The combined basic LP and affine IP relaxation for promise VCSPs on infinite domains*

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Convex relaxations have been instrumental in solvability of constraint satisfaction problems (CSPs), as well as in the three different generalisations of CSPs: valued CSPs, infinite-domain CSPs, and most recently promise CSPs. In this work, we extend an existing tractability result to the three generalisations of CSPs combined: We give a sufficient condition for the combined basic linear programming and affine integer programming relaxation for exact solvability of promise valued CSPs over infinite-domains. This extends a result of Brakensiek and Guruswami [SODA'20] for promise (non-valued) CSPs (on finite domains).

CCS Concepts: • **Theory of computation → Design and analysis of algorithms**.

Additional Key Words and Phrases: promise constraint satisfaction, valued constraint satisfaction, convex relaxations, polymorphisms

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1 INTRODUCTION

Constraint satisfaction. Constraint satisfaction problems (CSPs) are a wide class of computational decision problems. An instance of a CSP is defined by finitely many relations (constraints) that must hold among finitely many given variables; the computational task is to decide whether it is possible to find an assignment of labels from a fixed set (the domain) to the variables so that all the constraints are satisfied. Many problems in computer science (e.g., from artificial intelligence, scheduling, computational linguistic, computational biology and verification) can be modelled as CSPs by choosing an appropriate set of constraints. However, there are many other problems in which some of the constraints may be violated at a cost or in which there are satisfying assignments which are preferable to others. These situations are captured by valued constraint satisfaction problems.

Valued constraint satisfaction. An instance of a valued constraint satisfaction problem (VCSP) is defined by finitely many cost functions (valued constraints) depending on finitely many given variables and a (rational) threshold; the computational task is to decide whether it is possible to find an assignment of labels from the domain to the variables so that the value of the sum of the cost functions is at most the given threshold. In VCSP instances, cost functions can take on rational or infinite values. VCSPs not only capture optimisation problems but are also a generalisation of

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CSPs: the non-feasibility of an assignment is modelled by allowing the cost functions to evaluate to $+\infty$. A CSP can thus be seen as a VCSP in which the cost functions take values in $\{0, +\infty\}$.

Finite domains. In the case in which the domain (i.e., the fixed set of possible labels for the variables) is a *finite* set the computational complexity of both CSPs and VCSPs have been completely classified. Moreover, in both frameworks a dichotomy theorem holds: every CSP and VCSP is either in P or is NP-complete, depending on some algebraic condition of the underlying set of allowed relations and cost functions, respectively. A dichotomy theorem for CSPs was conjectured by Feder and Vardi [30]. The attempt to prove the conjecture motivated the introduction of the so-called universal algebraic approach [22] for CSPs, which was later extended to VCSPs in [27] and [43], where an analogue of the complexity dichotomy was conjectured for VCSPs. The dichotomy conjectures for finite-domain CSPs and VCSPs inspired an intensive line of research. A complexity classification of finite-domain VCSPs for sets of cost functions taking finite (rational-only) values was established in [52]. A complexity classification of VCSPs was consequently established in [39], assuming a dichotomy for CSPs, which was proved independently in [21] and [58].

Infinite domains. Although most research on CSPs and VCSPs in the past two decades focused on finite-domain problems, the literature is full of problems (studied independently of CSPs and VCSPs) that can be modelled as CSPs or VCSPs *only* if infinite domains are allowed. For instance, solvability of linear Diophantine equations [26, 37] and the model-checking problem for Kozen's modal μ-calculus [42] are examples of problems that can be modelled as infinite-domain CSPs. Linear Programming, Linear Least Square Regression [16], and Minimum Correlation Clustering [4] are examples of problems that can be modelled as infinite-domain VCSPs. The classes of infinite-domain CSPs and infinite-domain VCSPs are huge! In fact, *every* computational problem over a finite alphabet is polynomial-time Turing-equivalent to an infinite-domain CSP [8]. Therefore, only by focussing on special classes of infinite-domain CSPs (and VCSPs) is it possible to obtain general complexity results. There is a rich literature on the computational complexity of special classes of infinite-domain CSPs, e.g., [7, 9–11, 13–15, 36].

Promise constraint satisfaction. Both infinite-domains CSPs and VCSPs are extensions of the original (finite-domain) CSPs. Promise constraint satisfaction problems (PCSPs) are a third, recently introduced extension of CSPs [5, 17, 23, 31]. Informally, in a PCSP the goal is to find an approximately good solution to a problem under the assumption (the promise) that the problem has a solution. The difference between CSPs and PCSPs is that in a PCSP instance each constraint comes with two relations (not necessarily on the same domain), a “strict” and a “weak” relation. The computational task is then to distinguish between being able to satisfy all the strict constraints versus not being able to satisfy all the weak constraints. A CSP can be seen as a PCSP in which the strict and weak constraints coincide. Perhaps the most well-known example of a PCSP is the approximate graph colouring problem, in which the task is to distinguish k-colourable graphs from graphs that are not c-colourable, for some $c > k$. (For $c = k$, we get the standard k-colouring problem.) Kazda recently introduced the framework of promise VCSPs on finite domains [38], where he generalised some of the algebraic reductions from (finite-domain) promise CSPs to (finite-domain) promise VCSPs. As far as we are aware, the only other related work on (finite-domain) promise VCSPs is [3].

Convex relaxations. One of the most effective ways to design a polynomial-time algorithm for solving combinatorial and optimisation problems is to employ convex relaxations. The idea of convex relaxations is to transform the original problem to an integer program which is then relaxed to a polynomial-time solvable convex program [16], e.g. a linear program. In the context of CSPs, convex relaxations have been studied for robust solvability [6, 28, 29, 44]. Convex relaxations have been also successfully applied to the study of the three extensions of CSPs. For VCSPs, characterisations of the applicability of the basic linear programming relaxation [40], constant levels of the Sherali-Adams linear programming hierarchy [53], and a polynomial-size semidefinite programming relaxation [54] have been provided for exact solvability. In the PCSP framework, the polynomial-time tractability via a specific convex relaxation has been characterised for the basic linear programming relaxation [23], affine integer programming relaxation [23], and their combination [18–20]. For infinite-domain VCSPs, a sufficient condition has been identified for the solvability via a combination of the basic linear programming relaxation and an efficient sampling algorithm (that is, polynomial-time many-one reduction to a finite-domain VCSP) [12, 56].

1.1 Contributions

We initiate the study of convex relaxations for the three generalisations of CSPs *combined*; that is, convex relaxations for promise valued constraint satisfaction problems on infinite-domains. We focus on the combined basic linear programming (BLP) and affine integer programming (AIP) relaxation introduced by Brakensiek and Guruswami [19]. This relaxation is stronger than both the BLP and AIP relaxations individually in the sense that if a class of promise VCSPs is solved by, say, the BLP relaxation then it is also solved by the combined relaxation (and the same holds true for the AIP relaxation). The power of the combined relaxation for (finite-domain) promise CSPs was established in [20]. Rather surprisingly, the combined relaxation gives an algorithm that solves all tractable (non-promise) CSPs on Boolean domains, identified in Schaefer's work [50], thus giving a unified algorithm.

By extending the argument from [19], we establish a sufficient algebraic condition on the combined relaxation for the solvability of promise VCSPs in which the domain of the "weak cost functions" is possibly infinite (Theorem 8). The proof of this result draws on ideas introduced in [19] but requires a non-trivial amount of technical machinery to make it work in the infinite-domain valued setting. While our relaxation is inspired by [19], it is appropriately modified to work in the optimisation setting (of valued (P)CSPs). We remark that the condition we give is known to be necessary already in special cases of our setting, namely for finite-domain non-valued PCSPs [20]. As an application of our main result, we derive an algebraic condition under which an *infinite-domain* promise VCSP admitting an efficient sampling algorithm can be solved in polynomial time using the combined relaxation (Theorem 16). We emphasise that our main results (Theorems 8 and 16) are appreciatively general, and in particular hold for various special cases of our framework; e.g., for finite-domain promise VCSPs and infinite-domain promise CSPs.

Approximability of Max-CSPs. PCSPs are approximability problems in which we require that all constraints should be satisfied, although only in a weaker sense. Another very natural and well-studied form of relaxation is to try to maximise the number of satisfied constraints. Convex relaxations have played a crucial role in this research direction on approximability of (finite-domain) Max-CSPs, going back to the work of Goemans and Williamson [33], e.g., [24, 25, 32, 41, 46, 48, 55].

2 PRELIMINARIES

Throughout the paper, we denote by x_i the i-th component of a tuple x. We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{Q}_{\geq 0}$ the set of whole numbers, integer numbers, rational numbers, and nonnegative rational numbers, respectively. For every $m \in N$, we denote by $[m]$ the set $\{1, \ldots, m\} \subset \mathbb{N}$. Finally, for every $k \in \mathbb{Q}$ we use the $\lceil k \rceil$ and $\lfloor k \rfloor$ to denote the minimum natural number that is at least k and the maximum natural number that is at most k, respectively.

2.1 Valued Constraint Satisfaction Problems

A *valued structure* Γ (over D) consists of a signature τ consisting of function symbols f, each equipped with an arity $\mathrm{ar}(f)$; a set $D = \mathrm{dom}(\Gamma)$ (the *domain*); and, for each $f \in \tau$, a *cost function*, i.e., a function $f^\Gamma \colon D^{\mathrm{ar}(f)} \to \mathbb{Q} \cup \{+\infty\}$. Here, $+\infty$ is an extra element with the expected properties that for all $c \in \mathbb{Q} \cup \{+\infty\}$, we have $(+\infty) + c = c + (+\infty) = +\infty$ and $c < +\infty$ for every $c \in \mathbb{Q}$. Given a valued structure Γ with signature τ, for every $f \in \tau$ we define $\mathrm{dom}(f) := \{t \in D^{\mathrm{ar}(f)} \mid f^\Gamma(t) < +\infty\}$.

Let Γ be a valued structure with domain D and signature τ. The *valued constraint satisfaction problem* for Γ, denoted by VCSP(Γ), is the following computational problem.

An *instance* of VCSP(Γ) is a triple $I := (V, \phi, u)$ where V is a finite set of variables; ϕ is an expression of the form $\sum_{i=1}^m f_i(v_1^i, \ldots, v_{\mathrm{ar}(f_i)}^i)$, where $f_1, \ldots, f_m \in \tau$ and all the v_j^i are variables from V (each summand is called a τ-term); and u is a value from $\mathbb{Q}$. The task is to decide whether there exists an assignment $s \colon V \to D$, whose *cost*, defined as

$$\phi^\Gamma(s(v_1), \ldots, s(v_{|V|})) := \sum_{i=1}^m f_i^\Gamma(s(v_1^i), \ldots, s(v_{\mathrm{ar}(f_i)}^i))$$

is finite, and if so, whether there is one whose cost is at most u.

We remark that, given a valued structure Γ over a *finite* signature, the representation of the structure Γ is inessential for computational complexity as Γ is not part of the input.

2.2 Fractional Homomorphisms and Fractional Polymorphisms

Let X be a set. A *discrete probability measure* on X is a map $\mu \colon \mathcal{P}(X) \to [0, 1]$ such that $\mu(X) = 1$ and μ satisfies the countable additivity property; i.e., for every countable collection $\{X_n\}_{n\in\mathbb{N}}$ of pairwise disjoint subsets $X_n \subseteq X$, it holds that $\mu\big(\bigcup_{n\in\mathbb{N}} X_n\big) = \sum_{n\in\mathbb{N}} \mu(X_n)$.

Given a probability measure μ on a countable set X, we define its *support* as the set $\mathrm{Supp}(\mu) := \{x \in X \mid \mu(\{x\}) > 0\}$. In the reminder, given a probability measure μ on a set X, we use the notation $\mu(x) := \mu(\{x\})$. The following proposition is a well-known corollary of countable additivity (proved in Appendix A for completeness).

Proposition 1. *Let μ be a discrete probability measure on a set X. Then* $\mathrm{Supp}(\mu)$ *is a countable set. Furthermore, if X is countable then $\sum_{x\in\mathrm{Supp}(\mu)} \mu(x) = 1$; that is,* $\mathrm{Supp}(\mu)$ *is non-empty.*

Proposition 1 will guarantee that all supports in this paper are countable sets.

Let μ be a discrete probability measure on a countable set X and let Y be a random variable with countably many possible outcomes $y_1, y_2, \ldots$ occurring with probabilities $\mu(x_1), \mu(x_2), \ldots$, respectively. The expectation of Y associated with μ is $\mathbb{E}_{x\sim\mu}[Y] = \sum_{n\in\mathbb{N}} \mu(x_n) y_n$.

Let C and D be two sets. A map $g \colon D^m \to C$ is called an m-ary *operation*. For any $m \in \mathbb{N}$, we denote by C^{D^m} the set of all maps $g \colon D^m \to C$.

Let Γ and Δ be valued structures with the same signature τ with domains C and D, respectively. A *fractional homomorphism* [51] from Δ to Γ is a discrete probability measure χ with a non-empty support on C^D such that for every function symbol $\gamma \in \tau$ and tuple $a \in D^{\mathrm{ar}(\gamma)}$, it holds that

$$\mathbb{E}_{h\sim\chi}[\gamma^\Gamma(h(a))] = \sum_{h\in\mathrm{Supp}(\chi)} \chi(h)\gamma^\Gamma(h(a)) \leq \gamma^\Delta(a), \tag{1}$$

where the functions h are applied component-wise. We write $\Delta \to_f \Gamma$ to indicate the existence of a fractional homomorphism from Δ to Γ.

The following proposition, proved for completeness in Appendix A, is adapted from [51], where it was proved in the case of finite-domain valued structure, and appears in [12], where it was stated for valued structures with arbitrary domains and for fractional homomorphisms with finite supports.

Proposition 2. *Let Γ and Δ be valued structures over the same signature τ with domains C and D, respectively. Assume $\Delta \to_f \Gamma$. Let $V = \{v_1, \dots, v_n\}$ be a set of variables and ϕ a sum of finitely many τ-terms with variables from V. For every $u \in \mathbb{Q}$, if there exists an assignment $s \colon V \to D$ such that $\phi^\Delta(s(v_1, \dots, s(v_n)) \leq u$, then there exists an assignment $s' \colon V \to C$ such that $\phi^\Gamma(s'(v_1), \dots, s'(v_n)) \leq u$. In particular, it holds that $\inf_C \phi^\Gamma \leq \inf_D \phi^\Delta$.*

Let Γ be a valued structure with domain C and signature τ. An m-ary *fractional polymorphism* of Γ is a discrete probability measure on C^{C^m} with a non-empty support such that for every $f \in \tau$ and tuples $a^1, \dots, a^m \in C^{\mathrm{ar}(f)}$ it holds that

$$\mathbb{E}_{g \sim \omega}[f^\Gamma(g(a^1, \dots, a^m))] = \sum_{g \in C^{C^m}} \omega(g) f^\Gamma(g(a^1, \dots, a^m)) \leq \frac{1}{m} \sum_{i=1}^{m} f^\Gamma(a^i)$$

(where g is applied component-wise).

2.3 Promise VCSPs

Let Γ and Δ be two valued structures over the same signature τ with domains C and D, respectively. We say that (Δ, Γ) is a *promise valued template* if there exists a fractional homomorphism from Δ to Γ. Given a promise valued template (Δ, Γ), the *promise valued constraint satisfaction problem* [38] for (Δ, Γ), denoted by PVCSP(Δ, Γ), is the following computational problem.

An *instance* I of PVCSP(Δ, Γ) is a triple $I := (V, \phi, u)$ where V is a finite set of variables; ϕ is an expression of the form $\sum_{i=1}^m f_i(v_1^i, \dots, v_{\mathrm{ar}(f_i)}^i)$, where $f_1, \dots, f_m \in \tau$ and all the v_j^i are variables from V; and u is a value from $\mathbb{Q}$.

The task is to output yes if there exists an assignment $s \colon V \to D$ with cost

$$\phi^\Delta(s(v_1), \dots, s(v_{|V|})) := \sum_{i=1}^{m} f_i^\Delta(s(v_1^i), \dots, s(v_{\mathrm{ar}(f_i)}^i)) \leq u$$

and output no if *every* assignment $s' \colon V \to C$ has cost

$$\phi^\Gamma(s'(v_1), \dots, s'(v_{|V|})) := \sum_{i=1}^{m} f_i^\Gamma(s'(v_1^i), \dots, s'(v_{\mathrm{ar}(f_i)}^i)) \not\leq u.$$

Note that every valued structure Γ is fractionally homomorphic to itself and thus VCSP(Γ) is the same as PVCSP(Γ, Γ).

Let (Δ, Γ) be a promise valued template. We remark that if the common signature τ is finite then the representation of the template is inessential for the computational complexity of PVCSP(Δ, Γ) as (Δ, Γ) is not part of the input.

Let $e_i^{(m)} \colon D^m \to D$ denote the m-ary projection on D onto the i-th coordinate. Let $\mathscr{J}_D^{(m)} := \{e_1^{(m)}, \dots, e_m^{(m)}\}$, i.e., the set of all m-ary projections on D.

An m-ary *promise fractional polymorphism*[1] of a promise valued template (Δ, Γ) is a pair $\omega := (\omega_I, \omega_O)$ where ω_O is a discrete probability measure on C^{D^m} with a non-empty support and ω_I is a discrete probability measure with (finite) support $\mathrm{Supp}(\omega_I) = \mathscr{J}_D^{(m)}$ such that for every $f \in \tau$ and

[1]These are called weighted polymorphisms in [38].

tuples $a^1, \ldots, a^m \in D^{\mathrm{ar}(f)}$ it holds that

$$\mathbb{E}_{g\sim\omega_O}[f^\Gamma(g(a^1,\ldots,a^m))] = \sum_{g\in\mathrm{Supp}(\omega)} \omega_O(g) f^\Gamma(g(a^1,\ldots,a^m)) \leq \sum_{i=1}^{m} \omega_I(e_i^{(m)}) f^\Delta(a^i) = \mathbb{E}_{e\sim\omega_I}[f^\Delta(e(a^1,\ldots,a^m)). \quad (2)$$

Remark 3. An m-ary fractional polymorphism ω of a valued structure Γ with domain C can be seen as an m-ary promise fractional polymorphism $\mu = (\mu_I, \mu_O)$ of (Γ, Γ) such that $\mu_O = \omega$ and $\mu_I(e_i^{(m)}) = \frac{1}{m}$ for $1 \leq i \leq m$.

2.4 Block-Symmetric Maps

Let S_m be the symmetric group on $\{1, \ldots, m\}$. An m-ary map g is *fully symmetric* if for every permutation $\pi \in S_m$, we have $g(x_1, \ldots, x_m) = g(x_{\pi(1)}, \ldots, x_{\pi(m)})$.

An m-ary map g is *block-symmetric* if there exists a partition of the coordinates of g into blocks $B_1 \cup \cdots \cup B_k = [m]$ such that g is permutation-invariant within each block B_i. Let $\mathcal{P}_{sym}(g)$ be the set of all partitions into symmetric blocks of g. For $B_1 \cup \cdots \cup B_k \in \mathcal{P}_{sym}(m)$, we define $w(g, B_1 \cup \cdots \cup B_k) := \min_{1\leq i\leq k} |B_i|$ and we define the *width* of g to be

$$w(g) := \max\{w(g, B_1 \cup \cdots \cup B_k) \mid B_1 \cup \cdots \cup B_k \in \mathcal{P}_{sym}(g)\}.$$

Block-symmetric operation with width 1 are fully symmetric operations. The next example shows operations that are block-symmetric but not fully symmetric.

Example 4. We consider so-called "moving averages" (see, e.g., [35, 49]). For $k \in \mathbb{N}$, a k-ary 2*-period weighted centred moving average* is the operation $\mathrm{wMA}^{(k)} \colon \mathbb{Q}^k \to \mathbb{Q}$ defined by

$$\mathrm{wMA}^{(k)}(x^1, \ldots, x^k) := \frac{1}{3k}\left(\sum_{i=1}^{\lfloor \frac{k}{4} \rfloor} x^i + 2\sum_{i=\lfloor \frac{k}{4} \rfloor+1}^{\lfloor \frac{3}{4}k \rfloor} x^i + \sum_{i=\lfloor \frac{3}{4}k \rfloor+1}^{k} x^i\right).$$

It can be verified that a k-ary 2-period weighted centred moving average, where $k = 2m+1$, is a block-symmetric operation whose symmetric blocks are $B_1 := \{1, \ldots, \lfloor \frac{k}{4} \rfloor, \lfloor \frac{3}{4}k \rfloor + 1, \ldots, k\}$ and $B_2 := \{\lfloor \frac{k}{4} \rfloor + 1, \ldots, \lfloor \frac{3}{4}k \rfloor\}$. Observe that $|B_1| = \lfloor \frac{k}{2} \rfloor$ and $|B_2| = \lceil \frac{k}{2} \rceil$. △

An m-ary fractional polymorphism ω of a valued structure Γ is *block-symmetric* if there exists a partition of the coordinates of g into blocks $B_1 \cup \cdots \cup B_k = [m]$ such that every operation in $\mathrm{Supp}(\omega)$ is permutation-invariant within each coordinate block B_i.

Example 5. A k-ary *average* is a map $A : \mathbb{Q}^k \to \mathbb{Q}$ such that, for every $x \in \mathbb{Q}$, $A(x, \ldots, x) = x$ (i.e., A is idempotent) and, for every $x^1, \ldots, x^k \in \mathbb{Q}$, we have $\min(x^1, \ldots, x^k) \leq A(x^1, \ldots, x^k) \leq \max(x^1, \ldots, x^k)$, and for every $\lambda \in \mathbb{Q}$ it holds $\lambda A(x^1, \ldots, x^k) = A(\lambda x^1, \ldots, \lambda x^k)$. Examples of averages include the arithmetic average $\mathrm{avg}^{(k)}$ defined by $\mathrm{avg}^{(k)}(x^1, \ldots, x^k) = \frac{1}{k}(x^1 + \ldots + x^k)$ and the 2-period centred moving average (cf. Example 4).

Let A_1 and A_2 be two different averages. A function $f \colon \mathbb{Q} \to \mathbb{Q} \cup \{+\infty\}$ is called (A_1, A_2)*-convex*[2] if for every $k \in \mathbb{N}$ and every $x^1, \ldots, x^k \in \mathbb{Q}^n$ it holds $f(A_1(x^1, \ldots, x^k)) \leq A_2(f(x^1), \ldots, f(x^k))$.

A valued structure with domain $\mathbb{Q}$ containing only (wMA, avg)-convex functions has, for every $m \in \mathbb{N}$, a $(2m+1)$-ary block-symmetric fractional polymorphism $\omega^{(2m+1)}$ defined by $\omega^{(2m+1)}(g) = 1$ if $g = \mathrm{wMA}^{(2m+1)}$ and 0 otherwise. △

[2] In [1, 2], the notion of (A_1, A_2)-convexity is defined for A_1, A_2 symmetric averages. An average A is symmetric if for every $k \in \mathbb{N}$, and every $x^1, \ldots, x^k \in \mathbb{Q}$, it holds $A(x^1, \ldots, x^k) = A(x^{\pi(1)}, \ldots, x^{\pi(k)})$ for every $\pi \in S_k$.

Given a promise valued template (Δ, Γ), an m-ary promise fractional polymorphism $\omega = (\omega_I, \omega_O)$ of (Δ, Γ) is *block-symmetric* if

- there exists a partition of $[m]$ into blocks $B_1 \cup \cdots \cup B_k$ such that every map in $\mathrm{Supp}(\omega_O)$ is s permutation-invariant within each coordinate block B_i, and
- $\sum_{i \in B_j} \omega_I(e_i^{(m)}) = \frac{|B_j|}{m}$ for every $j \in \{1, \ldots, k\}$.

The proof of the following lemma can be found in Appendix A.

Lemma 6. *Let (Δ, Γ) be a promise valued template and let $m \in \mathbb{N}$. If $\omega = (\omega_I, \omega_O)$ is an m-ary block symmetric promise fractional polymorphism of (Δ, Γ), then also $\omega' = (\omega'_I, \omega_O)$, where $\omega'_I(e_i^{(m)}) = \frac{1}{m}$ for $1 \leq i \leq m$, is an m-ary block-symmetric promise fractional polymorphism of (Δ, Γ).*

In view of Lemma 6, we will assume without loss of generality that any m-ary block-symmetric promise fractional polymorphism $\omega = (\omega_I, \omega_O)$ is such that ω_I assign $\frac{1}{m}$ to each m-ary projection on the domain of Δ and we will identify ω with ω_O.

2.5 The Basic Linear Programming Relaxation

Every VCSP over a finite domain has a natural linear programming relaxation. Let Δ be a valued structure with finite domain D and signature τ. Let I be an instance of VCSP(Δ) with set of variables $V = \{x_1, \ldots, x_d\}$, objective function $\phi(x_1, \ldots, x_d) = \sum_{j \in J} f_j(x_1^j, \ldots, x_{n_j}^j)$, with $f_j \in \tau$, $x^j = (x_1^j, \ldots, x_{n_j}^j) \in V^{n_j}$, for all $j \in J$ (the set J is finite and indexing the cost functions that are summands of ϕ), and a threshold $u \in \mathbb{Q}$.[3] Define the sets of variables as follows: $W_1 := \{\lambda_j(t) \mid j \in J \text{ and } t \in D^{n_j}\}$, $W_2 := \{\mu_{x_i}(a) \mid x_i \in V \text{ and } a \in D\}$, and $W := W_1 \cup W_2$. Then the *basic linear programming* (BLP) relaxation associated to I (see [51], [40], and references therein) is a linear program with variables W and is defined in Figure 1.

$$\mathrm{BLP}(I, \Delta) := \min \sum_{j \in J} \sum_{t \in D^{n_j}} \lambda_j(t) f_j^{\Delta}(t)$$

subject to

$$\begin{aligned}
\sum_{t \in D^{n_j} : t_\ell = a} \lambda_j(t) &= \mu_{x_\ell^j}(a) && \text{for all } j \in J, \ell \in \{1, \ldots, n_j\}, a \in D, \\
\sum_{a \in D} \mu_{x_i}(a) &= 1 && \text{for all } x_i \in V, \\
\lambda_j(t) &= 0 && \text{for all } j \in J, t \notin \mathrm{dom}(f_j), \\
0 \leq \lambda_j(t), \mu_{x_i}(a) &\leq 1 && \text{for all } \lambda_j(t) \in W_1,\ \mu_{x_i}(a) \in W_2.
\end{aligned}$$

Fig. 1. BLP

We remark that a solution to the BLP also satisfies the constraints $\sum_{t \in D^{n^j}} \lambda_j(t) = 1$ for all $j \in J$. If there is no feasible solution to the BLP then $\mathrm{BLP}(I, \Delta) = +\infty$. For a finite-domain VCSP instance, the corresponding BLP relaxation can be computed in polynomial time.

We note that it is not difficult to lift the existing results characterising the power of BLP (in terms of fully symmetric operations) for CSPs [44], VCSPs [40], promise CSPs [5, 23], and infinite-domain VCSPs [12, 56] to our setting of promise VCSPs with infinite domains, cf. Theorem 20 in Appendix B. Our focus, however, is on the stronger combined relaxation presented in Section 3.

[3] Note that the BLP relaxation does not depend on the threshold u.

2.6 The Affine Integer Programming Relaxation

Let Δ be a valued structure with finite domain D and signature τ. Let I be an instance of VCSP(Δ) with set of variables $V = \{x_1, \ldots, x_d\}$, and objective function $\phi(x_1, \ldots, x_d) = \sum_{j \in J} f_j(x_1^j, \ldots, x_{n_j}^j)$, with $f_j \in \tau$, $x^j = (x_1^j, \ldots, x_{n_j}^j) \in V^{n_j}$, for all $j \in J$ (the set J is finite and indexing the cost functions that are summands of ϕ), and a threshold $u \in \mathbb{Q}$.[4] Define the sets of variables as follows: $R_1 := \{q_j(t) \mid j \in J \text{ and } t \in D^{n_j}\}$, $R_2 := \{r_{x_i}(a) \mid x_i \in V \text{ and } a \in D\}$, and $R := R_1 \cup R_2$. Then the *affine integer programming* (AIP) relaxation associated to I [18, 19] is an integer program with variables R and is defined in Figure 2.

$$\mathrm{AIP}(I, \Delta) := \min \sum_{j \in J} \sum_{t \in D^{n_j}} q_j(t) f_j^{\Delta}(t)$$

subject to

$$\begin{aligned}
\sum_{t \in D^{n_j} : t_\ell = a} q_j(t) &= r_{x_\ell^j}(a) && \text{for all } j \in J, \ell \in \{1, \ldots, n_j\}, a \in D,\\
\sum_{a \in D} r_{x_i}(a) &= 1 && \text{for all } x_i \in V,\\
q_j(t) &= 0 && \text{for all } j \in J, t \notin \mathrm{dom}(f_j),\\
q_j(t), r_{x_i}(a) &\in \mathbb{Z} && \text{for all } q_j(t) \in R_1,\ r_{x_i}(a) \in R_2.
\end{aligned}$$

Fig. 2. AIP

We remark that a solution to the AIP also satisfies the constraints $\sum_{t \in D^{n_j}} \lambda_q(t) = 1$ for all $j \in J$. If there is no feasible solution to the AIP then $\mathrm{AIP}(I, \Delta) = +\infty$. For a finite-domain VCSP instance, the corresponding AIP relaxation can be computed in polynomial time. Since the feasibility version of AIP can be solved in polynomial time [18, 37], (the optimisation version of) AIP can be solved in (oracle) polynomial time using an oracle for the feasibility version of the problem (see [34, Theorem 6.4.9]).

3 THE COMBINED BLP AND AIP RELAXATION FOR PVCSPS

Let (Δ, Γ) be a promise valued template such that the domain of Δ is a *finite* set. We may solve PVCSP(Δ, Γ) by using a *combination* of the BLP relaxation and the AIP relaxation of Δ, as proposed (for finite-domain promise non-valued) CSPs in [19], appropriately modified to the valued setting.

To describe such an algorithm, we need the following definition.

Definition 7. Let Δ be a valued structure with finite domain D and signature τ. Let us consider an instance $I := (V, \phi, u)$ of VCSP(Δ) such that $\phi(x_1, \ldots, x_d) = \sum_{j \in J} f_j(x_1^j, \ldots, x_{n_j}^j)$. Assume that $\mathrm{BLP}(I, \Delta) \leq u$. We define $(\lambda^\star, \mu^\star)$ as follows.

- If there exists a relative interior point of the rational feasibility polytope of $\mathrm{BLP}(I, \Delta)$ with cost at most u,[5] then $(\lambda^\star, \mu^\star)$ is such a point;
- otherwise, $(\lambda^\star, \mu^\star)$ is defined to be a point from the relative interior of the optimal polytope of $\mathrm{BLP}(I, \Delta)$.[6]

[4] Note that the AIP relaxation does not depend on the threshold u.

[5] There is a polynomial-time algorithm [19, 34] that decides the existence of a relative interior point in the rational feasibility polytope of $\mathrm{BLP}(I, \Delta)$ with cost at most u and, in the case it exists, finds it.

[6] Such a point can be found in polynomial time by applying the algorithm in [19, 34] to the feasibility linear program defined by adding to the constraints defining the feasibility polytope of $\mathrm{BLP}(I, \Delta)$ the additional constraint $\sum_{j \in J} \sum_{t \in D^{n_j}} \lambda_j(t) f_j^{\Delta}(t) = u$.

The *refinement of* AIP(I, Δ) *with respect to* $(\lambda^\star, \mu^\star)$ is the integer program AIP$^\star(I, \Delta)$ obtained by adding to AIP(I, Δ) the constraints

$$q_j(t) = 0 \qquad \text{for every } j \in J, t \in D^{n_j} \text{ such that } \lambda^\star_j(t) = 0,$$
$$r_{x_i}(a) = 0 \qquad \text{for every } x_i \in V, a \in D \text{ such that } \mu^\star_{x_i}(a) = 0.$$

ALGORITHM 1: The combined BLP + AIP Relaxation Algorithm for PVCSP(Δ, Γ)

Input:

$I := (V, \phi, u)$, a valid instance of PVCSP(Δ, Γ)

Output:

YES if there exists an assignment $s \colon V \to \mathrm{dom}(\Delta)$ such that $\phi^\Delta(s(x_1), \ldots, s(x_{|V|})) \leq u$

NO if there is no assignment $s \colon V \to \mathrm{dom}(\Gamma)$ such that $\phi^\Gamma(s(x_1), \ldots, s(x_{|V|})) \leq u$

BLP(I, Δ);
if BLP$(I, \Delta) \nleq u$ **then**
  output NO;
else
  $(\lambda^\star, \mu^\star)$, as in Definition 7;
  AIP$^\star(I, \Delta)$:= refinement of AIP(I, Δ) with respect to $(\lambda^\star, \mu^\star)$, as in Definition 7;
  if AIP$^\star(I, \Delta) \nleq u$ **then**
    output NO;
  else
    output YES;
  end
end

As our main result, we now present a sufficient condition under which Algorithm 1 correctly solves PVCSP(Δ, Γ).

Theorem 8. *Let* (Δ, Γ) *be a promise valued template such that* Δ *has a* finite domain. *Assume that for all* $L \in \mathbb{N}$ *there exists a block-symmetric promise fractional polymorphism of* (Δ, Γ) *with arity* $2L + 1$ *having two symmetric blocks of size* $L + 1$ *and* L*, respectively. Then Algorithm 1 correctly solves* PVCSP(Δ, Γ) *(in polynomial time).*

Note that in Theorem 8 the domain of the valued structure Γ can be finite or (countably) infinite.

To prove Theorem 8 we need to use a preliminary lemma and the notion of a *bimultiset-structures.* Let Δ be a valued τ-structure with domain D, let $L \in \mathbb{N}$, and let $B_1 \cup B_2$ any partition of $[2L + 1]$ such that $|B_1| = L + 1$ and $|B_2| = L$. The *bimultiset-structure* $\mathcal{B}^{2L+1}_{B_1,B_2}(\Delta)$ is the valued structure with domain $\left(\!\binom{D}{L+1}\!\right) \times \left(\!\binom{D}{L}\!\right)$ i.e., the set whose elements (α, β) are pairs of multisets of elements from D of size $L + 1$ and of size L, respectively. For every k-ary function symbol $f \in \tau$, and $(\alpha_1, \beta_1) \ldots, (\alpha_k, \beta_k) \in \left(\!\binom{D}{L+1}\!\right) \times \left(\!\binom{D}{L}\!\right)$ the function $f^{\mathcal{B}^{2L+1}_{B_1,B_2}(\Delta)}$ is defined as follows

$$f^{\mathcal{B}^{2L+1}_{B_1,B_2}(\Delta)}((\alpha_1, \beta_1) \ldots, (\alpha_k, \beta_k)) := \frac{1}{2L+1} \min_{\substack{t^1,\ldots,t^k \in D^{2L+1}: \\ \{t^\ell\}_{B_1} = \alpha_\ell, \{t^\ell\}_{B_2} = \beta_\ell}} \sum_{i=1}^{2L+1} f^\Delta(t^1_i, \ldots, t^k_i),$$

where $\{t^\ell\}_{B_1}$ denotes the multiset $\{t^\ell_i \mid i \in B_1\}$ and $\{t^\ell\}_{B_2}$ denotes the multiset $\{t^\ell_i \mid i \in B_2\}$.

Lemma 9. *Let* (Δ, Γ) *be a promise valued template such that* Δ *has a* finite domain. *Let* $L \in \mathbb{N}$ *and assume that* (Δ, Γ) *has a block-symmetric promise fractional polymorphism of arity* $2L + 1$ *with* *two symmetric blocks B_1 and B_2 of size $L+1$ and L, respectively. Then $\mathcal{B}^{2L+1}_{B_1,B_2}(\Delta)$ is fractionally homomorphic to* Γ.

Proof. Let C be the (possibly infinite) domain of Γ, let D be the finite domain of Δ, and let τ be the common signature of Γ and Δ. Let ω be the $(2L+1)$-ary block-symmetric promise fractional polymorphism of (Δ,Γ) with symmetric blocks B_1 and B_2 of size $L+1$ and L, respectively. For every $g \in \mathrm{Supp}(\omega) \subseteq C^{D^{2L+1}}$ we define $\tilde{g} \colon \left(\!\!\binom{D}{L+1}\!\!\right) \times \left(\!\!\binom{D}{L}\!\!\right) \to C$ by setting, for every $\alpha = \{\xi^j \mid j \in B_1\} \in \left(\!\!\binom{D}{L+1}\!\!\right)$ and every $\beta = \{\xi^j \mid j \in B_2\} \in \left(\!\!\binom{D}{L}\!\!\right)$,

$$\tilde{g}((\alpha,\beta)) = g(\xi^1,\xi^2,\dots,\xi^{2L+1}).$$

Observe that $\tilde{g}$ is well defined as g is block-symmetric with symmetric blocks B_1 and B_2 (the order of the coordinates from a same block does not matter). We define the discrete probability measure χ on $C^{\left(\!\binom{D}{L+1}\!\right)\times\left(\!\binom{D}{L}\!\right)}$ as follows

$$\chi(Y) = \sum_{\substack{g\in\mathrm{Supp}(\omega):\\ \tilde{g}\in Y}} \omega(g), \qquad \text{for every } Y \subseteq \left(\!\!\binom{D}{L+1}\!\!\right) \times \left(\!\!\binom{D}{L}\!\!\right).$$

Observe that χ satisfies the countable additivity property, since ω does. Furthermore, $\mathrm{Supp}(\chi) = \{h \in C^{D^{2L+1}} \mid h = \tilde{g} \text{ for some } g \in \mathrm{Supp}(\omega)\} = \{\tilde{g} \in C^{D^{(2L+1)}} \mid g \in \mathrm{Supp}(\omega)\}$ is countable by Proposition 1 and it holds that

$$\sum_{h\in\mathrm{Supp}(\chi)} \chi(h) = \sum_{g\in\mathrm{Supp}(\omega)} \omega(g) = 1.$$

We claim that χ is a fractional homomorphism from $\mathcal{B}^{2L+1}_{B_1,B_2}(\Delta)$ to Γ. Indeed, for every $f \in \tau$ and every tuple $((\alpha_1,\beta_1)\dots,(\alpha_k,\beta_k)) \in \left(\left(\!\!\binom{D}{L+1}\!\!\right) \times \left(\!\!\binom{D}{L}\!\!\right)\right)^k$ with $\alpha_i := \{\xi_i^j \mid j \in B_1\}$, $\beta_i := \{\xi_i^j \mid j \in B_2\}$, and $k := \mathrm{ar}(f)$, it holds that

$$\begin{aligned}
&\sum_{h\in C^{\left(\!\binom{D}{L+1}\!\right)\times\left(\!\binom{D}{L}\!\right)}} \chi(h) f^{\Gamma}(h((\alpha_1,\beta_1))\dots,h((\alpha_k,\beta_k)))\\
&= \sum_{g\in\mathrm{Supp}(\omega)} \omega(g) f^{\Gamma}(g(\xi_1^1,\dots,\xi_1^{2L+1}),\dots,g(\xi_k^1,\dots,\xi_k^{2L+1}))\\
&= \sum_{g\in\mathrm{Supp}(\omega)} \omega(g) f^{\Gamma}(g(\xi_1^{\pi_1(1)},\dots,\xi_1^{\pi_1(2L+1)}),\dots,g(\xi_k^{\pi_k(1)},\dots,\xi_k^{\pi_k(2L+1)})) &(3)\\
&\le \frac{1}{2L+1}\sum_{i=1}^{2L+1} f^{\Delta}(\xi_1^{\pi_1(i)},\dots,\xi_k^{\pi_k(i)}) &(4)
\end{aligned}$$

for every $\pi_1,\dots,\pi_k \in S_{2L+1}$ such that π_i is permutation-invariant within B_1 and within B_2 for $1 \le i \le k$. Equality (3) holds because the maps $g \in \mathrm{Supp}(\omega)$ are block-symmetric with symmetric blocks B_1 and B_2. Inequality (4) holds because ω is a promise fractional polymorphism of (Δ,Γ). Then, in particular, we obtain

$$\begin{aligned}
&\sum_{h\in C^{\left(\!\binom{D}{L+1}\!\right)\times\left(\!\binom{D}{L}\!\right)}} \chi(h) f^{\Gamma}(h((\alpha_1,\beta_1)\dots,(\alpha_k,\beta_k)))\\
\le &\frac{1}{2L+1} \min_{\substack{t^1,\dots,t^k\in D^{2L+1}:\\ \{t^\ell\}_{B_1}=\alpha_\ell,\{t^\ell\}_{B_2}=\beta_\ell}} \sum_{i=1}^{2L+1} f^{\Delta}(t_i^1,\dots t_i^k) = f^{\mathcal{B}^{2L+1}_{B_1,B_2}(\Delta)}((\alpha_1,\beta_1),\dots,(\alpha_k,\beta_k)).
\end{aligned}$$ □

Remark 10. Although we do not need it for our main result, we note that the converse of Lemma 9 holds true. More precisely, let (Δ, Γ) be a promise valued template such that Δ has a finite domain. Let $L \in \mathbb{N}$ and let $B_1 \cup B_2$ be a partition of $[2L+1]$ such that $|B_1| = L+1$ and $|B_2| = L$. If $\mathcal{B}^{2L+1}_{B_1,B_2}(\Delta)$ is fractionally homomorphic to Γ, then (Δ, Γ) has a block-symmetric promise fractional polymorphism of arity $2L+1$ with symmetric blocks B_1 and B_2. To show this we reason as in the proof of [51, Lemma 2.2]. Let D be the domain of Δ, and let χ be a fractional homomorphism from $\mathcal{B}^{2L+1}_{B_1,B_2}(\Delta)$ to Γ. Let us define $h \colon D^{2L+1} \to \left(\!\binom{D}{L+1}\!\right) \times \left(\!\binom{D}{L}\!\right)$ such that it maps a tuple $a = (a_1, \ldots, a_m) \in D^{2L+1}$ to the bimultiset $(\{a\}_{B_1}, \{a\}_{B_2})$. Then, it is easily verified that the discrete probability measure ω on C^{D^m} such that

$$\omega(g') = \sum_{\substack{g \in \mathrm{Supp}(\chi): \\ g \circ h = g'}} \omega(g)$$

is the desired promise fractional polymorphism.

Proof of Theorem 8. Let C be the (possibly infinite) domain of Γ and let D be the finite domain of Δ. Let τ be the common signature of Δ and Γ. Let I be an instance of PVCSP(Δ, Γ) with variables $V = \{x_1, \ldots, x_n\}$, objective function $\phi(x_1, \ldots, x_n) = \sum_{j \in J} \gamma_j(x^j)$ where J is a finite set of indices, $\gamma_j \in \Gamma$, and $x^j \in V^{\mathrm{ar}(j)}$, and threshold u.

Assume that $\min_D \phi^\Delta \leq u$. Our goal is to show that Algorithm 1 outputs yes. Since $\min_D \phi^\Delta \leq u$ we have $\mathrm{BLP}(I, \Delta) \leq u$ and in particular we have that either $\mathrm{BLP}(I, \Delta) < u$, which by linearity implies the existence of a relative interior point in the feasibility polytope of $\mathrm{BLP}(I, \Delta)$ with value at most u; or $\mathrm{BLP}(I, \Delta) = u = \min_D \phi^\Delta$. In the first case, each coordinate of $(\lambda^\star, \mu^\star)$ is positive if and only if the same coordinate is positive at some point in the feasibility polytope of the BLP. Therefore, the feasibility lattice of $\mathrm{AIP}^\star(I, \Delta)$ includes every possible assignment which is in the support of some feasible solution to $\mathrm{BLP}(I, \Delta)$, including integral solutions and as a consequence $\mathrm{AIP}^\star(I, \Delta) \leq \min_D \phi^\Delta \leq u$. In the second case, each coordinate of $(\lambda^\star, \mu^\star)$ is positive if and only if the same coordinate is positive at some point in the optimal polytope of the BLP. Therefore, the feasibility lattice of $\mathrm{AIP}^\star(I, \Delta)$ includes every possible assignment which is in the support of some optimal solution to $\mathrm{BLP}(I, \Delta)$, including integral solutions and as a consequence $\mathrm{AIP}^\star(I, \Delta) \leq \min_D \phi^\Delta = u$. Thus, in both cases, $\mathrm{BLP}(I, \Delta) \leq u$ and $\mathrm{AIP}^\star(I, \Delta) \leq u$ and hence Algorithm 1 indeed outputs yes, as required.

In the other direction, we want to show (by contrapositive) that if Algorithm 1 outputs yes then $\inf_C \phi^\Gamma \leq u$. Thus, assume that $\mathrm{BLP}(I, \Delta) \leq u$ and $\mathrm{AIP}^\star(I, \Delta) \leq u$. Let $(\lambda^\star, \mu^\star)$ be as in Definition 7 and denote $\mathrm{BLP}^\star(I, \Delta) := \sum_{j \in J} \sum_{t \in D^{n_j}} \lambda^\star_j(t) f^\Delta_j(t)$; observe that $\mathrm{BLP}^\star(I, \Delta) \leq u$ by the definition of $(\lambda^\star, \mu^\star)$. Let $(q^\star, r^\star)$ be a solution to $\mathrm{AIP}^\star(I, \Delta)$ with objective value at most u. Let ℓ be a positive integer such that $\ell \cdot \lambda^\star$, and $\ell \cdot \mu^\star$ are both integral, and let M be the maximum of the absolute values of the coordinates of both $q^\star$ and $r^\star$. Let us set $L := (M+1)\ell$. From $\mathrm{BLP}^\star(I, \Delta) \leq u$ and $\mathrm{AIP}^\star(I, \Delta) \leq u$ it immediately follows that

$$\frac{2(M+1)\ell}{2(M+1)\ell + 1} \mathrm{BLP}^\star(I, \Delta) + \frac{1}{2(M+1)\ell + 1} \mathrm{AIP}^\star(I, \Delta) \leq u. \tag{5}$$

We claim that

$$\min_{\left(\!\binom{D}{L+1}\!\right) \times \left(\!\binom{D}{L}\!\right)} \phi^{\mathcal{B}^{2L+1}_{B_1,B_2}(\Delta)} \leq \frac{2(M+1)\ell}{2(M+1)\ell + 1} \mathrm{BLP}^\star(I, \Delta) + \frac{1}{2(M+1)\ell + 1} \mathrm{AIP}^\star(I, \Delta) \tag{6}$$

for all the partitions $B_1 \cup B_2 = [2L+1]$ such that $|B_1| = L+1$ and $|B_2| = L$. To prove the claim, let us define, for every $i \in \{1, \dots, n\}$ and for every $a \in D$, the following nonnegative integers

$$\begin{aligned} W_{x_i,B_1}(a) &:= (M+1)\ell\mu^\star_{x_i}(a) + r^\star_{x_i}(a), \\ W_{x_i,B_2}(a) &:= (M+1)\ell\mu^\star_{x_i}(a). \end{aligned}$$

(To check that $W_{x_i,B_1}(a)$ and $W_{x_i,B_2}(a)$ are nonnegative it is enough to observe that if $\mu^\star_{x_i}(a)$ is 0 then, by Definition 7, $r^\star_{x_i}(a)$ is also 0, otherwise $\mu^\star_{x_i}(a)$ is at least $\frac{1}{\ell}$, and the positivity of $W_{x_i,B_1}(a)$, $W_{x_i,B_2}(a)$ follows by the choice of M.) Observe that for every $i \in \{1, \dots, n\}$ we have that

$$\begin{aligned} \sum_{a\in D} W_{x_i,B_1}(a) &= (M+1)\ell \sum_{a\in D} \mu^\star_{x_i}(a) + \sum_{a\in D} r^\star_{x_i}(a) = (M+1)\ell + 1 = L+1, \\ \sum_{a\in D} W_{x_i,B_2}(a) &= (M+1)\ell \sum_{a\in D} \mu^\star_{x_i}(a) = (M+1)\ell = L. \end{aligned}$$

Let $\nu \colon V \to \left(\!\binom{D}{L+1}\!\right) \times \left(\!\binom{D}{L}\!\right)$ be the map defined, for every $x_i \in V$, by $\nu(x_i) = (\alpha_i, \beta_i)$, where α_i is the multiset of $\left(\!\binom{D}{L+1}\!\right)$ that contains $W_{x_i,B_1}(a)$ many occurrences of a, for every $a \in D$, and β_i is the multiset of $\left(\!\binom{D}{L}\!\right)$ that contains $W_{x_i,B_2}(a)$ many occurrences of a, for every $a \in D$.

Let f_j be a k-ary function symbol appearing as a term of the objective function ϕ. Let us define, for every $t \in D^k$, the following nonnegative integers

$$\begin{aligned} P_{j,B_1}(t) &:= (M+1)\ell\lambda^\star_j(t) + q^\star_j(t), \\ P_{j,B_2}(t) &:= (M+1)\ell\lambda^\star_j(t). \end{aligned}$$

Observe that

$$\begin{aligned} \sum_{t\in D^k} P_{j,B_1}(t) &= (M+1)\ell \sum_{t\in D^k} \lambda^\star_j(t) + \sum_{t\in D^k} q^\star_j(t) = (M+1)\ell + 1 = L+1, \\ \sum_{t\in D^k} P_{j,B_2}(t) &= (M+1)\ell \sum_{t\in D^k} \lambda^\star_j(t) = (M+1)\ell = L. \end{aligned}$$

We write now

$$\sum_{t\in D^k} P_{j,B_1}(t) f^\Delta_j(t) = \sum_{h=1}^{(M+1)\ell+1} f^\Delta_j(\zeta^h_1, \dots, \zeta^h_k)$$

where $\zeta^1, \dots, \zeta^{(M+1)\ell+1}$ are defined to be $(M+1)\ell+1$ elements of D^k such that $P_{j,B_1}(t)$ many of them are equal to t, for every $t \in D^k$; and

$$\sum_{t\in D^k} P_{j,B_2}(t) f^\Delta_j(t) = \sum_{h=1}^{(M+1)\ell} f^\Delta_j(\xi^h_1, \dots, \xi^h_k)$$

where $\xi^1, \dots, \xi^{(M+1)\ell}$ are defined to be $(M+1)\ell$ elements of D^k such that $P_{j,B_2}(t)$ many of them are equal to t, for every $t \in D^k$.

We obtain

$$
\begin{aligned}
&\frac{2(M+1)\ell}{2(M+1)\ell+1}\lambda_j^\star(t)f_j^\Delta(t)+\frac{1}{2(M+1)\ell+1}q_j^\star(t)f_j^\Delta(t)\\
=&\frac{1}{2(M+1)\ell+1}\left(\sum_{t\in D^k}P_{j,B_1}(t)f_j^\Delta(t)+\sum_{t\in D^k}P_{j,B_2}(t)f_j^\Delta(t)\right)\\
=&\frac{1}{2L+1}\left(\sum_{h=1}^{L+1}f_j^\Delta(\zeta_1^h,\dots,\zeta_k^h)+\sum_{h=1}^{L}f_j^\Delta(\xi_1^h,\dots,\xi_k^h)\right)\\
\geq&\frac{1}{2L+1}\min_{t^1,\dots,t^k\in D^m:\{t^\ell\}_{B_1}=\zeta_\ell,\{t^\ell\}_{B_2}=\xi_\ell}\sum_{i=1}^{2L+1}f^\Delta(t_i^1,\dots,t_i^k)\\
=&f_j^{\mathcal{B}^{2L+1}_{B_1,B_2}(\Delta)}((\zeta_1,\xi_1),\dots,(\zeta_k,\xi_k))=f_j^{\mathcal{B}^{2L+1}_{B_1,B_2}(\Delta)}(\nu(x_1^j),\dots,\nu(x_k^j)),
\end{aligned}
$$

where the last equality follows because, for every $a\in D$, the number of a's in ζ_h is

$$
\begin{aligned}
&\sum_{t\in D^k:t_h=a}P_{j,B_1}(t)=(M+1)\ell\sum_{t\in D^k:t_h=a}\lambda_j^\star(t)+\sum_{t\in D^k:t_h=a}q_j^\star(t)\\
=&(M+1)\ell\mu^\star_{x_h^j}(a)+r^\star_{x_h^j}(a)=W_{x_h^j,B_1}(a),
\end{aligned}
$$

and, for every $a\in D$, the number of a's in ξ_h is

$$
\sum_{t\in D^k:t_h=a}P_{j,B_2}(t)=(M+1)\ell\sum_{t\in D^k:t_h=a}\lambda_j^\star(t)=(M+1)\ell\mu^\star_{x_h^j}(a)=W_{x_h^j,B_2}(a).
$$

This proves the claim.

From Inequalities (5) and (6) it follows that for all partitions $B_1\cup B_2=[2L+1]$ such that $|B_1|=L+1$ and $|B_2|=L$ it holds

$$
\min_{\left(\!\binom{D}{L+1}\!\right)\times\left(\!\binom{D}{L}\!\right)}\phi^{\mathcal{B}^{2L+1}_{B_1,B_2}(\Delta)}\leq u.
$$

Moreover, since there exists a block-symmetric promise fractional polymorphisms of (Δ,Γ) of arity $2L+1$ having two symmetric blocks B_1 and B_2 with respective size $L+1$ and L, Lemma 9 implies the existence of a fractional homomorphism from $\mathcal{B}^{2L+1}_{B_1,B_2}(\Delta)$ to Γ. From Proposition 2 it follows that

$$
\inf_C\phi^\Gamma\leq\min_{\left(\!\binom{D}{L+1}\!\right)\times\left(\!\binom{D}{L}\!\right)}\phi^{\mathcal{B}^{2L+1}_{B_1,B_2}(\Delta)}\leq u
$$

and this concludes the proof. □

We conclude the section with a couple of remarks showing that the number of symmetric blocks and their size do not play a crucial role in the proofs of Theorem 8 and Lemma 9. The notion of bimultiset-structure can be straightforwardly generalised to the notion of *k-multiset-structure with blocks of size $b_1,\dots,b_k$*.

Remark 11. The proof of Lemma 9 can be easily adapted to the case in which the promise template (Δ,Γ) has a block-symmetric promise fractional polymorphism ω of arity $L\in\mathbb{N}$ with an arbitrary number k of symmetric blocks of size $b_1,\dots,b_k$ (given that the blocks are the same for each map in $\mathrm{Supp}(\omega)$). In this case, the k-multiset structure $\mathcal{B}_{B_1,\dots,B_k}(\Delta)$ is fractionally homomorphic to Γ. The converse (corresponding to Remark 10) is also true.

Remark 12. Let (Δ, Γ) be a promise valued template such that Δ has a *finite domain*. Assume that (Δ, Γ) has block-symmetric promise fractional polymorphisms with arbitrarily many blocks of arbitrarily large size. Then Algorithm 1 correctly solves PVCSP(Δ, Γ). This statement can be proved by a slight modification of the proof of Theorem 8 employing a block-symmetric promise fractional polymorphism whose symmetric blocks $B_1, \dots, B_k$ have each size at least $M\ell^2$ and where the coefficients $W_{x_i, B_b(a)}$, and $P_{j,B_b}(t)$ are defined as in the proof of [20, Theorem 4.1].

Finally, we point out that for finite-domain PCSPs, the condition of having a block-symmetric promise polymorphism of arity $2L+1$ is equivalent to the one of having a block-symmetric promise polymorphism of arity $2L+1$ with two symmetric blocks of arity $L+1$ and L, for every $L \in \mathbb{N}$ [20].

4 THE COMBINED RELAXATION WITH SAMPLING FOR PVCSPS

We use the notion of a sampling algorithm for a valued structure from [12].

Definition 13. Let Γ be a valued structure with domain C and finite signature τ. A *sampling algorithm* for Γ takes as input a positive integer d and computes a finite-domain valued structure Δ fractionally homomorphic to Γ such that, for every finite sum ϕ of τ-terms having at most d distinct variables, $V = \{x_1, \dots, x_d\}$, and every $u \in \mathbb{Q}$, there exists a solution $s \colon V \to C$ with $\phi^\Gamma(s(x_1), \dots, s(x_d)) \leq u$ if and only if there exists a solution $h' \colon V \to D$ with $\phi^\Delta(h'(x_1), \dots, h'(x_d)) \leq u$. A sampling algorithm is called *efficient* if its running time is bounded by a polynomial in d. The finite-domain valued structure computed by a sampling algorithm is called a *sample*.

Example 14. A valued structure Γ with domain $\mathbb{Q}$ and signature τ is called *piecewise linear homogeneous (PLH)* if, for every $\gamma \in \tau$, the cost function γ^Γ is first-order definable over the structure $\mathfrak{L} = (\mathbb{Q}; \leq, 1, \{c\cdot\}_{c\in\mathbb{Q}})$ where

- $<$ is a relation symbol (i.e., a $\{0, \infty\}$-valued function symbol) of arity 2 and $<^\mathfrak{L}$ is the strict linear order of $\mathbb{Q}$,
- 1 is a constant symbol and $1^\mathfrak{L} := 1 \in \mathbb{Q}$, and
- $c\cdot$ is a unary function symbol for every $c \in \mathbb{Q}$ such that $(c\cdot)^\mathfrak{L}$ is the function $x \mapsto cx$, i.e., the multiplication by c.

If Γ is a PLH valued structure with a finite signature, then it admits an efficient sampling algorithm [12]. $\triangle$

Let Γ be a valued structure with a finite signature that admits an efficient sampling algorithm. Observe that for every finite-domain valued structure Δ_d computed by such an efficient sampling algorithm, the pair (Δ_d, Γ) is a promise valued template. The following lemma follows from the definition of a sampling algorithm for a valued structure.

Lemma 15. *Let (Γ_1, Γ_2) be a promise valued template with a finite signature. Assume that Γ_1 admits an efficient sampling algorithm. If* PVCSP(Δ_d, Γ_2) *is polynomial-time solvable for every finite-domain valued structure Δ_d computed by an efficient sampling algorithm for Γ_1, then* PVCSP(Γ_1, Γ_2) *is polynomial-time solvable.*

Proof. It can be verified that (Δ_d, Γ_2) is a valid promise valued template, for every $d \in \mathbb{N}$; that is, the existence of a fractional homomorphism from Δ_d to Γ_1 and a fractional homomorphism from Γ_1 to Γ_2 implies the existence of a fractional homomorphism from Δ_d to Γ_2, which is obtained by a composition of the two fractional homomorphisms in the same fashion as in the proof of Theorem 16.

Let $I := (V, \phi, u)$ be an instance of PVCSP(Γ_1, Γ_2) and let $d := |V|$. Let C_1, C_2 and D_d be the domains of Γ_1, Γ_2 and Δ_d, respectively. Assume that there is a polynomial-time algorithm solving PVCSP(Δ_d, Γ_2). If the output of such an algorithm is no then clearly the answer to PVCSP(Γ_1, Γ_2)

is NO. Otherwise, if the output of such an algorithm is YES, then by the definition of sampling algorithm the answer to PVCSP(Γ_1, Γ_2) is YES. □

THEOREM 16. *Let* (Γ_1, Γ_2) *be a promise valued template with a finite signature. Assume that* Γ_1 *admits an efficient sampling algorithm. Moreover, assume that* (Γ_1, Γ_2) *has a block-symmetric promise fractional polymorphism of arity* $2L+1$ *with two symmetric blocks of size* $L+1$ *and* L*, respectively, for all* $L \in \mathbb{N}$*. Then* PVCSP(Γ_1, Γ_2) *is polynomial-time solvable.*

Note that in Theorem 16 (and in Lemma 15) *both* the respective domains of the valued structures Γ_1 and Γ_2 have arbitrary (countable) cardinality, that is, each of Γ_1 and Γ_2 can have a finite or an infinite domain.

PROOF. For every positive integer d let Δ_d be the finite-domain valued structure Δ_d computed by an efficient sampling algorithm for Γ_1 on input d. By Lemma 15, to prove that PVCSP(Γ_1, Γ_2) is polynomial-time solvable it is enough to prove that PVCSP(Δ_d, Γ_2) is polynomial-time solvable for every positive integer d. We claim that for every $d, L \in \mathbb{N}$ there exists a block-symmetric promise fractional polymorphism of (Δ_d, Γ_2) with arity $2L+1$ having two symmetric blocks of size $L+1$ and L, respectively. By Theorem 8, proving our claim will imply that PVCSP(Δ_d, Γ_2) is polynomial-time solvable and therefore, by Lemma 15, that PVCSP(Γ_1, Γ_2) is polynomial-time solvable.

We now prove the claim. Let C_1, C_2 and D_d be the domains of Γ_1, Γ_2 and Δ_d, respectively, and let τ be the signature of Γ_1 and Γ_2 (by the definition of sampling algorithm it is also the signature of Δ_d). By the definition of sampling algorithm, there exits a fractional homomorphism χ from Δ_d to Γ_1 and, by assumption, there exists a block-symmetric promise fractional polymorphism $\omega := (\omega_I, \omega_O)$ of (Γ_1, Γ_2) with two symmetric blocks B_1 and B_2 having size $L+1$ and L, respectively. For every $h \in \mathrm{Supp}(\chi) \subseteq C_1^{D_d}$ and every $g \in \mathrm{Supp}(\omega_O) \subseteq C_2^{C_1^{2L+1}}$ we define the map $g \circ h \colon D_d^{2L+1} \to C_2$ by setting, for every $a^1, \dots, a^m \in D_d^{2L+1}$,

$$g \circ h(a^1, \dots, a^{2L+1}) = g\left(h(a^1), \dots, h(a^{2L+1})\right).$$

Observe that $g \circ h$ is a block-symmetric map with symmetric block B_1 and B_2, because so is g. We define the discrete probability measure μ_O on $C_2^{D_d^{2L+1}}$ as follows

$$\mu_O(Y) = \sum_{h \in \mathrm{Supp}(\chi)} \sum_{\substack{g \in \mathrm{Supp}(\omega_O): \\ g \circ h \in Y}} \chi(h)\omega_O(g), \qquad \text{for every } Y \subseteq D_d^{2L+1}.$$

Since $\sum_{h \in \mathrm{Supp}(\chi)} \chi(h)$ and $\sum_{g \in \mathrm{Supp}(\omega_O)} \omega_O(g)$ are absolutely convergent series, it follows by Martens' Theorem (see, e.g., [45, Theorem 6.57]) that μ_O is well defined. Observe that μ_O satisfies the countable additivity property, since ω_O and χ do. Furthermore, $\mathrm{Supp}(\mu_O) = \{(g \circ h) \mid h \in \mathrm{Supp}(\chi), g \in \mathrm{Supp}(\omega_O)\}$ is countable and, again by Martens' Theorem, it holds

$$\sum_{g' \in \mathrm{Supp}(\mu_O)} \mu_O(g') = \sum_{h \in \mathrm{Supp}(\chi)} \chi(h) \sum_{g \in \mathrm{Supp}(\omega_O)} \omega_O(g) = 1.$$

Let μ_I be the discrete probability measure in $\mathscr{J}_D^{(2L+1)}$ such that $\mu_I(e_i^{(2L+1)}) = \frac{1}{2L+1}$. We now show that $\mu := (\mu_I, \mu_O)$ is a promise fractional polymorphism of (Δ_d, Γ_1) with arity $2L+1$. By definition, μ is block-symmetric with two symmetric blocks of respective size $L+1$ and L.

For every $f \in \tau$ and every $a^1, \ldots, a^{2L+1} \in D_d^{\mathrm{ar}(f)}$ it holds that

$$\begin{aligned}
&\sum_{g' \in \mathrm{Supp}(\mu_O)} \mu_O(g') f^{\Gamma_2}\left(g(a^1, \ldots, a^{2L+1})\right) \\
&= \sum_{h \in \mathrm{Supp}(\chi)} \chi(h) \sum_{g \in \mathrm{Supp}(\omega_O)} \omega_O(g) f^{\Gamma_2}\left(g(h(a^1), \ldots, h(a^{2L+1}))\right) \\
&\leq \sum_{h \in \mathrm{Supp}(\chi)} \chi(h) \frac{1}{2L+1} \sum_{i=1}^{2L+1} f^{\Gamma_1}\left(g(h(a^i))\right) && (7) \\
&= \frac{1}{2L+1} \sum_{i=1}^{2L+1} \sum_{h \in \mathrm{Supp}(\chi)} \chi(h) f^{\Gamma_1}\left(h(a^i)\right) \\
&\leq \frac{1}{2L+1} \sum_{i=1}^{2L+1} f^{\Delta_d}(a^i), && (8)
\end{aligned}$$

where Inequality (7) holds because ω is a promise fractional polymorphism of (Γ_1, Γ_2) and Inequality (8) holds because χ is a fractional homomorphism from Δ_d to Γ_1. We obtained that μ is a promise fractional polymorphism of (Δ_d, Γ_2) and this concludes the proof. □

In the particular case in which $\Gamma_1 = \Gamma_2$, from Theorem 16 we obtain the following result for infinite-domain (non-promise) VCSPs.

Corollary 17. *Let* Γ *be a valued structure with a finite signature that admits an efficient sampling algorithm. Assume that* Γ *has a block-symmetric fractional polymorphism of arity* $2L + 1$ *with two symmetric blocks of size* $L + 1$ *and* L*, respectively, for all* $L \in \mathbb{N}$*. Then* $\mathrm{VCSP}(\Gamma)$ *is polynomial-time solvable.*

Corollary 17 and the existence of an efficient sampling algorithm for PLH valued structures (see Example 14) imply the following tractability result for (wMA, avg)-convex PLH valued structures.

Corollary 18. *Let* wMA *be the* 2*-period centred weighted moving average. Let* Γ *be a PLH valued structure with a finite signature such that every cost function from* Γ *is a* (wMA, avg)*-convex cost function (see Example 5). Then* $\mathrm{VCSP}(\Gamma)$ *is polynomial-time solvable.*

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A OMITTED PROOFS FROM SECTION 2

Let μ be a discrete probability measure as defined in Section 2. We note that countable additivity implies that $\mu(\emptyset) = 0$ and that $\mu(X_1) \leq \mu(X_2)$ for every $X_1 \subseteq X_2$, i.e., μ is monotone. We also remark that while the set X might be countable or uncountable, the terminology *discrete probability measure* refers to the fact that the σ-algebra is $\mathcal{P}(X)$, i.e., the discrete topology on X (see [47, Examples 2.8 and 2.9]).

Proposition (Proposition 1 restated)**.** Let μ be a discrete probability measure on a set X. Then $\mathrm{Supp}(\mu)$ is a countable set. Furthermore, if X is countable then $\sum_{x \in \mathrm{Supp}(\mu)} \mu(x) = 1$; that is, $\mathrm{Supp}(\mu)$ is non-empty.

Proof. Let us define, for all $n \in \mathbb{N}$, the subset $A_n := \left\{x \in X \mid \mu(\{x\}) \geq \frac{1}{n}\right\}$. We can write the support of μ as $\mathrm{Supp}(\mu) = \bigcup_{n\in\mathbb{N}} A_n$. Observe that, for every $n \in \mathbb{N}$, the set A_n is finite. If this was not the case, then there would exist a sequence $(x_i)_{i\in\mathbb{N}} \in (A_n)^{\mathbb{N}}$ of pairwise distinct elements and, by the countable additivity property, we would obtain that $\mu(\{x_i \mid i \in \mathbb{N}\}) = \sum_{i\in\mathbb{N}} \mu(\{x_i\}) \geq \sum_{i\in\mathbb{N}} \frac{1}{n} = +\infty$, which contradicts the assumption that $\mu(\{x_i \mid i \in \mathbb{N}\}) \leq \mu(X) = 1$. Therefore, $\mathrm{Supp}(\mu)$ is countable, because it is the countable union of finite sets. □

Proposition (Proposition 2 restated)**.** Let Γ and Δ be valued structures over the same signature τ with domains C and D, respectively. Assume $\Delta \to_f \Gamma$. Let $V = \{v_1, \dots, v_n\}$ be a set of variables and ϕ a sum of finitely many τ-terms with variables from V. For every $u \in \mathbb{Q}$, if there exists an assignment $s \colon V \to D$ such that $\phi^\Delta(s(v_1, \dots, s(v_n)) \leq u$, then there exists an assignment $s' \colon V \to C$ such that $\phi^\Gamma(s'(v_1), \dots, s'(v_n)) \leq u$. In particular, it holds that

$$\inf_C \phi^\Gamma \leq \inf_D \phi^\Delta.$$

Proof. Let $\phi(v_1, \dots, v_n) := \sum_{j\in J} \gamma_j(v^j)$, where $\gamma_j \in \tau$ and $v^j \in V^{ar(\gamma_j)}$; and let $u \in \mathbb{Q}$. Let χ be a fractional homomorphism from Δ to Γ and let $s \colon V \to D$ be an assignment with cost $\phi^\Delta(s(v_1), \dots, s(v_n)) \leq u$. Then, by the definition of fractional homomorphism,

$$\sum_{h\in\mathrm{Supp}(\chi)} \chi(h) \sum_{j\in J} \gamma_j^\Gamma(h(s(v^j))) = \sum_{j\in J} \sum_{h\in\mathrm{Supp}(\chi)} \chi(h)\gamma_j^\Gamma(h(s(v^j))) \leq \sum_{j\in J} \gamma_j^\delta(s(v^j)) \leq u.$$

Therefore, there exists at least one map $h\colon D \to C$ from $\mathrm{Supp}(\chi)$, such that $\phi^\Gamma(h \circ s(v_1), \ldots, h \circ s(v_n)) \leq \phi^\Delta(s(v_1), \ldots, s(v_n)) \leq u$, that is, $h \circ s\colon V \to C$ is an assignment with cost at most u. □

Lemma (Lemma 6 restated)**.** Let (Δ, Γ) be a promise valued template and let $m \in \mathbb{N}$. If $\omega = (\omega_I, \omega_O)$ is an m-ary block symmetric promise fractional polymorphism of (Δ, Γ), then also $\omega' = (\omega_I', \omega_O)$, where $\omega_I'(e_i^{(m)}) = \frac{1}{m}$ for $1 \leq i \leq m$, is an m-ary block-symmetric promise fractional polymorphism of (Δ, Γ).

Proof. Let C and D be the domains of Γ and Δ, respectively, and let τ be the common signature of the two valued structures. By the definition of block-symmetric promise fractional polymorphism, there exists a partition $B_1 \cup \cdots \cup B_k$ of $[m]$ such that for every permutation $\pi \in S_m$ that preserves the blocks $B_1, \ldots, B_k$, every $f \in \tau$ and every $a^1, \ldots, a^m \in D^{ar(f)}$ it holds that

$$\sum_{g \in \mathrm{Supp}(\omega_O)} \omega_O(g) f^\Gamma(g(a^1, \ldots, a^m)) \leq \sum_{i=1}^m \omega_I(e_i^{(m)}) f^\Delta(a^{\pi(i)}).$$

By summing the last inequality over all $\pi \in S_m$ that preserve all the blocks $B_1, \ldots, B_k$, i.e., over all $\pi = \pi^1 \cdots \pi^k$ such that π^j is a permutation of the elements in B_j for $1 \leq i \leq k$, we obtain

$$\begin{aligned}\prod_{j=1}^k |B_j|! \sum_{g \in \mathrm{Supp}(\omega_O)} f^\Gamma(g(a^1, \ldots, a_m)) \;&\leq\; \sum_{j=1}^k \prod_{h \neq j} |B_h|! \sum_{i \in B_j} \omega_I(e_i^{(m)}) \left((|Bj| - 1)! \sum_{\ell \in B_j} f^\Delta(a^\ell) \right) \\ &= \prod_{j=1}^k |B_j|! \sum_{j=1}^k \frac{1}{|B_j|} \sum_{i \in B_j} \omega_I(e_i^{(m)}) \sum_{\ell \in B_j} f^\Delta(a^\ell).\end{aligned}$$

Since, by the definition of block-symmetric promise fractional polymorphism, $\sum_{i \in B_j} \omega_I(e_i^{(m)}) = \frac{|B_j|}{m}$ for every $j \in \{1, \ldots, k\}$, we obtain

$$\sum_{g \in \mathrm{Supp}(\omega_O)} f^\Gamma(g(a^1, \ldots, a^m)) \leq \frac{1}{m} \sum_{i=1}^m f^\Delta(a^\ell),$$

that is, ω' is a promise fractional polymorphism of (Δ, Γ). □

B THE BLP RELAXATION

A fractional polymorphism of a valued structure Γ is *fully symmetric* if every operation in its support is a fully symmetric map. Given a promise valued template (Δ, Γ), a promise fractional polymorphism $\omega = (\omega_I, \omega_O)$ of (Δ, Γ) is *fully symmetric* if every map in $\mathrm{Supp}(\omega_O)$ is fully symmetric.

The following lemma is a corollary of Lemma 6.

Lemma 19. *Let (Δ, Γ) be a promise valued template and let $m \in \mathbb{N}$. If $\omega = (\omega_I, \omega_O)$ is an m-ary fully symmetric promise fractional polymorphism of (Δ, Γ), then also $\omega' = (\omega_I', \omega_O)$, where $\omega_I'(e_i^{(m)}) = \frac{1}{m}$ for $1 \leq i \leq m$, is an m-ary fully symmetric promise fractional polymorphism of (Δ, Γ).*

In view of Lemma 19, we will assume without loss of generality that any m-ary fully symmetric promise fractional polymorphism $\omega = (\omega_I, \omega_O)$ is such that ω_I assign $\frac{1}{m}$ to each m-ary projection on the domain of Δ and we will identify ω with ω_O.

Recall the BLP relaxation from Section 2.5. Let (Δ, Γ) be a promise valued template such that the domain of Δ is a *finite* set. We may solve $\mathrm{PVCSP}(\Delta, \Gamma)$ by using the following algorithm that computes a BLP relaxation of Δ.

ALGORITHM 2: BLP Relaxation Algorithm for PVCSP(Δ, Γ)

Input:
$I := (V, \phi, u)$, a valid instance of PVCSP(Δ, Γ)

Output:
YES if there exists an assignment $s\colon V \to \mathrm{dom}(\Delta)$ such that $\phi^\Delta(s(x_1), \dots, s(x_{|V|})) \leq u$
NO if there is no assignment $s\colon V \to \mathrm{dom}(\Gamma)$ such that $\phi^\Gamma(s(x_1), \dots, s(x_{|V|})) \leq u$

BLP(I, Δ);
if BLP$(I, \Delta) \nleq u$ **then**
    output NO;
else
    output YES;
end

Algorithm 2 runs in polynomial time in I, and that if it outputs NO, then indeed the answer to PVCSP(Δ, Γ) is NO, without further assumptions.

We now present a sufficient condition under which Algorithm 2 correctly solves PVCSP(Δ, Γ).

THEOREM 20. *Let (Δ, Γ) be a promise valued template such that Δ has a* finite domain. *Assume that for all $m \in \mathbb{N}$ there exists a fully symmetric promise fractional polymorphisms of (Δ, Γ) with arity m. Then Algorithm 2 correctly solves* PVCSP(Δ, Γ) *(in polynomial time).*

To prove Theorem 20 we need to use a preliminary lemma and the notion of a *multiset-structures*.

Let Δ be a valued τ-structure with domain D and let $m \in \mathbb{N}$. The *multiset-structure* $\mathcal{P}^m(\Delta)$ is the valued structure with domain $\left(\!\binom{D}{m}\!\right)$ i.e., the set whose elements α are multisets of elements from D of size m. For every k-ary function symbol $f \in \tau$, and $\alpha_1, \dots, \alpha_k \in \left(\!\binom{D}{m}\!\right)$ the function $f^{\mathcal{P}^m(\Delta)}$ is defined as follows

$$f^{\mathcal{P}^m(\Delta)}(\alpha_1, \dots, \alpha_k) := \frac{1}{m} \min_{\substack{t^1, \dots, t^k \in D^m: \\ \{t^\ell\} = \alpha_\ell}} \sum_{i=1}^{m} f^\Delta(t_i^1, \dots, t_i^k),$$

where $\{t^\ell\}$ is the multiset of the coordinates of t^ℓ, i.e., the multiset $\{t_i^\ell \mid 1 \leq i \leq m\}$.

LEMMA 21. *Let (Δ, Γ) be a promise valued template such that Δ has a* finite domain. *Let $m \in \mathbb{N}$ and let us assume that (Δ, Γ) has a fully symmetric promise fractional polymorphism of arity m. Then $\mathcal{P}^m(\Delta)$ is fractionally homomorphic to Γ.*

Lemma 21 is a generalisation of [51, Lemma 2.2] to promise valued templates (Δ, Γ) where Δ is a finite-domain valued structure. Furthermore, it already appeared in a similar form in [12, Lemma 6.9].[7]

PROOF OF LEMMA 21. Let C be the (possibly infinite) domain of Γ, let D be the finite domain of Δ, and let τ be the common signature of Γ and Δ. Let ω be the m-ary fully symmetric promise fractional polymorphism of (Δ, Γ). For every $g \in \mathrm{Supp}(\omega) \subseteq C^{D^m}$ we define $\tilde{g}\colon \left(\!\binom{D}{m}\!\right) \to C$ by setting, for every $\alpha = \{\xi^1, \dots, \xi^m\} \in \left(\!\binom{D}{m}\!\right)$,

$$\tilde{g}(\alpha) = g(\xi^1, \xi^2, \dots, \xi^m).$$

[7]In [12] the assumption of having an m-ary fully-symmetric promise fractional polymorphism of (Δ, Γ) is replaced by the hypothesis of having an m-ary fully-symmetric fractional polymorphism of Γ. Another difference is that in [12] the notion of fractional polymorphism employed allows only finitely supported probability measures.

Observe that $\tilde{g}$ is well defined as g is fully symmetric. We define the discrete probability measure χ on $C^{\left(\!\binom{D}{m}\!\right)}$ as follows

$$\chi(Y) = \sum_{\substack{g\in \mathrm{Supp}(\omega):\\ \tilde{g}\in Y}} \omega(g), \qquad \text{for every } Y \subseteq \left(\!\!\binom{D}{m}\!\!\right).$$

Observe that χ satisfies the countable additivity property, since ω does. Furthermore, $\mathrm{Supp}(\chi) = \{h \in C^{D^m} \mid h = \tilde{g} \text{ for some } g \in \mathrm{Supp}(\omega)\} = \{\tilde{g} \in C^{D^m} \mid g \in \mathrm{Supp}(\omega)\}$ is countable and it holds that

$$\sum_{h\in \mathrm{Supp}(\chi)} \chi(h) = \sum_{g\in \mathrm{Supp}(\omega)} \omega(g) = 1.$$

We claim that χ is a fractional homomorphism from $\mathcal{P}^m(\Delta)$ to Γ. Indeed, for every $f \in \tau$ and every tuple $(\alpha_1, \dots, \alpha_k) \in \left(\!\binom{D}{m}\!\right)^k$ with $\alpha_i := \{\xi_i^1, \dots, \xi_i^m\}$ and $k := \mathrm{ar}(f)$, it holds that

$$\begin{aligned}
&\sum_{h\in C^{\left(\!\binom{D}{m}\!\right)}} \chi(h) f^\Gamma(h(\alpha_1)\dots, h(\alpha_k))\\
=&\sum_{g\in \mathrm{Supp}(\omega)} \omega(g) f^\Gamma(g(\xi_1^1,\dots,\xi_1^m),\dots,g(\xi_k^1,\dots,\xi_k^m))\\
=&\sum_{g\in \mathrm{Supp}(\omega)} \omega(g) f^\Gamma(g(\xi_1^{\pi_1(1)},\dots,\xi_1^{\pi_1(m)}),\dots,g(\xi_k^{\pi_k(1)},\dots,\xi_k^{\pi_k(m)})) &(9)\\
\leq&\frac{1}{m}\sum_{i=1}^m f^\Delta(\xi_1^{\pi_1(i)},\dots,\xi_k^{\pi_k(i)}) &(10)
\end{aligned}$$

for every $\pi_1, \dots, \pi_k \in S_m$. Equality (9) holds because the maps $g \in \mathrm{Supp}(\omega)$ are fully symmetric and Inequality (10) holds because ω is a promise fractional polymorphism of (Δ, Γ). Then, in particular, we obtain

$$\begin{aligned}
&\sum_{h\in C^{\left(\!\binom{D}{m}\!\right)}} \chi(h) f^\Gamma(h(\alpha_1,\dots,\alpha_k)) \leq \frac{1}{m}\min_{\substack{t^1,\dots,t^k\in D^m:\\ \{t^\ell\}=\alpha_\ell}} \sum_{i=1}^m f^\Delta(t_i^1,\dots t_i^k)\\
=&f^{\mathcal{P}^m(\Delta)}(\alpha_1,\dots,\alpha_k).
\end{aligned}$$

□

With Lemma 21, the proof of Theorem 20 follows the same argument as in the finite-domain non-promise case [51, Theorem 3.2]. We include the proof here for completeness.

Proof of Theorem 20. Let C be the (possibly infinite) domain of Γ and let D be the finite domain of Δ. Let τ be the common signature of Δ and Γ. Let I be an instance of PVCSP(Δ, Γ) with variables $V = \{x_1, \dots, x_n\}$, objective function $\phi(x_1, \dots, x_n) = \sum_{j\in J} \gamma_j(x^j)$ where J is a finite set of indices, $\gamma_j \in \Gamma$, and $x^j \in V^{\mathrm{ar}(j)}$, and threshold u. Note that if $\mathrm{BLP}(I, \Delta) \not\leq u$ (this also includes the case $\mathrm{BLP}(I, \Delta) = +\infty$, i.e., the case that I is not feasible) then $\min_D \phi^\Delta \not\leq u$. We may therefore safely output no. Otherwise, $\mathrm{BLP}(I, \Delta) \leq u$ implies that $\inf_C \phi^\Gamma \leq \mathrm{BLP}(I, \Delta)$. The proof of this last statement is contained in the first part of the proof of [51, Theorem 3.2]; we include it here for completeness. Let $(\lambda^\star, \mu^\star)$ be an optimal solution to $\mathrm{BLP}(I, \Delta)$ and let M be a positive integer such that $M \cdot \lambda^\star$, and $M \cdot \mu^\star$ are both integral. Let $\nu\colon V \to \left(\!\binom{D}{M}\!\right)$ be defined by mapping the variable x_i to the multiset in which the elements are distributed accordingly to $\mu^\star_{x_i}$, i.e., for every $a \in D$ the number of occurrences of a in $v(x_i)$ is equal to $M\mu^\star_{x_i}(a)$. Let f_j be a k-ary function symbol in τ that occurs in a term $f_j(x^j)$ of the objective function ϕ. Now we write

$$M \cdot \sum_{t \in D^k} \lambda^\star_j(t) f^\Delta_j(t) = f^\Delta_j(\alpha^1) + \cdots + f^\Delta_j(\alpha^M),$$

where the $\alpha^i \in D^k$ are such that $\lambda^\star_j(t)$-fractions are equal to t. Let us define $\alpha'_\ell := (\alpha^1_i, \ldots, \alpha^M_i)$ for $1 \leq i \leq k$. We get

$$\begin{aligned}
&\sum_{t \in D^k} \lambda^\star_j(t) f^\Delta_j(t) = \frac{1}{M} \sum_{i=1}^{M} f^\Delta_j(\alpha^i) = \frac{1}{M} \sum_{i=1}^{M} f^\Delta_j(\alpha^i_1, \ldots, \alpha^i_k) \\
\geq &\frac{1}{M} \min_{\substack{t^1, \ldots, t^k \in D^M: \\ \{t^\ell\} = \{\alpha'_\ell\}}} \sum_{i=1}^{M} f^\Delta_j(t^1_i, \ldots, t^k_i) = f_j^{\mathcal{P}^M(\Delta)}(\alpha'_1, \ldots, \alpha'_k) \\
= &f_j^{\mathcal{P}^M(\Delta)}(v(x)),
\end{aligned}$$

where the last equality follows as the number of a's in α'_i is

$$M \cdot \sum_{t \in D^k : t^i = a} \lambda^\star_j(t) = M \cdot \mu^\star_{x_i}(a).$$

Then

$$\begin{aligned}
\mathrm{BLP}(I, \Delta) &= \sum_{j \in J} \sum_{t \in D^{\mathrm{ar}(f_j)}} \lambda^\star_j(t) f^\Delta_j(t) = \sum_{j \in J} \left(\sum_{t \in D^{\mathrm{ar}(f_j)}} \lambda^\star_j(t) f^\Delta_j(t) \right) \\
&\geq \sum_{j \in J} \left(f_j^{\mathcal{P}^M(\Delta)}(v(x)) \right) \geq \min_{\left(\!\binom{D}{m}\!\right)} \phi^{\mathcal{P}^M(\Delta)}.
\end{aligned}$$

Since we assumed $\mathrm{BLP}(I, \Delta) \leq u$, we obtain $\min_{\left(\!\binom{D}{m}\!\right)} \phi^{\mathcal{P}^M(\Delta)} \leq u$. Moreover, since Γ has fully symmetric fractional polymorphisms of all arities, Lemma 21 implies the existence of a fractional homomorphism ω from $\mathcal{P}^M(\Delta)$ to Γ. From Proposition 2 it follows that

$$\inf_C \phi^\Gamma \leq \min_{\left(\!\binom{D}{m}\!\right)} \phi^{\mathcal{P}^M(\Delta)} \leq \mathrm{BLP}(I, \Delta) \leq u.$$

□