Ex 2.1

(a) First, lets show $=$. We'll do this by decomposing the LHS as:

$R_1 =$ $:: a \mapsto \{(b,b,a) \mid b \in A\}$

$R_2 =$ $:: (a,b,c) \mapsto \begin{cases} a & \text{if } b = c \\ \{\} & \text{otherwise} \end{cases}$

Then: $= R_2 \circ R_1$

$:: a \xmapsto{R_1} \{(b,b,a) \mid b \in A\} \xmapsto{R_2} \{b \mid b = a\} = \{a\}$

and both map $a \mapsto \{a\}$.

Hence $=$.

For $=$, we have:

LHS $:: * \xmapsto{\cup} (a,a)$

RHS $:: * \xmapsto{\cup} (a,a) \xmapsto{\times} (a,a)$.

The other two equations in 4.11 are proven by flipping the relations above.

(Extra stuff on this page)

n.b. We can also prove $\cup\cap = |$ by writing the diagram formula for the LHS and following the recipe from PQP (p.65).

Step 1 Write LHS as diagram formula:

Let $R = \cup\cap$, then: $R^{A_3}_{A_1} = \cup^{A_3 A_2} \cap_{A_2 A_1}$

Step 2 Replace labels with set elems:

$$R^{a_3}_{a_1} = \cup^{a_3 a_2} \cap_{a_2 a_1} \quad \text{for } a_1, a_2, a_3 \in A$$

Step 3:

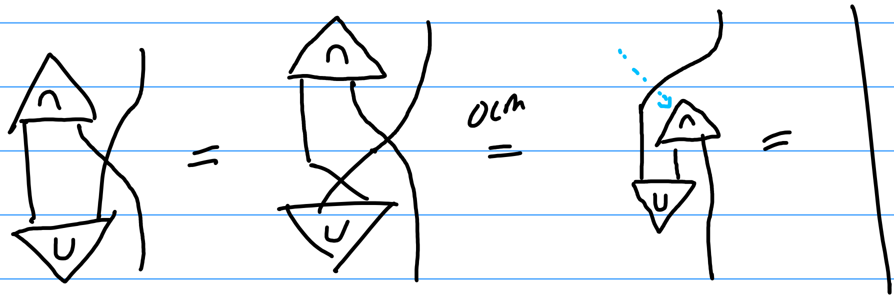
$$\cup\cap :: a_1 \mapsto a_3 \iff \exists a_2 \in A. \left(\begin{matrix} \cup :: (a_3, a_2) \mapsto * \\ \cap :: * \mapsto (a_2, a_1) \mapsto * \end{matrix} \right)$$

$$\iff \exists a_2.\ a_1 = a_2 = a_3$$

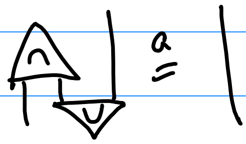
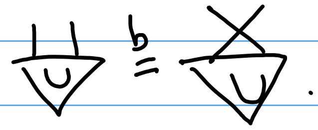
$$\iff a_1 = a_3$$

Since $\cup\cap :: a \mapsto a \ \forall a \in A$, $\cup\cap = |$.

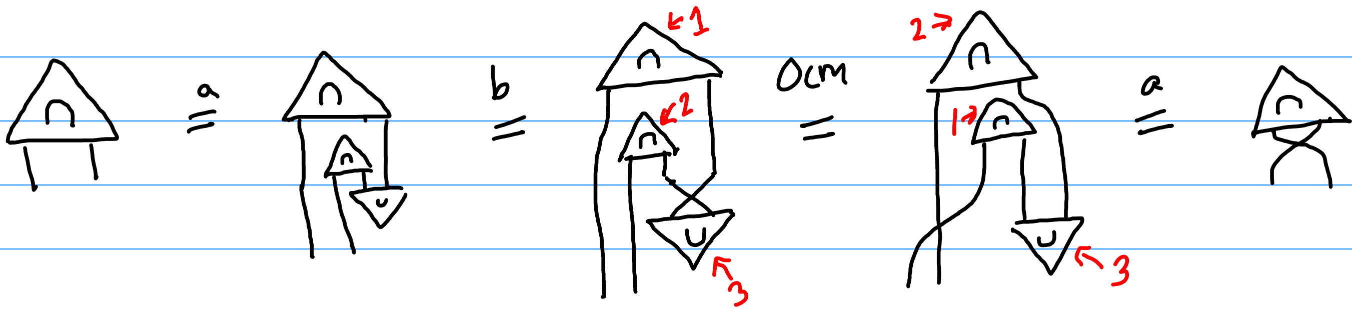
Ex 2.2

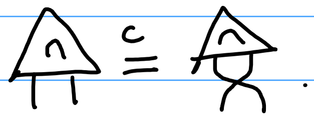


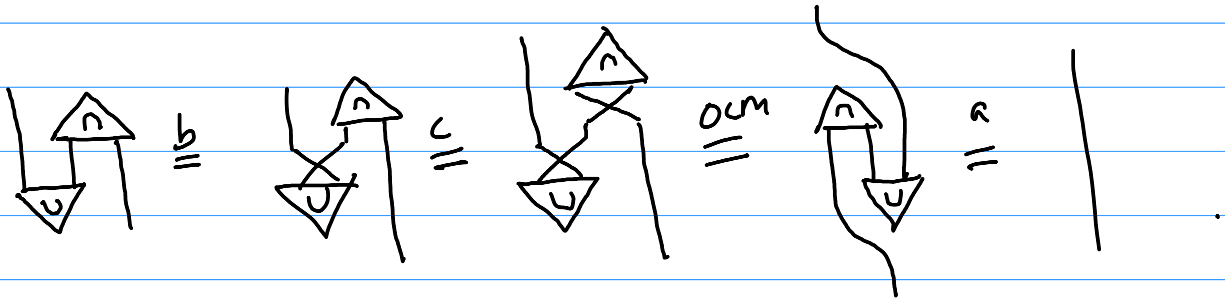
Ex 2.3

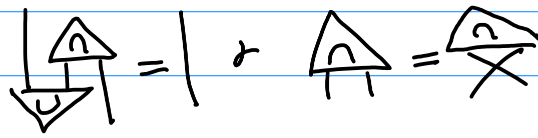
Assume (i)  and .

Then:



so we have . From this, we can show:



So we have shown (ii)  follows from (i). The proof that (ii) $\Rightarrow$ (i) is the same, but with all diagrams flipped vertically (or horizontally!).

Ex 2.4

Thm The following are equivalent:

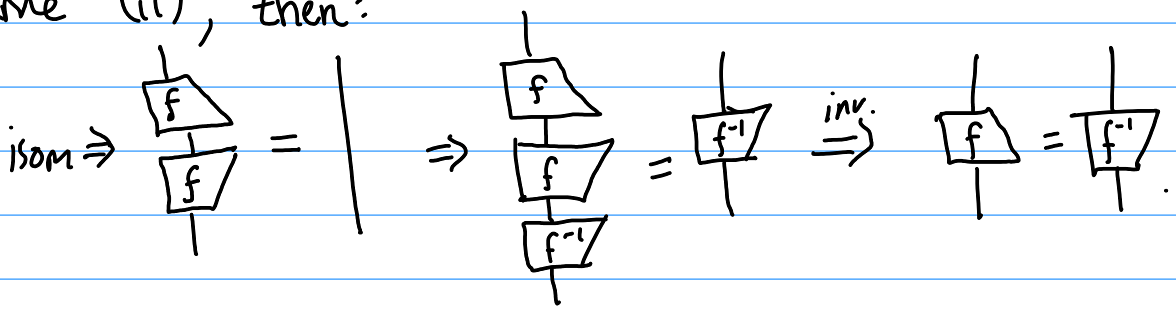
(i) f is a unitary

(ii) f is an isometry & has an inverse

(iii) $f^\dagger$ is an isometry & has an inverse

Pf If f is a unitary, then it is an isometry (by definition) & $f^{-1} = f^\dagger$, so (i) $\Rightarrow$ (ii).

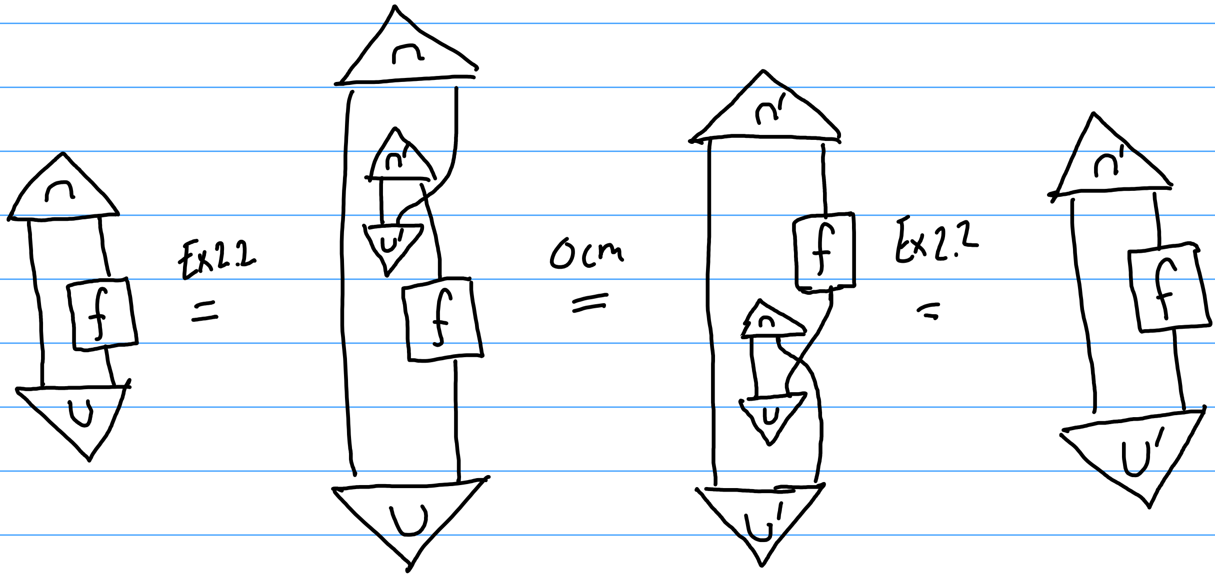
Assume (ii), then:



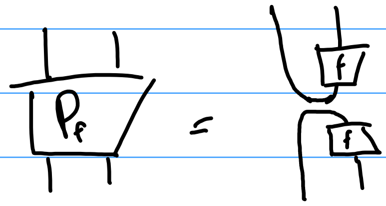
Since $f^\dagger = f^{-1}$, f is a unitary. Hence (i) $\Leftrightarrow$ (ii).

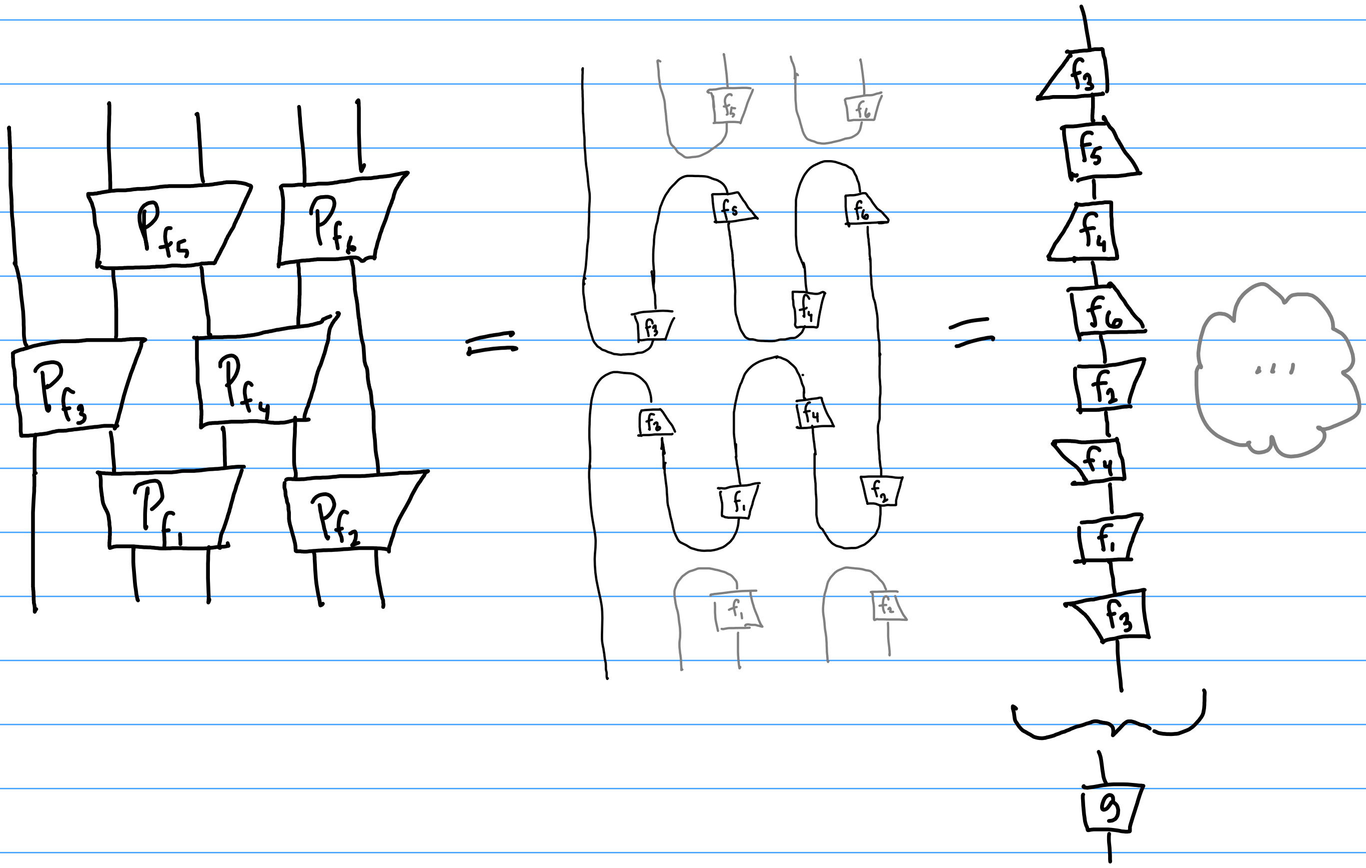
Now, f is a unitary iff $f^\dagger$ is unitary. Hence (i) $\Leftrightarrow$ (iii) by the same proof. ☒

Ex 2.5

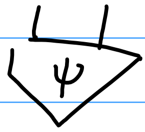


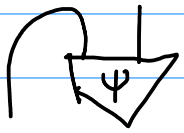
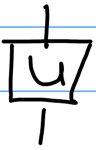
Ex 2.6

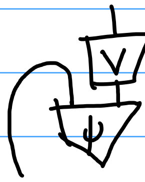
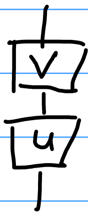
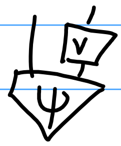

Pf
=
f
f


Pf5
Pf6
Pf3
Pf4
Pf1
Pf2
=
f5
f6
f3
f4
f1
f2
=
f3
f5
f4
f6
f2
f4
f1
f3
g

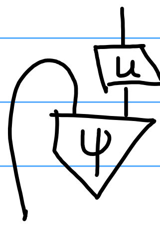
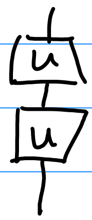
Ex 2.7

Suppose  is maximally non-sep.

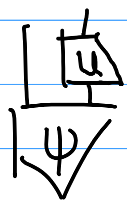
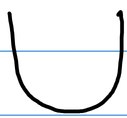
Then  $\approx$  for some unitary U.

For any unitary V,  $\approx$  is also unitary. Hence  is also maximally non-sep.

If we let $V := \overbrace{U^\dagger}^{\text{also unitary}}$, then:

 $\approx$  $=$  .

By bending the input wire up, we get:

 $\approx$  .