu) + \widehat{\gamma}\left(\widehat{\eta}\right) \leq \underline{\alpha}\left(\varepsilon^q\right), \tag{5.11}$$

the relation

$$R = \left\{(x_\tau, x_{\mathsf{q}}) \in X_\tau \times X_{\mathsf{q}} \mid \mathbb{E}\left[V(x_\tau, H_{\mathsf{q}}(x_{\mathsf{q}}))\right] \leq \underline{\alpha}\left(\varepsilon^q\right)\right\}$$

is an ε-approximate bisimulation relation between $S_{\mathsf{q}}(\Sigma)$ and $S_\tau(\Sigma)$.

THEOREM 5.9. *Consider a δ-ISS-M_q stochastic control system Σ. Let $\widehat{\eta}$ be given by* (5.6). *For any $\varepsilon\in\mathbb{R}^+$, and any tuple $\mathsf{q} = (\tau, \mu, N, x_s)$ of parameters satisfying $\mu \leq span(\mathsf{U})$ and*

$$\left(\beta\left(\varepsilon^q, \tau\right) + \gamma(\mu)\right)^{\frac{1}{q}} + \widehat{\eta} \leq \varepsilon, \tag{5.12}$$

the relation

$$R = \left\{(x_\tau, x_{\mathsf{q}}) \in X_\tau \times X_{\mathsf{q}} \mid \left(\mathbb{E}\left[\|x_\tau - H_{\mathsf{q}}(x_{\mathsf{q}})\|^q\right]\right)^{\frac{1}{q}} \leq \varepsilon\right\}$$

is an ε-approximate bisimulation relation between $S_{\mathsf{q}}(\Sigma)$ and $S_\tau(\Sigma)$.

REMARK 5.10. *The symbolic model $S_{\mathsf{q}}(\Sigma)$, computed using the parameter q provided in Theorem 5.8 (resp. Theorem 5.9), has fewer (or at most equal number of) states than the symbolic model $\overline{S}_{\mathsf{q}}(\Sigma)$, computed by using the parameter q provided in Theorem 5.6 (resp. Theorem 5.7) while having the same precision. However, the symbolic model $S_{\mathsf{q}}(\Sigma)$ has states with probabilistic output values, rather than non-probabilistic ones, which is likely to require more involved control synthesis procedures (cf. Subsection 5.3).*

REMARK 5.11. *Although we assume that the set U is infinite, Theorems 5.6, 5.7, 5.8, and 5.9 still hold when the set U is finite, with the following modifications. First, the system Σ is required to satisfy the property (3.1) for $\upsilon = \upsilon'$. Second, take $U_{\mathsf{q}} = \mathsf{U}$ in the definition of $\overline{S}_{\mathsf{q}}(\Sigma)$ (resp. $S_{\mathsf{q}}(\Sigma)$). Finally, in the conditions (5.9), (5.10),* (5.11)*, and* (5.12) *set $\mu = 0$.*

Finally, we establish the results on the existence of symbolic model $\overline{S}_{\mathsf{q}}(\Sigma)$ (resp. $S_{\mathsf{q}}(\Sigma)$) such that $\overline{S}_{\mathsf{q}}(\Sigma) \cong^{\varepsilon}_{\mathcal{S}} S_\tau(\Sigma)$ (resp. $S_{\mathsf{q}}(\Sigma) \cong^{\varepsilon}_{\mathcal{S}} S_\tau(\Sigma)$).

THEOREM 5.12. *Consider the results in Theorem 5.6. If we select*

$$X_{\tau 0} = \left\{ x \in \mathbb{R}^n \,|\, \|x - \overline{H}_{\mathsf{q}}(x_{\mathsf{q}0})\| \leq \left(\overline{\alpha}^{-1}\left(\underline{\alpha}\left(\varepsilon^q\right)\right)\right)^{\frac{1}{q}}, \forall x_{\mathsf{q}0} \in X_{\mathsf{q}0} \right\},$$

then we have $\overline{S}_{\mathsf{q}}(\Sigma) \cong^{\varepsilon}_{\mathcal{S}} S_\tau(\Sigma)$.

PROOF. We start by proving that $S_\tau(\Sigma) \preceq^{\varepsilon}_{\mathcal{S}} \overline{S}_{\mathsf{q}}(\Sigma)$. For every $x_{\tau 0} \in X_{\tau 0}$ there always exists $x_{\mathsf{q}0} \in X_{\mathsf{q}0}$ such that $\|x_{\tau 0} - \overline{H}_{\mathsf{q}}(x_{\mathsf{q}0})\| \leq \left(\overline{\alpha}^{-1}\left(\underline{\alpha}\left(\varepsilon^q\right)\right)\right)^{\frac{1}{q}}$. Then,

$$\begin{aligned}\mathbb{E}\left[V\left(x_{\tau 0}, \overline{H}_{\mathsf{q}}(x_{\mathsf{q}0})\right)\right] &= V\left(x_{\tau 0}, \overline{H}_{\mathsf{q}}(x_{\mathsf{q}0})\right) \\ &\leq \overline{\alpha}(\|x_{\tau 0} - \overline{H}_{\mathsf{q}}(x_{\mathsf{q}0})\|^q) \leq \underline{\alpha}\left(\varepsilon^q\right),\end{aligned}$$

since $\overline{\alpha}$ is a $\mathcal{K}_\infty$ function. Hence, $(x_{\tau 0}, x_{\mathsf{q}0}) \in R$ implying that $S_\tau(\Sigma) \preceq^{\varepsilon}_{\mathcal{S}} \overline{S}_{\mathsf{q}}(\Sigma)$. In a similar way, we can show that $\overline{S}_{\mathsf{q}}(\Sigma) \preceq^{\varepsilon}_{\mathcal{S}} S_\tau(\Sigma)$ which completes the proof. □

The next theorem provides a similar result in line with the one of previous theorem, but by using a different relation.

THEOREM 5.13. *Consider the results in Theorem 5.7. If we select*

$$X_{\tau 0} = \left\{ x \in \mathbb{R}^n \mid \|x - \overline{H}_{\mathsf{q}}(x_{\mathsf{q}0})\| \leq \varepsilon, \ \forall x_{\mathsf{q}0} \in X_{\mathsf{q}0} \right\},$$

then we have $\overline{S}_{\mathsf{q}}(\Sigma) \cong^{\varepsilon}_{\mathcal{S}} S_\tau(\Sigma)$.

PROOF. We start by proving that $S_\tau(\Sigma) \preceq^{\varepsilon}_{\mathcal{S}} \overline{S}_{\mathsf{q}}(\Sigma)$. For every $x_{\tau 0} \in X_{\tau 0}$ there always exists $x_{\mathsf{q}0} \in X_{\mathsf{q}0}$ such that $\|x_{\tau 0} - \overline{H}_{\mathsf{q}}(x_{\mathsf{q}0})\| \leq \varepsilon$ and $\left(\mathbb{E}\left[\left\|x_{\tau 0} - \overline{H}_{\mathsf{q}}(x_{\mathsf{q}0})\right\|^q\right]\right)^{\frac{1}{q}} \leq \varepsilon$. Hence, $(x_{\tau 0}, x_{\mathsf{q}0}) \in R$ implying that $S_\tau(\Sigma) \preceq^{\varepsilon}_{\mathcal{S}} \overline{S}_{\mathsf{q}}(\Sigma)$. In a similar way, we can show that $\overline{S}_{\mathsf{q}}(\Sigma) \preceq^{\varepsilon}_{\mathcal{S}} S_\tau(\Sigma)$ which completes the proof. □

The next two theorems provide similar results as the ones of Theorems 5.12 and 5.13, but by using the symbolic model $S_{\mathsf{q}}(\Sigma)$.

THEOREM 5.14. *Consider the results in Theorem 5.8. Let $\mathcal{A}$ denote the set of all $\mathbb{R}^n$-valued random variables, measurable over $\mathcal{F}_0$. If we select*

$$X_{\tau 0} = \left\{ a \in \mathcal{A} | \left(\mathbb{E}\left[\|a - H_{\mathsf{q}}(x_{\mathsf{q}0})\|^q\right]\right)^{\frac{1}{q}} \leq \left(\overline{\alpha}^{-1}\left(\underline{\alpha}\left(\varepsilon^q\right)\right)\right)^{\frac{1}{q}}, \forall x_{\mathsf{q}0} \in X_{\mathsf{q}0} \right\},$$

then we have $S_{\mathsf{q}}(\Sigma) \cong^{\varepsilon}_{\mathcal{S}} S_\tau(\Sigma)$.

PROOF. The proof is similar to the one of Theorem 5.12. □

THEOREM 5.15. *Consider the results in Theorem 5.9. Let $\mathcal{A}$ denote the set of all $\mathbb{R}^n$-valued random variables, measurable over $\mathcal{F}_0$. If we select*

$$X_{\tau 0} = \left\{ a \in \mathcal{A} \mid \left(\mathbb{E}\left[\|a - H_{\mathsf{q}}(x_{\mathsf{q}0})\|^q\right]\right)^{\frac{1}{q}} \leq \varepsilon, \ \forall x_{\mathsf{q}0} \in X_{\mathsf{q}0} \right\},$$

then we have $S_{\mathsf{q}}(\Sigma) \cong^{\varepsilon}_{\mathcal{S}} S_\tau(\Sigma)$.

PROOF. The proof is similar to the one of Theorem 5.13. □

5.3 Control synthesis over $S_{\mathsf{q}}(\Sigma)$

Note that both $\overline{S}_{\mathsf{q}}(\Sigma)$ and $S_{\mathsf{q}}(\Sigma)$ are finite systems. The only difference is that the outputs of the former system are always non-probabilistic points, whereas those of the latter can be non-degenerate random variables. Let us describe the control synthesis for these systems over the safety formula $\Box A$, for $A \subset \mathbb{R}^n \subset Y$, which has already been used in Subsection 5.1. Clearly, since the original system $S_\tau(\Sigma)$ is stochastic in the sense that its outputs are non-degenerate random variables similarly to $S_{\mathsf{q}}(\Sigma)$, it would be too conservative to require that it satisfies the formula exactly. Thus, we are rather interested in an input policy that makes $S_\tau(\Sigma)$ satisfy $\Box A_\varepsilon$ with some $\varepsilon > 0$: recall from Subsection 5.1 that the latter LTL formula can be satisfied by non-degenerate random variables, in contrast to $\Box A$. Let us recap how to use abstractions for this task, and let us start with $\overline{S}_{\mathsf{q}}(\Sigma)$ belonging to a more familiar type of systems whose outputs are non-probabilistic.

We label a state x_{q} of $\overline{S}_{\mathsf{q}}(\Sigma)$ with A if $\overline{H}_{\mathsf{q}}(x_{\mathsf{q}}) \in A$ and, say, with B otherwise. As a result, we obtain a transition system with labels over the states and can synthesize a control strategy that makes an output run of $\overline{S}_{\mathsf{q}}(\Sigma)$ satisfy $\Box A$. After that, we can exploit ε-approximate bisimilarity to guarantee that the refined input policy makes the corresponding output run of the original system satisfy $\Box A_\varepsilon$.

The main subtlety in the case of $S_{\mathsf{q}}(\Sigma)$ is how to label its states. We cannot do this as for $\overline{S}_{\mathsf{q}}(\Sigma)$, since $H_{\mathsf{q}}(x_{\mathsf{q}})$ may never be an element of A for any $x_{\mathsf{q}} \in X_{\mathsf{q}}$: indeed, the latter is a set of non-probabilistic points, whereas all the outputs of $S_{\mathsf{q}}(\Sigma)$ can happen to be non-degenerate random variables. In order to cope with this issue, we propose to relax the original problem and at the same time to strengthen the quality of the abstraction. Namely, we can consider a relaxed problem $\Box A_\delta$ over the abstraction $S_{\mathsf{q}}(\Sigma)$, for some $\delta \in (0, \varepsilon)$, where the latter is now required to be $(\varepsilon - \delta)$-approximate (rather than just ε-approximate) bisimilar to the original system. Clearly $(A_\delta)_{\varepsilon - \delta} \subseteq A_\varepsilon$, so that whenever the control policy for $\Box A_\delta$ is synthesized over $S_{\mathsf{q}}(\Sigma)$, its refined version is guaranteed to force $\Box A_\varepsilon$ over the original system. Thanks to the fact that A_δ contains non-degenerate random variables, we eliminate the conservativeness presented before in the sense that it is likely that there are now points $x_{\mathsf{q}} \in X_{\mathsf{q}}$

in $S_{\mathsf{q}}(\Sigma)$ such that $H_{\mathsf{q}}(x_{\mathsf{q}}) \in A_\delta$. The only remaining question is how to check whether $H_{\mathsf{q}}(x_{\mathsf{q}}) \in A_\delta$. To answer this question, we check that the distance

$$\mathbf{d}\left(H_{\mathsf{q}}(x_{\mathsf{q}}), A\right) = \inf_{a \in A} \left(\mathbb{E}\|\xi_{x_s x_{\mathsf{q}}}(N\tau) - a\|^q\right)^{1/q} \tag{5.13}$$

is smaller than δ, which involves both computing the expectation over the solution of the SDE, and optimizing the value of this expectation. Clearly, such a computation in general cannot be done analytically, and the evaluation of the expectation itself is a highly non-trivial task unless the SDE has a very special form.

We propose a Monte Carlo approach to compute an approximation of the quantity in (5.13) by means of empirical expectations. Using such an approach, we can estimate $\mathbf{d}\left(H_{\mathsf{q}}(x_{\mathsf{q}}), A\right)$ only up to some precision, say ϵ. If the estimated distance is less than $\delta - \epsilon$, we are safe to label x_{q} with A, whereas all other states are labeled by B. Furthermore, since this result is based on a Monte Carlo method, it holds true only with a certain confidence level $1 - \pi$ where $\pi \in [0, 1]$. The benefit of our approach is that it is not only valid asymptotically (as the number of samples tends to infinity), but we are also able to provide a number of simulations that is sufficient to estimate $\mathbf{d}\left(H_{\mathsf{q}}(x_{\mathsf{q}}), A\right)$ with any given precision ϵ and with any given confidence $1 - \pi$. This can be considered as an extension of the well-known Hoeffding's inequality [5] to the case when one has to deal with an optimization problem. Note that regardless of the specification of interest, the main task over $S_{\mathsf{q}}(\Sigma)$ is always to compute some distance as in (5.13) for any set that appears in the specification, so the method below applies not only to the safety formula $\Box A$, but also to more general formulae, which are left as object of future research.

Due to space limitations, here we only consider the case $q = 1$. For $q \geq 2$, similar results can be derived. Suppose that A as in (5.13) is a compact subset of $\mathbb{R}^n$, and let A^r be the smallest subset of $[\mathbb{R}^n]_r$ such that $A \subseteq \bigcup_{p \in A^r} \mathcal{B}_{\frac{r}{2}}(p)$. Let M be the number of samples and let

$$\mathbf{d}_M^r := \min_{a \in A^r} \frac{1}{M} \sum_{i=1}^{M} \left\| \xi^i_{x_s x_{\mathsf{q}}}(N\tau) - a \right\|,$$

where the superscript i denotes the number of samples. Now we have the following result.

THEOREM 5.16. *Consider a stochastic control system Σ and suppose that we are interested in its dynamics over a compact set D. It holds that $|\mathbf{d}\left(H_{\mathsf{q}}(x_{\mathsf{q}}), A\right) - \mathbf{d}_M^r| \leq \epsilon$ with confidence of at least $1 - \pi$ given that $r/2 < \epsilon$ and that*

$$M \geq \frac{D^2}{2\left(\epsilon - r/2\right)^2} \cdot \log \frac{2|A^r|}{\pi},$$

where $D = \sup\{\|x - y\| \mid x, y \in \mathsf{D}\}$.

Let us make some remarks regarding Theorem 5.16. First of all, no matter how many distances one has to evaluate, one can always use the same samples ξ^i and there is no need to generate new samples. Second, the number of samples is quadratic in the precision ϵ and is only logarithmic in the lack of confidence π, thus it is fairly fast and easy to satisfy the desired degree of accuracy with very high confidence.

5.4 Comparison with existing results in the literature

Note that given any precision ε and sampling time τ, one can always use the results in Theorem 5.12 to construct a symbolic model $\overline{S}_{\mathsf{q}}(\Sigma)$ that is ε-approximate bisimilar to $S_\tau(\Sigma)$. However, the results in Theorem 5.1 in [19] cannot be applied for any sampling time τ if the precision ε is lower than the thresholds introduced in inequality (5.5) in [19]. Furthermore, while the results in [19] only provide symbolic models with non-probabilistic output values, the ones in this work provide symbolic models with probabilistic output values as well, which can result in less conservative symbolic models (cf. Remark 5.10 and the example section).

One can compare the results provided in Theorems 5.6 (corr. 5.12) and 5.7 (corr. 5.13) with the results provided in Theorems 5.1 and 5.3 in [19] in terms of the size of the generated symbolic models. One can readily verify that the precisions of the symbolic models proposed here and the ones proposed in [19] are approximately the same as long as both use the same input set quantization parameter μ and the state space quantization parameter, called ν, in [19] is equal to the parameter η in (5.3), i.e. $\nu \leq \left(\underline{\alpha}^{-1}\left(\mathsf{e}^{-\kappa N\tau}\eta_0\right)\right)^{1/q}$, where $\eta_0 = \max_{u_{\mathsf{q}} \in U_{\mathsf{q}}} V\left(\overline{\xi}_{x_s u_{\mathsf{q}}}(\tau), x_s\right)$. The reason their precisions are approximately (rather than exactly) the same is because we use $h_{x_s}(\sigma, (N+1)\tau)$ in conditions (5.9) and (5.10) in this paper rather than $h(\sigma, \tau) = \sup_{x \in \mathsf{D}} h_x(\sigma, \tau)$ that is being used in conditions 5.4 and 5.14 in [19] for a compact set $\mathsf{D} \subset \mathbb{R}^n$. By assuming that $h_{x_s}(\sigma, (N+1)\tau)^{\frac{1}{q}}$ and $h(\sigma, \tau)^{\frac{1}{q}}$ are much smaller than η and ν, respectively, or $h_{x_s}(\sigma, (N+1)\tau) \approx h(\sigma, \tau)$, one should expect to obtain the same precisions for the symbolic models provided here and those provided in [19] under the aforementioned conditions.

The number of states of the proposed symbolic model in this paper is $\left|[\mathsf{U}]_\mu\right|^N$. Assume that we are interested in the dynamics of Σ on a compact set $\mathsf{D} \subset \mathbb{R}^n$. Since the set of states of the proposed symbolic model in [19] is $[\mathsf{D}]_\nu$, its size is $\left|[\mathsf{D}]_\nu\right| = \frac{K}{\nu^n}$, where K is a positive constant proportional to the volume of D. Hence, it is more convenient to use the proposed symbolic model here rather than the one proposed in [19] as long as:

$$\left|[\mathsf{U}]_\mu\right|^N \leq \frac{K}{\left(\underline{\alpha}^{-1}\left(\mathsf{e}^{-\kappa N\tau}\eta_0\right)\right)^{n/q}}.$$

Without loss of generality, one can assume that $\underline{\alpha}(r) = r$ for any $r \in \mathbb{R}_0^+$. Hence, for sufficiently large value of N, it is more convenient to use the proposed symbolic model here in comparison with the one proposed in [19] as long as:

$$\left|[\mathsf{U}]_\mu\right| \mathsf{e}^{\frac{-\kappa\tau n}{q}} \leq 1. \tag{5.14}$$

Note that the methodology proposed in this paper allows us to construct less conservative symbolic models with probabilistic output values (see the example section) while the proposed one in [19] only provides conservative symbolic models with non-probabilistic output values.

6. EXAMPLE

We show the effectiveness of the results of the paper by constructing a bisimilar symbolic model for a simple 6-dimensional linear stochastic control system Σ, aiming mostly at elucidating the details. The model of Σ is described by:

$$\Sigma : \{\mathrm{d}\,\xi = (A\xi + B\upsilon)\,\mathrm{d}\,t + 0.5\xi\,\mathrm{d}\,W_t, \tag{6.1}$$

where

$$A = \begin{bmatrix} -20.73 & 0.45 & -0.77 & 0.92 & 0.68 & 1.28 \\ 0.95 & -22.41 & -1.73 & -0.14 & 0.47 & 0.77 \\ 0.57 & -0.74 & -23.57 & 0.37 & 0.58 & 0.57 \\ -0.71 & 0.07 & 1.04 & -21.41 & -1 & 0.14 \\ -0.95 & 0.47 & 0.96 & -1.34 & -23.96 & 0.11 \\ 1.72 & 0.37 & -0.21 & -0.43 & 0.89 & -22.91 \end{bmatrix}$$

$B^T = [\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 100\ ]^T$.

We assume that $\mathsf{U} = [-1, 1]$ and that $\mathcal{U}_\tau$ contains curves taking values in $[\mathsf{U}]_1$. Hence, as explained in Remark 5.11, $\mu = 0$ is to be used in (5.9), (5.10), (5.11), and (5.12). One can readily verify that the function $V(x, x') = \sqrt{(x - x')^T I_6 (x - x')}$, for any $x, x' \in \mathbb{R}^6$, satisfies conditions (i)-(iii) in Definition 3.2 with $q = 1$, $\underline{\alpha}(r) = r$, $\overline{\alpha}(r) = \sqrt{6}r$, $\rho(r) = \sqrt{6}\|B\|r$, $\forall r \in \mathbb{R}_0^+$, and $\kappa = 19.5$. Hence, Σ is δ-ISS-M_1, equipped with the δ-ISS-M_1 Lyapunov function V. Using the results of Theorem 3.3, provided in [19], one gets that functions $\beta(r, s) = \sqrt{6}\mathsf{e}^{-\kappa s} r$ and $\gamma(r) = \frac{\sqrt{6}\|B\|r}{\mathsf{e}\kappa}$ satisfy property (3.1) for Σ. Given the Lyapunov function V, we solve the optimization problem in (5.8) using the function *fminimax* in Matlab and obtain $x_s \approx 0_6$.

For a given precision $\varepsilon = 1$ and fixed sampling time $\tau = 0.01$, the parameter N for $\overline{S}_{\mathsf{q}}(\Sigma)$, based on inequality (5.9) in Theorem 5.6, is obtained as 10. Therefore, the resulting cardinality of the set of states for $\overline{S}_{\mathsf{q}}(\Sigma)$ is $\left|[\mathsf{U}]_1\right|^{10} = 3^{10} = 59049$. Using the aforementioned parameters, one gets $\eta \leq 0.127$, where η is given in (5.3). Note that the results in Theorems 5.7 and 5.9 cannot be applied here because $(\beta(\varepsilon^q, \tau))^{\frac{1}{q}} > \varepsilon$. Using criterion (5.14), one has $\left|[\mathsf{U}]_\mu\right| \mathsf{e}^{\frac{-\kappa\tau n}{q}} = 0.93$, implying that the approach proposed in this paper is more appropriate in terms of the size of the abstraction than the one proposed in [19]. We elaborate more on this at the end of the section.

REMARK 6.1. *By considering the dynamics of Σ over the subset $\mathsf{D} = [-4, 4]^6$ of $\mathbb{R}^6$, at least $1 - 10^{-5}$ confidence level, and precision $\epsilon = 0.01$ and using Hoeffding's inequality [5], one can verify that the number of samples should be at least 3.9059×10^6 to empirically compute the upper bound of $\widehat{\eta}$ in (5.5). We compute $\widehat{\eta} \leq 0.1287$ when $N = 10$, $\tau = 0.01$, and $x_s \approx 0_6$. Using the results in Theorem 5.8 and the same parameters q as the ones in $\overline{S}_{\mathsf{q}}(\Sigma)$, one obtains $\varepsilon = 0.73$ in (5.11). Therefore, $S_{\mathsf{q}}(\Sigma)$, with confidence at least $1 - 10^{-5}$, provides less conservative precision than $\overline{S}_{\mathsf{q}}(\Sigma)$ while having the same size as $\overline{S}_{\mathsf{q}}(\Sigma)$.*

Now, consider that the objective is to design a control policy forcing the trajectories of Σ, starting from the initial condition $x_0 = 0_6$, to first sequentially visit (in the 1st moment metric) two regions of interest $W_1 = \{x \in \mathbb{R}^6 \mid x_6 = 0.3\}$ and $W_2 = \{x \in \mathbb{R}^6 \mid x_6 = -0.3\}$; then once the system has visited these regions, to reach the region $W_3 = \{x \in \mathbb{R}^6 \mid x_6 = 0.2\}$ in finite time and remain there forever (in the 1st moment metric). The LTL formula[2] representing this goal is $\Diamond\Box W_3 \wedge \Diamond(W_1 \wedge \Diamond W_2)$. Figure 2 displays a few realizations of the closed-loop solution process $\xi_{x_0\upsilon}$ along the 6th dimension, as well as the corresponding evolution of the input signal υ. In Figure 3, we show the average value (over 1000 experiments) of the distance in time of the solution process $\xi_{x_0\upsilon}$ to the sets W_1, W_2, and W_3, namely $\|\xi_{x_0\upsilon}(t)\|_{W_1}$, $\|\xi_{x_0\upsilon}(t)\|_{W_2}$, and $\|\xi_{x_0\upsilon}(t)\|_{W_3}$, where the point-to-set distance is defined as $\|x\|_W = \inf_{w \in W} \|x - w\|$.

[2]Note that the semantics of LTL are defined over the output behaviors of $S_{\mathsf{q}}(\Sigma)$.

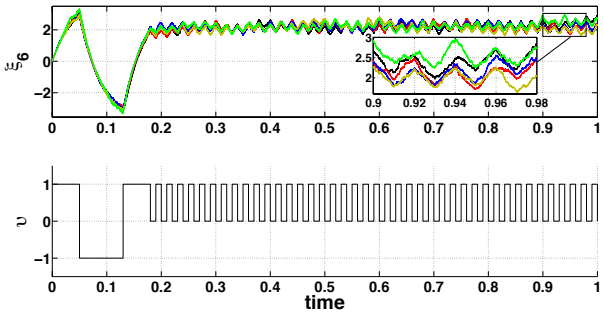

Figure 2: A few realizations of the closed-loop solution process $\xi_{x_0\upsilon}$ along the 6th dimension (top panel) and the corresponding evolution of the obtained input signal υ (bottom panel).

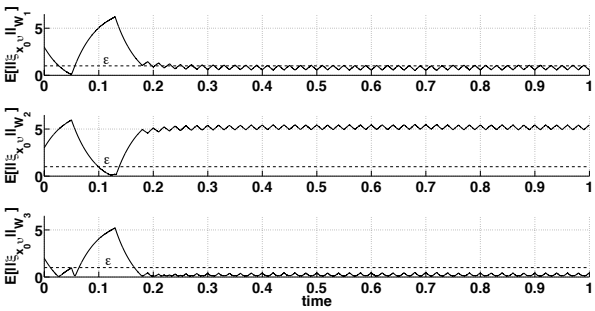

Figure 3: The average values (over 1000 experiments) of the distance of the solution process $\xi_{x_0\upsilon}$ to the sets W_1 (top panel), W_2 (middle panel), and W_3 (bottom panel).

To compute exactly the size of the symbolic model, proposed in Theorem 5.1 in [19], we consider the dynamics of Σ over the subset $\mathsf{D} = [-4, 4]^6$ of $\mathbb{R}^6$. Note that Theorem 5.3 in [19] cannot be applied here because $(\beta(\varepsilon^q, \tau))^{\frac{1}{q}} > \varepsilon$. Using the same precision $\varepsilon = 1$ and sampling time $\tau = 0.01$ as the ones here, and the inequalities (5.3) and (5.4) in [19], we obtain the state space quantization parameter as $\nu \leq 0.01$. Therefore, if one uses $\nu = 0.01$, the cardinality of the state set of the symbolic model, provided by the results in Theorem 5.1 in [19], is equal to $\left(\frac{8}{0.01}\right)^6 = 2.62 \times 10^{17}$ which is much higher than the one proposed here, which amounts to 59049 points.

7. DISCUSSION

In this paper we have proposed a symbolic abstraction technique for incrementally stable stochastic control systems with only discretization of the input sets. The proposed approach is potentially more scalable than that proposed in [18, 19] for higher dimensional stochastic control systems.

Future work will concentrate on efficient implementations of the symbolic models proposed here, using Binary Decision Diagrams (BDD's) or Algebraic Decision Diagrams (ADD's) as well as efficient controller synthesis techniques.

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Appendix

PROOF. of Lemma 5.1: Let $x_{\mathsf{q}} \in X_{\mathsf{q}}$, where $x_{\mathsf{q}} = (u_1, u_2, \ldots, u_N)$, and $u_{\mathsf{q}} \in U_{\mathsf{q}}$. Using the definition of $\overline{S}_{\mathsf{q}}(\Sigma)$, one obtains $x'_{\mathsf{q}} = (u_2, \ldots, u_N, u_{\mathsf{q}}) \in \mathbf{Post}_{u_{\mathsf{q}}}(x_{\mathsf{q}})$. Since V is a δ-ISS-M$_q$ Lyapunov function for Σ, we have:

$$\begin{aligned}
&\underline{\alpha}\left(\left\|\overline{\xi}_{\overline{H}_{\mathsf{q}}(x_{\mathsf{q}})u_{\mathsf{q}}}(\tau) - \overline{H}_{\mathsf{q}}\left(x'_{\mathsf{q}}\right)\right\|^q\right) \leq V\left(\overline{\xi}_{\overline{H}_{\mathsf{q}}(x_{\mathsf{q}})u_{\mathsf{q}}}(\tau), \overline{H}_{\mathsf{q}}\left(x'_{\mathsf{q}}\right)\right)\\
&= V\left(\overline{\xi}_{\overline{\xi}_{x_s x_{\mathsf{q}}}(N\tau)u_{\mathsf{q}}}(\tau), \overline{\xi}_{x_s x'_{\mathsf{q}}}(N\tau)\right)\\
&= V\left(\overline{\xi}_{\overline{\xi}_{x_s u_1}(\tau)(u_2,\ldots,u_N,u_{\mathsf{q}})}(N\tau), \overline{\xi}_{x_s(u_2,\ldots,u_N,u_{\mathsf{q}})}(N\tau)\right)\\
&\leq \mathsf{e}^{-\kappa N\tau} V\left(\overline{\xi}_{x_s u_1}(\tau), x_s\right). \qquad (8.1)
\end{aligned}$$

We refer the interested readers to the proof of Theorem 3.3 in [19] to see how we derived the inequality (8.1). Hence, one gets

$$\left\|\overline{\xi}_{\overline{H}_{\mathsf{q}}(x_{\mathsf{q}})u_{\mathsf{q}}}(\tau) - \overline{H}_{\mathsf{q}}\left(x'_{\mathsf{q}}\right)\right\| \leq \left(\underline{\alpha}^{-1}\left(\mathsf{e}^{-\kappa N\tau} V\left(\overline{\xi}_{x_s u_1}(\tau), x_s\right)\right)\right)^{1/q}, \qquad (8.2)$$

because of $\underline{\alpha} \in \mathcal{K}_\infty$. Since the inequality (8.2) holds for all $x_{\mathsf{q}} \in X_{\mathsf{q}}$ and $u_{\mathsf{q}} \in U_{\mathsf{q}}$, and $\underline{\alpha} \in \mathcal{K}_\infty$, inequality (5.2) holds. □

PROOF. of Lemma 5.2: Let $x_{\mathsf{q}} \in X_{\mathsf{q}}$, where $x_{\mathsf{q}} = (u_1, u_2, \ldots, u_N)$, and $u_{\mathsf{q}} \in U_{\mathsf{q}}$. Using the definition of $\overline{S}_{\mathsf{q}}(\Sigma)$, one obtains $x'_{\mathsf{q}} = (u_2, \ldots, u_N, u_{\mathsf{q}}) \in \mathbf{Post}_{u_{\mathsf{q}}}(x_{\mathsf{q}})$. Since Σ is δ-ISS-M$_q$ and using inequality (3.1), we have:

$$\begin{aligned}
&\left\|\overline{\xi}_{\overline{H}_{\mathsf{q}}(x_{\mathsf{q}})u_{\mathsf{q}}}(\tau) - \overline{H}_{\mathsf{q}}\left(x'_{\mathsf{q}}\right)\right\|^q = \left\|\overline{\xi}_{\overline{\xi}_{x_s x_{\mathsf{q}}}(N\tau)u_{\mathsf{q}}}(\tau) - \overline{\xi}_{x_s x'_{\mathsf{q}}}(N\tau)\right\|^q\\
&= \left\|\overline{\xi}_{\overline{\xi}_{x_s u_1}(\tau)(u_2,\ldots,u_N,u_{\mathsf{q}})}(N\tau) - \overline{\xi}_{x_s(u_2,\ldots,u_N,u_{\mathsf{q}})}(N\tau)\right\|^q\\
&\leq \beta\left(\left\|\overline{\xi}_{x_s u_1}(\tau) - x_s\right\|^q, N\tau\right).
\end{aligned}$$

Hence, one gets

$$\left\|\overline{\xi}_{\overline{H}_{\mathsf{q}}(x_{\mathsf{q}})u_{\mathsf{q}}}(\tau) - \overline{H}_{\mathsf{q}}\left(x'_{\mathsf{q}}\right)\right\| \leq \left(\beta\left(\left\|\overline{\xi}_{x_s u_1}(\tau) - x_s\right\|^q, N\tau\right)\right)^{1/q}. \qquad (8.3)$$

Since the inequality (8.3) holds for all $x_{\mathsf{q}} \in X_{\mathsf{q}}$ and all $u_{\mathsf{q}} \in U_{\mathsf{q}}$, and β is a $\mathcal{K}_\infty$ function with respect to its first argument when the second one is fixed, inequality (5.4) holds. □